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Elina Shishkina

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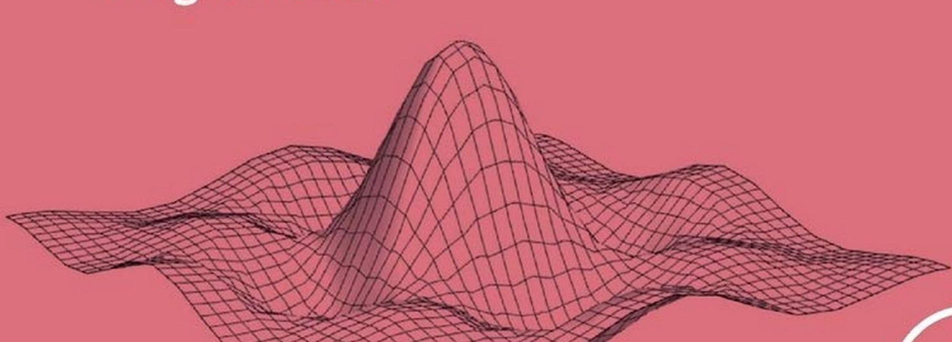

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Contents

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Introduction

Transmutation operators theory is one of the attempts to create a general approach to different mathematical problems. Let us start with the main definition.

Definition 1. *For a given pair of operators* (A, B)*, an operator* T *is called a transmutation (or intertwining) operator if on elements of some functional spaces the following property is valid:*

$$T A = B T. \tag{1}$$

It is obvious that the notion of transmutation is a direct and far reaching generalization of the matrix similarity from linear algebra. But transmutations do not reduce to similar operators because intertwining operators often are not bounded in classical spaces and the inverse operator may not exist or not be bounded in the same space. As a consequence, spectra of intertwining operators are not the same as a rule. Moreover, transmutations may be unbounded. It is the case for the Darboux transformations which are defined for a pair of differential operators and are differential operators themselves; in this case all three operators are unbounded in classical spaces. But the theory of the Darboux transformations is included in transmutation theory too. Also, a pair of intertwining operators may not be differential ones. In transmutation theory there are problems for the following various types of operators: integral, integro-differential, difference-differential (e.g., the Dunkl operator), differential or integro-differential of infinite order (e.g., in connection with Schur's lemma), general linear operators in functional spaces, and pseudodifferential and abstract differential operators.

All classical integral transforms due to Definition 1 are also special cases of transmutations; they include the Fourier, Petzval (Laplace), Mellin, Hankel, Weierstrass, Kontorovich–Lebedev, Meijer, Stankovic, Obrechkoff, finite Grinberg, and other transforms.

In quantum physics, in the study of the Schrödinger equation and inverse scattering theory, the underlying transmutations are called wave operators.

The commuting operators are also a special class of transmutations. The most important class consists of operators commuting with derivatives. In this case transmutations as commutants are usually in the form of formal series or pseudodifferential or infinite order differential operators. Finding commutants is directly connected with finding all transmutations in the given functional space. For these problems works a theory of operator convolutions, including the Berg–Dimovski convolutions [89]. Also, more and more applications are developed that are connected with the transmutation theory for commuting differential operators; such problems are based on

classical results of J. L. Burchnall and T. W. Chaundy. The transmutations are also connected with factorization problems for integral and differential operators. A special class of transmutations are the so-called Dirichlet-to-Neumann and Neumann-to-Dirichlet operators, which link together solutions of the same equation but with different kinds of boundary conditions.

How do transmutations usually work? Suppose we study properties for a rather complicated operator A. But suppose also that we know the corresponding properties for a more simple model operator B and transmutation (1) readily exists. Then we usually may copy results for the model operator B to corresponding ones for the more complicated operator A. This is the main idea of transmutations.

Let us consider for example an equation $Au = f$. Then applying to it a transmutation with property (1), we consider a new equation $Bv = g$, with $v = Tu$, $g = Tf$. So if we can solve the simpler equation $Bv = g$, then the initial one is also solved and has solution $u = T^{-1}v$. Of course, it is supposed that the inverse operator exists and its explicit form is known. This is a simple application of the transmutation technique for finding and proving formulas for solutions of ordinary and partial differential equations.

The monographs [51,571] are completely devoted to transmutation theory and its applications (note also the author's survey [532]). Moreover, essential parts of monographs [56,89,252,259], among others, include material on transmutations; the complete list of books which investigate some transmutational problems is now near of 100 items.

We use the term "transmutation" due to [53]: "Such operators are often called transformation operators by the Russian school (Levitan, Naimark, Marchenko, etc.), but transformation seems a too broad term, and since some of the machinery seems 'magical' at times, we have followed Lions and Delsarte in using the word '*transmutation*'."

Now transmutation theory is a completely formed part of the mathematical world in which methods and ideas from different areas are used, i.e., differential and integral equations, functional analysis, function theory, complex analysis, special functions, and fractional integro-differentiation.

In the history of transmutation theory, three main periods can be distinguished. In the first initial period, basic ideas and definitions were formulated. Their source was the theory of similarity of finite matrices (see [175,212,573]), some ideas on similarity of operators, and some results for the simplest differential equations. It is believed that the idea of transmutations in the operator formulation was expressed by Friedrichs [153]. But in fact, the method of transmutation operators for obtaining representations of solutions to differential equations was developed and first applied much earlier in the 19th century in the works of A. V. Letnikov. In addition, it was essentially the first real application of fractional integro-differentiation as transmutations to problems of differential equations [273,498].

The second period conventionally continued during 1940–1980. This can be called the classic period. Numerous results in the theory of transmutation operators and their applications were obtained. We list the main directions and results of this period.

The methods of transmutations were successfully applied in the theory of inverse problems, defining the generalized Fourier transform, the spectral function, and

solutions of the famous Gelfand–Levitan equation (see the works by Z. S. Agranovich and V. A. Marchenko [4,368–374] and B. M. Levitan [316–318,321,322,325–327]). In scattering theory, the no less famous Marchenko equation was presented (see B. M. Levitan [316–318,321,322,325–327], V. A. Marchenko [4,373,374], and L. D. Fadeev [131,132]). For both classes of inverse problems, transmutation operators are the main tool, since the listed classical equations are written out for the kernels of the transmutation operators, and the values of the kernels on the diagonal reconstruct unknown potentials in the inverse problem from the spectral function (see [31,68,308,309,417,420,421,497]).

For the Sturm–Liouville operators, classical transmutations on the interval were constructed by A. Ya. Povzner [454] and on the half-axis by B. Ya. Levin [313]. In spectral theory, well-known trace formulas and the asymptotics of the spectral function were obtained by V. A. Marchenko [373,374] and B. M. Levitan [316–318,321,322,325–327]. Estimates of the kernels of transmutation operators responsible for the stability of inverse problems and scattering problems were given by V. A. Marchenko [4,373,374]. Estimates of Jost's solutions in quantum scattering theory were obtained by Z. S. Agranovich and V. A. Marchenko [4,373,374], B. M. Levitan [316–318,321,322,325–327], V. V. Stashevskaya [557,558], and A. S. Sokhin [546–549]. As a result of applying the transmutations, we can say that the theory of Sturm–Liouville operators with a variable coefficient was trivialized to the level of the simplest equation with trigonometric or exponential solutions. The Dirac system and other matrix systems of differential equations were studied by B. M. Levitan and I. S. Sargsyan [326].

The theory of generalized analytic functions was developed. Such theory can be interpreted as a part of the transmutation operators theory that intertwines the unperturbed and perturbed Cauchy–Riemann equations (see L. Bers [26,27], S. Bergman [25], I. N. Vekua [579,582], B. Boyarsky [36], and G. N. Polozhy [450–452]). It has applications to mechanics problems and the theory of elasticity and gas dynamics. Based on the methods of transmutation operators, a new part of harmonic analysis was created. This part contains various modifications of generalized translation operators and generalized convolutions (see J. Delsarte [83,84], I. I. Zhitomirsky [609,610], and B. M. Levitan [321,327]).

A deep connection between transmutation operators and Paley–Wiener type theorems was established (see V. V. Stashevskaya [557,558], A. I. Akhiezer [5], H. Chablis [62–65], and H. Trimesh [569,570]). The theory of transmutation operators allowed us to give a new classification of special functions and integral operators with special functions in kernels (see R. Carroll [51–53] and T. Corvinder [273]). Moreover, to find the transmutation operator kernels, the existence and explicit form of the Green or Riemann functions for various classes of differential equations are used [545,588,589], stimulating the finding of these functions for various problems.

In the theory of nonlinear differential equations, the Lax method was developed. This method uses transmutation operators to prove the existence of and construct solutions to nonlinear differential equations (see [1,54,605,617]). Darboux transformation has also been widely used as transformation operator in the case when both the intertwining and intertwined operators are differential [366]. For a connection between Darboux transformation theories and transmutation operators, see [11]. In quantum

physics, when considering the Schrödinger equation and problems of theory scattering, a special class of transmutation operators, so-called wave operators, were studied. General scattering problems and inverse problems were considered from the point of view of transmutations in [131,132,375]. In [213] the wave operators were constructed for problems of scattering theory with the Stark potential. Unfortunately this paper by V. P. Kachalov and Ya. V. Kurylyova (1989) is practically forgotten. For example, in the article [324] (1995), B. M. Levitan formulates the problem of constructing the corresponding transmutation operator as unsolved.

In the theory of transmutation operators, restrictions related to the order of the differential operator were discovered. It was shown that for differential operators of orders higher than the third, classical Volterra operators exist only in the case of analytic coefficients (see V. I. Matsaev [365], L. A. Sakhnovich [488–490], and M. M. Malamud [356–360]). In the general case, transmutations have a more complicated structure that requires access to the complex plane even for constructing real solutions (see A. F. Leontiev [328], Yu. N. Valitsky [578], I. G. Khachatryan [254, 255], M. M. Malamud [356–360], and A. P. Khromov [256]). At the same time, in the spaces of analytic functions, the equivalence of differential operators of the same order was proved and a number of problems were studied (see D. K. Fage [133–139], B. A. Marchenko [370–372], Yu. F. Korobeinik [269,270], and M. K. Fishman [144]). Operator theory was applied to the theory of solubility for the well-known Bianchi equation (see D. K. Fage [139]).

Transmutation theory is strongly connected with many applications in different fields of mathematics. Transmutation operators are applied in inverse problems via the generalized Fourier transform, the spectral function, and the famous Levitan equation; in scattering theory, the Marchenko equation is formulated in terms of transmutations; in spectral theory, transmutations help to prove trace formulas and asymptotics for the spectral function; estimates for transmutational kernels control stability in inverse and scattering problems; for nonlinear equations via the Lax method, transmutations for Sturm–Lioville problems lead to proving existence and explicit formulas for solutions. Special kinds of transmutations are the generalized analytic functions, generalized translations and convolutions, and Darboux transformations. In the theory of partial differential equations, the transmutations work for proving explicit correspondence formulas among solutions of perturbed and nonperturbed equations, singular and degenerate equations, pseudodifferential operators, problems with essential singularities at inner or corner points, and estimates of solution decay for elliptic and ultraelliptic equations. In function theory, transmutations are applied to embedding theorems and generalizations of Hardy operators, Paley–Wiener theory, and generalizations of harmonic analysis based on generalized translations. Methods of transmutations are used in many applied problems: investigation of Jost solutions in scattering theory, inverse problems, Dirac and other matrix systems of differential equations, integral equations with special function kernels, probability theory and random processes, stochastic random equations, linear stochastic estimation, inverse problems of geophysics, and transsound gas dynamics. Also a number of applications of the transmutations to nonlinear equations is permanently increased.

In fact, the modern transmutation theory originated from two basic examples (see [532]). The first is the transmutation T for Sturm–Liouville problems with some potential $q(x)$ and natural boundary conditions

$$T(D^2\, y(x) + q(x)y(x)) = D^2\,(T\,y(x)),\, D^2\, y(x) = y''(x).$$

In this book we pay a lot of attention to equations with the **singular Bessel differential operator** B_γ,

$$(B_\gamma)_t = \frac{\partial^2}{\partial t^2} + \frac{\gamma}{t}\frac{\partial}{\partial t} = \frac{1}{t^\gamma}\frac{\partial}{\partial t} t^\gamma \frac{\partial}{\partial t}, \qquad t > 0, \qquad \gamma \in \mathbb{R},$$

and the second example of transmutation is a problem of intertwining the Bessel operator B_γ and the second derivative:

$$T(B_\gamma) f = (D^2) T f.$$

This class of transmutations includes the Sonine–Poisson–Delsarte and Buschman–Erdélyi operators and their generalizations. Such transmutations found many applications for a special class of partial differential equations with singular coefficients.

It should be noted here that the first fundamental paper that began the study of degenerate and singular partial differential equations with variable coefficients is the article by M. V. Keldysh [235] (Problem E).

Let $u = u(x_1, ..., u_n)$, $f = f(x_1, ..., x_n)$. In accordance with I. A. Kipriyanov's terminology, the equation

$$\sum_{i=1}^{n} (B_{\gamma_i})_{x_i} u = f \tag{2}$$

is classified as B**-elliptic**, the equation

$$\frac{\partial}{\partial x_1} u - \sum_{i=2}^{n} (B_{\gamma_i})_{x_i} u = f$$

is classified as B**-parabolic**, and the equation

$$(B_{\gamma_1})_{x_1} u - \sum_{k=2}^{n} (B_{\gamma_i})_{x_i} u = f$$

is classified as B**-hyperbolic**.

Singular elliptic equations containing the Bessel operator are mathematical models of axial and multi-axial symmetry of the most diverse processes and phenomena of the world. Difficulties in the study of such equations are associated, inter alia, with the presence of singularities in the coefficients. The foundation of a systematic study of equations of B-elliptic type was laid in the works [592,594,596,598,599], where Weinstein's theory of generalized axially symmetric potential (GASPT) was created.

In his papers Weinstein made a link between the B-elliptic equation and Tricomi equations and their fundamental solutions. I. A. Kipriyanov, together with V. V. Katrakhov (see [225,247,364]), studied boundary value problems for elliptic equations, with singularities of the type of essential singularities of analytic functions at isolated boundary points. L. N. Lyakhov studied the questions regarding fractional powers of the B-elliptic operator (elliptic operator with the Bessel operator instead of all or some second derivatives) and realized a solution to the B-elliptic equation and other questions (see [343–347,351,352]). In the paper of M. B. Kapilevic [217], the theory of degenerate elliptic differential equations of Bessel class were considered.

The first who applied the Fourier–Bessel (Hankel) transformation to equations with the Bessel operator B_γ was Yakov Isaakovich Zhitomirsky. At the beginning of the 1950s, the rapid development of the theory of generalized functions by Gelfand and Shilov made it possible to establish the uniqueness class for the solution of the Cauchy problem for a system of linear partial differential evolution equations with constant coefficients that depend only on the order of the system. Further attempts to extend these results to equations with variable coefficients depending on spatial coordinates were made. In search of such equations, Ya. I. Zhitomirsky came to parabolic equations with the Bessel operator (B-parabolic equations). For such equations, he developed and used the theory of Fourier–Bessel (Hankel) integral transform in the corresponding function spaces to obtain results on uniqueness classes. These results were obtained in the thesis by Ya. I. Zhitomirsky (1954) and published in articles [609,610]. He found classes of the correct solvability of problems for parabolic systems with increasing coefficients [611]. Exact uniqueness classes were established for solving the Cauchy problem for linear evolutionary systems with variable coefficients, and a new boundary for the growth of coefficients was found that guarantees the stability of uniqueness classes in [612–615]. Subsequently, Ya. I. Zhitomirsky turned to questions of the existence and uniqueness of a solution to the Cauchy problem in terms of the general theory of differential equations.

A. B. Muravnik studied parabolic differential equations and their generalizations. In [390,392,394,396,397,399–402], the Cauchy problem for parabolic differential-difference equations are investigated. In [391,403], these investigations are extended to more general cases where second derivatives and translation operators act with respect to an arbitrary amount of nonspecial spatial variables, while Bessel operators and the corresponding generalized translation operators act with respect to an arbitrary amount of special spatial variables; thus, the considered functional-differential equations become differential-difference and integro-differential at the same time. In the monograph [9], the above investigations are summarized and developed. In [407–409,412], elliptic differential-difference equations in the half-plane are investigated. In [85,86,393,395,398,404–406,410,411], qualitative properties of solutions (including blow-up phenomena) are investigated for various quasilinear partial differential equations and inequalities (including singular and degenerate ones) with Kardar–Parisi–Zhang nonlinearities arising in numerous applications. In [386–389], specific properties of Fourier–Bessel transforms of measures are studied and applied to singular differential equations.

The class of B-hyperbolic equations was first studied by Euler, Poisson, and Darboux and this study was continued by Weinstein [593,595,597,599]. In [593,595] the Cauchy problem for (7.1) is considered with $k \in \mathbb{R}$, the first initial condition being nonzero and the second initial condition equaling zero. A solution of the Cauchy problem (7.1)–(7.2) in the classical sense was obtained in [595,596,599,602] and in the distributional sense in [38,56]. S. A. Tersenov in [564] solved the Cauchy problem for (7.1) in the general form where the first and the second conditions are nonzeros. Different problems for Eq. (7.1) with many applications to gas dynamics, hydrodynamics, mechanics, elasticity and plasticity, and so on, were also studied in [7,32,38,39,56,61,74,88,96,97,127,140,148–150,159,203,306,383,461,462, 539,550,552,553,559,564,581,602,616], and of course the above list of references is incomplete. Problems for operator-differential (abstract) equations including hyperbolics with Bessel operator appeared in the well-known monograph [56] A. V. Glushak studied abstract differential equations with a Bessel operator such as B-hyperbolic equations (see [182,185,188–190,192,193]). In particular, he investigated the stability of the property of uniform well-posedness of the Cauchy problem for the indicated equations and studied the solvability conditions for such problems with the Fredholm operator with derivatives.

In the most detailed and complete way, equations with Bessel operators were studied by the Voronezh mathematician Kipriyanov and his disciples Ivanov, Ryzhkov, Katrakhov, Arhipov, Baidakov, Bogachov, Brodskii, Vinogradova, Zaitsev, Zasorin, Kagan, Katrakhova, Kipriyanova, Kononenko, Kluchantsev, Kulikov, Larin, Leizin, Lyakhov, Muravnik, Polovinkin, Sazonov, Sitnik, Shatskii, and Yaroslavtseva. The essence of Kipriyanov's school results was published in [242]. For classes of equations with Bessel operators, Kipriyanov introduced special functional spaces which were named after him [243]. In this field, interesting results were investigated by Katrakhov and his disciples; now these problems are considered by Gadjiev, Guliev, Glushak, Lyakhov, and Shishkina with their coauthors and students. Abstract equations of the form (2) originating from the monograph [56] were considered by Egorov, Repnikov, Kononenko, Glushak, Shmulevich, and others. To describe the classes of solutions to the corresponding equations, I. A. Kipriyanov introduced and studied the functional spaces [243], later named after him (see the monographs by H. Tribel [568] and L. D. Kudryavtsev and S. M. Nikolsky [304] in which separate sections are devoted to Kipriyanov's spaces). Transmutations are still one of the basic tools for equations with Bessel operators; they are applied in the construction of solutions and fundamental solutions, the study of singularities, and new boundary value and other problems.

Transmutation operators for numerous generalizations of the Bessel operator were also considered. An important generalization of the Sonin–Poisson–Delsart operator is the transmutation operator for hyper-Bessel functions. The theory of such functions was originally laid down in the works of Kummer and Deleru. A complete study of hyper-Bessel functions, differential equations for them, and the corresponding transformation operators was exhaustively carried out by I. Dimovsky and his students [89,92,93]. The corresponding operators deservedly received in the literature the names of Sonin–Dimovsky and Poisson–Dimovsky operators; they were also stud-

ied by V. Kiryakova [92,93,252,253]. The central role in the theory of hyper-Bessel functions, differential equations, and transmutation operators for them is played by the famous Obreshkov integral transform, introduced by the Bulgarian mathematician N. Obreshkov. This transformation, whose core is expressed in the general case in terms of the Meijer G-function, is a simultaneous generalization of the Laplace, Mellin, sine- and cosine-Fourier, Hankel, Meijer, and other classical integral transforms. Various forms of hyper-Bessel functions, differential equations, and transformation operators for them, as well as special cases of the Obreshkov transform, were subsequently rediscovered many times. Obreshkov's transform was historically the first integral transform whose kernel is expressed in terms of the Meijer G-function but cannot be expressed in terms of one generalized hypergeometric function. Another important integral transformation, the Stankovic transform, was introduced by the Serbian mathematician B. Stankovic. The core of the Stankovic transform is expressed in terms of the Wright–Fox H-function, but is not expressed in terms of the simpler Meijer G-function. This transformation finds important applications in the study of fractional differential equations of the type of fractional diffusion [118,264, 265,459,460].

At the same time, similar theories were also constructed for some other model operators, such as [51–53,601]

$$
\begin{aligned}
&A = \frac{1}{v(x)}\frac{d}{dx}v(x)\frac{d}{dx}, \qquad (3)\\
&v(x) = \sin^{2\nu+1} x,\, \mathrm{sh}^{2\nu+1} x,\, (e^x - e^{-x})^{2\nu+1}(e^x + e^{-x})^{2\mu+1}.
\end{aligned}
$$

The importance of A operators of the form (3) for the theory lies in the fact that, according to the famous Gelfand formula, they represent the radial part of the Laplace operator on symmetric spaces [162]. Here the Bessel operator is obtained by choosing $v(x) = x^\gamma$ in (3). Another model operator for which the transmutations are constructed is the Airy operator $D^2 + x$. In [213] its perturbed version related to the Stark effect from quantum mechanics was also considered. We studied the shift operators with respect to the spectral parameter Erdélyi–Vekua–Lowndes [337–339].

Papers from the 1990s to the present can be attributed to the third period of development of transmutation theory, which can be called the modern period. In this period, many important studies have been received and continue to appear (see, for example, reviews [55,234,375,528,532,533]). We list some of them. The development of the theory of generalized analytic functions was continued (see A. P. Soldatov [551], S. B. Klimentov [260–263], and V. V. Kravchenko [277]). Applications of transmutation operators to embedding function spaces and a generalization of Hardy operators were found [522,524,525]. Various constructions of a generalized translation and the generalized versions of harmonic analysis based on them were studied by A. D. Gadzhiev, V. Guliev, and A. Serbetci [174,206,207], S. S. Platonov [442–444], and L. N. Lyakhov and E. L. Shishkina [351–353], as well as in [510]. The use of transmutation operators and related methods in the theory of inverse problems and scattering theory continued [59,439,467,604]. For differential equations, the development of the Darboux method and its modifications continues (see V. B. Matveev

[366]), new classes of problems for solutions with significant features on the part of the boundary at internal or corner points are considered (see V. V. Katrakhov [225,227] and I. A. Kipriyanov [248–250]), and exact estimates of the rate of decrease of solutions of some elliptic and ultraelliptic equations have been obtained (see V. Z. Meshkov and S. M. Sitnik [379,380,520]).

A separate topic is the use of operators in the study of various fractional integro-differentiation operators (see I. Dimovski and V. Kiryakova [89,93,252] and N. A. Virchenko [583,584]). Using the methods of transmutation operators, singular and degenerate boundary value problems, pseudodifferential operators (see V. V. Katrakhov [225,227], I. A. Kipriyanov [249,250], L. N. Lyakhov [343,344], and O. A. Repin [468]), and operator equations (see A. V. Glushak [184–186] and V. E. Fedorov [141,142]) were studied. The equations with Bessel operator and related questions were studied by A. V. Glushak [185,188,189], V. S. Guliev [208], L. N. Lyakhov, I. P. Polovinkin, and E. L. Shishkina [349,350,354], L. S. Pulkina [463], K. B. Sabitov [486], and V. V. Volchkov [587].

A separate class of problems is comprised of problems of the Dirichlet-to-Neumann and Neumann-to-Dirichlet types, under which the transformation operator acts on the boundary or initial conditions, preserving the differential expression; such tasks have found important applications in mechanics (see O. E. Yaremko [600]).

Enough completed modifications of harmonic analysis for Bessel operators were constructed in the works of S. S. Platonov [442–444]. For a perturbed Bessel type operator with variable coefficients, see Kh. Triméche [571,572]. Recently, harmonic analysis has been actively created for differential-difference operators of the Dunkl type [115–117,479,480] based on appropriate generalizations of Sonin–Poisson–Darboux operators.

The existence of transmutation operators corresponding to generalized translation also allows us to determine the generalized convolution and new algebraic and group structures, and consider various problems of approximating functions [69]. The ideas of M. K. Fage developed for the Bianchi equation in connection with the construction of transmutations for higher order differential equations found their continuation in the study of more general equations in the works of V. I. Zhegalov, A. N. Mironov, and E. A. Utkina [606,607]. In the theory of fractional order equations, papers that can be interpreted as considering the transmutations method for representing solutions of fractional order equations through solutions of integer order equations have appeared (see A. Pskhu [459,460], Ya. Pruss [458], and A. N. Kochubey [264,265]).

Transmutation operators find applications in the theory of the Radon transform and mathematical tomography [162,416,485], as well as in the expansion of functions in various series in special functions [214]. In the works of V. A. Marchenko, the application of transmutations to quantum theory continued [375,376].

An important section of transmutation theory is a special class of Bushman–Erdélyi operators. This is a class of transmutation which, with a certain choice of parameters, is a generalization of the Sonin–Poisson–Dardoux operators and their conjugates, the fractional integro-differentiation operators of Riemann–Liouville and Erdélyi–Kober, and the Mehler–Fock integral transforms.

Integral operators of the indicated form with Legendre functions in kernels were first encountered in the works of E. T. Copson according to the Euler–Poisson–Darboux equation in the late 1950s [70–72]. The first detailed study of the solvability and reversibility of these operators was started in the 1960s in the works of R. Bushman [41,42] and A. Erdélyi [123–127].

Bushman–Erdélyi operators and their analogues were also studied by T. P. Higgins [167], Ta Li [561,562], E. R. Love [335,336], G. M. Habibullah, K. N. Srivastava, V. I. Smirniv, B. Rubin, N. A. Virchenko, and I. Fedotova [584], A. A. Kilbas and O. B. Skoromnik [240], and others. Moreover, the problems of solving integral equations with these operators and their factorization and inversion were studied. The results are partially mentioned in the monograph [494], although the case of the integration limits chosen by us is considered special there and is not considered, with the exception of one set of composition formulas (see also [234,522,537]).

The term "Bushman–Erdélyi operators" is the most historically justified. It was introduced by S. M. Sitnik in [522,523], and later it was used by other authors. Earlier, in [494], the term "Bushman operators" was proposed by O. I. Marichev. The term "Chebyshev–Gegenbauer operators" [485] is also used in the theory of Radon transforms and mathematical tomography. The most complete study of the Bushman–Erdélyi operators, in our opinion, was carried out in the 1980s and 1990s [521–525] and then continued in [234,528,533–535]. It should be noted that the role of the Bushman–Erdélyi operators as transmutations before these works has never been noted or considered before.

Recently, V. Kravchenko and S. Torba together with their colleagues have taken up the problem of efficient construction of the integral kernels of the transmutation operators. They tried to use the fact that the result of application of a transmutation operator to any nonnegative integer power of the independent variable can be obtained without knowledge of the operator itself. The powers of the independent variable are transmuted to so-called formal powers arising in the spectral parameter power series (SPPS) method (see [257,276,277,283]). This mapping property of the transmutation operator [45,286] leads to the possibility to transmute any polynomial into a corresponding generalized polynomial in terms of formal powers. V. Kravchenko and S. Torba proved a completeness property of so-called wave polynomials in a class of solutions of the wave equation and, as a corollary, the completeness of the system of the transmuted wave polynomials in a class of solutions of the hyperbolic equation satisfied by the transmutation kernel [287,288]. This result led to a method of approximation of the transmutation kernels and consequently of solutions of the Sturm–Liouville equation. It was observed in [287,288] that an approximate representation of the solution based on an approximation of the transmutation kernel admits estimates independent of the real part of the square root of the spectral parameter. This feature makes such representations especially valuable for solving spectral problems and allows one to compute large sets of eigendata.

The next step was made in [281], where the authors managed to pass from approximation of the transmutation kernels to their exact representation in the form of functional series involving Legendre polynomials with easily computable expansion coefficients. As a corollary, new representations for solutions of the one-dimensional

Schrödinger equation and later on for the Sturm–Liouville equation [290] in the form of the Neumann series of Bessel functions were obtained revealing the same attractive feature: They admit truncation estimates independent of the real part of the square root of the spectral parameter and allow one to compute in practice thousands of eigendata applying minimal computational efforts.

It is worth mentioning that the mapping property of the transmutation operators allowing one to obtain the images of the powers of the independent variable without knowledge of the transmutation operator itself was used in several publications [45–48,258,274,275,282] for obtaining complete systems of solutions of partial differential equations and for using them when solving different boundary value problems.

Besides regular Sturm–Liouville equations, singular perturbed Bessel equations were studied in [284,291,293].

In [278] and [292] the authors explored the possibility to expand the transmutation kernels into series in terms of other systems of orthogonal polynomials obtaining different series representations for solutions of the one-dimensional Schrödinger equation.

In [280], V. Kravchenko found a way to obtain a functional series representation for the transmutation operator with a condition at infinity, the Levin transmutation operator arising in the Gelfand–Levitan–Marchenko scattering theory. In [82] this result was developed and led to an attractive representation of the Jost solutions and as a consequence to an efficient method of practical solution of spectral problems on infinite intervals, allowing one to compute spectral (or scattering) data corresponding not only to the discrete part of the spectrum but also to its continuous part, a computationally challenging problem.

In [279], V. Kravchenko discovered an application of the Fourier–Legendre series representation for the transmutation kernel from [281] to the solution of the classical inverse Sturm–Liouville problem on a finite interval. Their idea is based on the observation that the potential can be recovered from the very first coefficient of the Fourier–Legendre series, and to find this coefficient a system of linear algebraic equations can be obtained directly from the Gelfand–Levitan equation. In contrast to existing methods for solving inverse Sturm–Liouville problems, the method derived by V. Kravchenko is not iterative. The inverse spectral problem is reduced directly to a linear system of algebraic equations.

The same approach was developed in [280] for the inverse scattering problem on the line and in [81] for the inverse Sturm–Liouville problem on the half-line. Thus, as was shown by V. Kravchenko and his group, the transmutation operator method is an important tool for practical solution of forward and inverse spectral problems.

Thus, the methods of transmutation theory and related problems were applied to one degree or another in the works of many mathematicians. We list some of them: A. I. Aliev, H. Begehr, J. Betancor, A. Boumenir, B. Braaksma, L. Bragg, R. Carroll, H. Chebli, I. Dimovski, C. Dunkl, J. Delsarte, A. Fitouhi, R. Gilbert, V. Hristov, V. Hutson, G. K. Kalish, S. L. Kalla, T. H. Koornwinder, V. Kiryakova, J. Löffström, J. Lions, M. M. Moro, J. S. Pym, B. Rubin, F. Santosa, J. Siersma, H. S. V. de Snoo, K. Stempak, V. Thyssen, K. Triméche, M. Voit, Vu Kim Tuan, Z. S. Agranovich, A. A. Androshchuk, A. G. Baskakov, L. E. Britvina, Yu. N. Val-

itsky, V. Ya. Volk, V. V. Volchkov, A. D. Gadzhiev, A. V. Glushak, M. L. Gorbachuk, I. Ts. Gokhberg, V. S. Guliev, I. M. Huseynov, Ya. I. Zhytomyrskii, L. A. Ivanov, M. S. Eremin, D. B. Karp, V. V. Katrakhov, A. P. Kachalov, A. A. Kilbas, I. A. Kipriyanov, M. I. Klyuchantsev, V. I. Kononenko, Yu. F. Korobeinik, V. V. Kravchenko, M. G. Krein, P. P. Kulish, I. F. Kushnirchuk, G. I. Laptev, B. Ya. Levin, B. M. Levitan, A. F. Leontiev, N. E. Lynchuk, S. S. Lynchuk, L. N. Lyakhov, G. V. Lyakhovetsky, M. M. Malamud, V. A. Marchenko, V. I. Matsaev, A. B. Muravnik, N. I. Nagnibida, L. P. Nizhnik, M. N. Olevsky, S. S. Platonov, A. Ya. Povzner, B. Rubin, F. S. Rofe-Beketov, K. B. Sabitov, L. A. Sakhnovich, A. S. Sokhin, V. V. Stashevskaya, S. M. Torba, L. D. Faddeev, D. K. Fage, K. M. Fishman, I. G. Khachatryan, A. P. Khromov, E. L. Shishkina, S. D. Shmulevich, and V. Ya. Yaroslavtseva. Of course, this list is not complete and could be significantly expanded.

We must note that the term "operator" is used in this book for brevity in the broad and sometimes not exact meaning, so appropriate domains and function classes are not always specified. It is easy to complete and make strict for every special result.

Now let us list the content of the book briefly by chapter.

In Chapter 1, basic definitions and propositions are presented. First we give definitions of some special functions such as the gamma function, beta function, Pochhammer symbol, error function, Bessel functions, hypergeometric type functions, and some orthogonal polynomials. Next, some functional spaces and integral transforms are considered. Also Kipriyanov's classification of second order linear partial differential equations, the divergence theorem and Green's second identity for B-elliptic and B-hyperbolic operators, the Tricomi equation, and the abstract Euler–Poisson–Darboux equation are discussed.

In Chapter 2, we collect the basic facts about fractional calculus and fractional order differential equations. First we give a brief history of fractional calculus and fractional order differential equations, which include one-dimensional fractional derivatives and integrals, fractional derivatives in mechanics, fractional powers of multi-dimensional operators such as Riesz potentials, and differential equations of fractional order. We list some standard fractional order integro-differential operators, such as Riemann–Liouville fractional integrals and derivatives on a segment and a semiaxis, Gerasimov–Caputo fractional derivatives, Dzrbashian–Nersesyan fractional operators, sequential order fractional operators, and others. Also integral transforms and basic differential equations of fractional order are considered.

Chapter 3 contains information about transmutations. We give a definition of the transmutation operator, some examples of classical transmutations, transmutations for the Sturm–Liouville operator, and transmutations for the singular Bessel operator such as the Poisson operator, the generalized translation, and the weighted spherical mean.

Chapter 4 contains detailed studies of weighted generalized functions generated by quadratic forms. First we define the weighted generalized function associated with a positive quadratic form concentrated on a part of a cone and obtain its properties; next, we obtain the Hankel transform of weighted generalized functions generated by quadratic forms.

Chapter 5 covers one- and multi-dimensional Buschman–Erdélyi transmutation operators theory. It includes Buschman–Erdélyi transmutations of the first, second, and third kinds with properties, Sonine–Katrakhov and Poisson–Katrakhov transmutations, and generalizations to the multi-dimensional case.

In Chapter 6, we present the integral transform compositions method (ITCM) for constructing different transmutations. We give basic ideas, a background, and a definition of the ITCM and apply the ITCM to derive transmutations connected with the Bessel operator. Also some examples of the use of the ITCM to the solution of differential equations are given.

In Chapter 7, differential equations with Bessel operator without fractional power operators are considered. Firstly, hyperbolic and ultrahyperbolic equations with Bessel operator such as the general and generalized Euler–Poisson–Darboux equation and the singular Klein–Gordon equation are solved. The rest of Chapter 7 contains the solution to the problem for elliptic equations with Bessel operator. In this chapter we also give a short historical introduction on differential equations with Bessel operators and a rather detailed reference list of monographs and papers on mathematical theory and applications of this class of differential equations.

Chapter 8 introduces the applications of transmutations to different problems. It includes applications of Buschman–Erdélyi transmutation to the Copson lemma, norm estimates and embedding theorems in Kipriyanov spaces, and the Radon transform. Next, applications of the transmutation method to estimations of the solutions for differential equations with variable coefficients and the E. M. Landis problem and to the perturbed Bessel and the one-dimensional Schrödinger equation are given. Finally, we present identities for iterated weighted spherical means, which are necessary in various applied problems of tomography and integral geometry.

In Chapter 9, fractional powers of Bessel operators are studied. We consider fractional Bessel integrals and derivatives on a segment and on a semiaxis, and some of their integral transforms, such as the Mellin transform, the Hankel transform, and the generalized Whittaker transform. Moreover, resolvents for the right-sided fractional Bessel integral on a semiaxis and the generalized Taylor formula with powers of Bessel operators are given.

In Chapter 10, the theory of fractional powers of hyperbolic operators with Bessel operators instead of all or some second derivatives is developed. Such operators are called hyperbolic B-potentials. First, we give definitions of the hyperbolic B-potentials and prove their absolute convergence and boundedness. Next, using the idea of approximative inverse operators, we construct an inverse-to-hyperbolic B-potential operator. Also mixed hyperbolic Riesz B-potentials and their inversions are considered.

In Chapter 11, we solve fractional differential equations with singular coefficients. We apply the Meijer transform method for solution of homogeneous fractional equations with left-sided fractional Bessel derivatives on semiaxes of Gerasimov–Caputo type and the Mellin transform method for the solution of ordinary linear nonhomogeneous differential equations of fractional order on semiaxes. Next, we use the Riesz B-potential method for solution of nonhomogeneous hyperbolic equations with Bessel operators.

Basic definitions and propositions

1

1.1 Special functions

1.1.1 Gamma function, beta function, Pochhammer symbol, and error function

The gamma function, also called the Euler integral of the second kind, is one of the extensions of the factorial function (see [2], p. 255).

The *gamma function* $\Gamma(z)$ is defined via a convergent improper integral

$$\Gamma(z)=\int\limits_0^\infty y^{z-1}e^{-y}dy, \tag{1.1}$$

which converges for all $z\in\mathbb{C}$ such that $\operatorname{Re}z>0$. Function (1.1) is extended by analytic continuation to all complex numbers except the nonpositive integers (where the function has simple poles).

Integration by parts of expression (1.1) yields the recurrent formula

$$\Gamma(z+1)=z\Gamma(z). \tag{1.2}$$

Rewriting formula (1.2) in the form

$$\Gamma(z-1)=\frac{\Gamma(z)}{z-1}, \tag{1.3}$$

we get an expression that allows us to determine the gamma function of $z\in\mathbb{C}$ such that $\operatorname{Re}z\leq 0$, for which the definition (1.1) is unacceptable. Formula (1.3) shows that $\Gamma(z)$ has simple poles at $z=0,-1,-2,-3,....$ From (1.3) we get

$$\Gamma(z+m+1)=z(z+1)\cdots(z+m)\Gamma(z),\qquad m\in\mathbb{N}. \tag{1.4}$$

For the gamma function Euler's reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin z\pi}, \tag{1.5}$$

the formula

$$\Gamma\left(\frac{1}{2}+z\right)\Gamma\left(\frac{1}{2}-z\right)=\frac{\pi}{\cos(\pi z)}, \tag{1.6}$$

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and the Legendre duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{1.7}$$

are valid.

The beta function, also called the Euler integral of the first kind, is closely related to the gamma function (see [2], p. 258).

The *beta function* $B(z,w)$ for $z,w\in\mathbb{C}$, $\operatorname{Re} z>0$, $\operatorname{Re} w>0$ is a special function defined by

$$B(z,w) = \int\limits_0^1 t^{z-1}(1-t)^{w-1}dt. \tag{1.8}$$

The beta function is related to the gamma function by the formula

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{1.9}$$

The *Pochhammer symbol* $(z)_n$ for integer n is defined by

$$(z)_n = z(z+1)...(z+n-1), \qquad n=1,2,..., \qquad (z)_0\equiv 1 \tag{1.10}$$

(see [2], p. 256). The following equalities are true:

$$(z)_n = (-1)^n(1-n-z)_n, \qquad (1)_n = n!,$$

and

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}. \tag{1.11}$$

Equality (1.11) can be used to extend $(z)_n$ to real or complex values of n.

The *error function* (also called the probability integral) is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int\limits_0^x e^{-t^2}\,dt, \qquad x\in\mathbb{R}, \tag{1.12}$$

(see [2], p. 297).

The error function's Maclaurin series holds for every complex number z and has the form

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}}\left(z-\frac{z^3}{3}+\frac{z^5}{10}-\frac{z^7}{42}+\frac{z^9}{216}-\cdots\right).$$

The error function is an entire function.

1.1.2 Bessel functions

Bessel functions, named after the German astronomer Friedrich Bessel, are defined as solutions of the Bessel differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0,$$

where α is a complex number.

The Bessel functions of the first kind, denoted by $J_\alpha(x)$, are solutions of Bessel's differential equation that are finite at the origin $x=0$. The Bessel function $J_\alpha(x)$ can be defined by the series

$$J_\alpha(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}. \tag{1.13}$$

For noninteger α the functions $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent. If α is integer the following relationship is valid:

$$J_{-\alpha}(x)=(-1)^\alpha J_\alpha(x).$$

The Bessel functions of the second kind, denoted by $Y_\alpha(x)$, for noninteger α are related to $J_\alpha(x)$ by the formula

$$Y_\alpha(x)=\frac{J_\alpha(x)\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)}.$$

In the case of integer order n, the function $Y_\alpha(x)$ is defined by taking the limit as a noninteger α tends to n:

$$Y_n(x)=\lim_{\alpha\to n}Y_\alpha(x).$$

Functions $Y_\alpha(x)$ are also called Neumann functions and are denoted by $N_\alpha(x)$. The linear combination of the Bessel functions of the first and second kinds represents a complete solution of the Bessel equation:

$$y(x)=C_1J_\alpha(x)+C_2Y_\alpha(x).$$

Hankel functions of the first and second kind, denoted by $H_\alpha^{(1)}(x)$ and $H_\alpha^{(2)}(x)$, respectively, are defined by the equalities

$$H_\alpha^{(1)}(x)=J_\alpha(x)+iY_\alpha(x) \tag{1.14}$$

and

$$H_\alpha^{(2)}(x)=J_\alpha(x)-iY_\alpha(x). \tag{1.15}$$

Modified Bessel functions (or occasionally the hyperbolic Bessel functions) *of the first and second kind* $I_\alpha(x)$ and $K_\alpha(x)$ are defined as

$$I_\alpha(x) = i^{-\alpha} J_\alpha(ix) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}, \tag{1.16}$$

$$K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}, \tag{1.17}$$

where α is noninteger.

In the case of integer order α, the functions $I_\alpha(x)$ and $K_\alpha(x)$ are defined by taking the limit as a noninteger α tends to $n \in \mathbb{Z}$:

$$I_n(x) = \lim_{\alpha \to n} I_\alpha(x), \qquad K_n(x) = \lim_{\alpha \to n} K_\alpha(x).$$

It is obvious that $K_\alpha(x) = K_{-\alpha}(x)$.

Function $I_\nu(r)$ is exponentially growing when $r \to \infty$ and $K_\nu(r)$ is exponentially decaying when $r \to \infty$ for real r and ν:

$$I_\nu(z) \propto \frac{e^z}{\sqrt{2\pi z}} \left(1 + O\left(\frac{1}{z}\right)\right), \qquad |Arg(z)| < \frac{\pi}{2}, |z| \to \infty,$$
$$K_\nu(z) \propto \sqrt{\frac{\pi}{2}} \frac{e^{-z}}{\sqrt{z}} \left(1 + O\left(\frac{1}{z}\right)\right), \qquad |z| \to \infty.$$

For small arguments $0 < |r| \ll \sqrt{\nu+1}$, we have

$$I_\nu(r) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{r}{2}\right)^\nu,$$
$$K_\nu(r) \sim \begin{cases} -\ln\left(\frac{r}{2}\right) - \vartheta & \text{if } \nu = 0, \\ \frac{\Gamma(\nu)}{2^{1-\nu}} r^{-\nu} & \text{if } \nu > 0, \end{cases} \tag{1.18}$$

where

$$\vartheta = \lim_{n\to\infty} \left(-\ln n + \sum_{k=1}^{n} \frac{1}{k}\right) = \int\limits_1^\infty \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor}\right) dx$$

is the Euler–Mascheroni constant [121].

Here are some of the important particular cases of Bessel functions:

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin(z), \qquad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos(z),$$
$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sinh(z), \qquad I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cosh(z),$$
$$K_{\frac{1}{2}}(z) = K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$

The normalized Bessel function of the first kind j_ν is defined by the formula (see [242], p. 10, [317])

$$j_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu} J_\nu(x), \tag{1.19}$$

where J_ν is a Bessel function of the first kind. Operator function of the type (1.19) was considered in [183,187].

The normalized modified Bessel function of the first kind i_ν is defined by the formula

$$i_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu} I_\nu(x), \tag{1.20}$$

where I_ν is a modified Bessel function of the first kind.

The normalized modified Bessel function of the second kind k_ν is defined by the formula

$$k_\nu(x) = \frac{1}{2^\nu \Gamma(1+\nu) x^\nu} K_\nu(x), \tag{1.21}$$

where K_ν is a modified Bessel function of the second kind. We have

$$\frac{dk_\nu(x)}{dx} = -\frac{1}{2^\nu \Gamma(1+\nu) x^\nu} K_{\nu+1}(x). \tag{1.22}$$

Here are some of the important particular cases of normalized Bessel functions:

$$j_{\frac{1}{2}}(z) = \frac{\sin(z)}{z}, \qquad j_{-\frac{1}{2}}(z) = \cos(z),$$
$$i_{\frac{1}{2}}(z) = \frac{\sinh(z)}{z}, \qquad i_{-\frac{1}{2}}(z) = \cosh(z),$$
$$k_{\frac{1}{2}}(z) = \frac{e^{-z}}{z}, \qquad k_{-\frac{1}{2}}(z) = e^{-z}.$$

Using formulas (9.1.27) from [2] we obtain that $j_\nu(t)$ is an eigenfunction of operator $(B_\nu)_t = \frac{d^2}{dt^2} + \frac{\nu}{t}\frac{d}{dt}$:

$$(B_\nu)_t j_{\frac{\nu-1}{2}}(\tau t) = -\tau^2 j_{\frac{\nu-1}{2}}(\tau t), \tag{1.23}$$
$$(B_\nu)_t i_{\frac{\nu-1}{2}}(\tau t) = \tau^2 i_{\frac{\nu-1}{2}}(\tau t), \tag{1.24}$$
$$(B_\nu)_t k_{\frac{\nu-1}{2}}(\tau t) = \tau^2 k_{\frac{\nu-1}{2}}(\tau t). \tag{1.25}$$

Normalized Bessel functions have the following properties:

$$j_\nu(0) = 1, \qquad j'_\nu(0) = 0, \qquad i_\nu(0) = 1, \qquad i'_\nu(0) = 0,$$
$$\lim_{x\to 0} x^{2\nu} k_\nu(x) = \frac{1}{2\nu}, \qquad \nu > 0, \tag{1.26}$$

$$\lim_{x\to 0} k_\nu(x) = \frac{\Gamma(-\nu)}{2^{2\nu+1}\Gamma(1+\nu)}, \qquad \nu < 0, \qquad -\nu \notin \mathbb{N}, \tag{1.27}$$

$$\lim_{x\to 0} x^\alpha k_0(x) = 0, \qquad \alpha > 0, \qquad \lim_{x\to 0} \frac{1}{\ln x} k_0(x) = -1, \tag{1.28}$$

$$\lim_{x\to 0} x^{2\nu+1} \frac{dk_\nu(x)}{dx} = -1, \qquad \nu > -1. \tag{1.29}$$

We will use notations

$$\mathbf{j}_\gamma(x,\xi) = \prod_{i=1}^{n} j_{\frac{\gamma_i-1}{2}}(x_i\xi_i) \tag{1.30}$$

and

$$\mathbf{i}_\gamma(x,\xi) = \prod_{i=1}^{n} i_{\frac{\gamma_i-1}{2}}(x_i\xi_i), \tag{1.31}$$

where $\gamma = (\gamma_1, ..., \gamma_n)$, $\gamma_1 > 0, ..., \gamma_n > 0$.

Information about the Bessel functions is taken from [591].

The Struve function is a solution $y(x)$ of the nonhomogeneous Bessel differential equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \alpha^2\right) y = \frac{4\left(\frac{x}{2}\right)^{\alpha+1}}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}.$$

Struve functions, denoted as $\mathbf{H}_\alpha(x)$, have the power series form

$$\mathbf{H}_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma\left(m+\frac{3}{2}\right)\Gamma\left(m+\alpha+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2m+\alpha+1}. \tag{1.32}$$

Another definition of the Struve function, for values of α satisfying $\operatorname{Re}\alpha > -\frac{1}{2}$, is possible using an integral representation:

$$\mathbf{H}_\alpha(x) = \frac{2\left(\frac{x}{2}\right)^\alpha}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \sin(x\cos\tau)\sin^{2\alpha}(\tau)\,d\tau.$$

1.1.3 Hypergeometric type functions

The hypergeometric Gauss function inside the circle $|z|<1$ is determined as the sum of the hypergeometric series (see [2], p. 373, formula (15.3.1))

$${}_2F_1(a,b;c;z) = F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^k}{k!}, \tag{1.33}$$

and for $|z| \geq 1$ it is obtained by analytic continuation of this series. In (1.33) parameters a, b, c and variable z can be complex, and $c \neq 0, -1, -2, \ldots$. The multiplier $(a)_k$ is the Pochhammer symbol (1.10).

Since the hypergeometric series (1.33) converges only in the unit circle of the complex plane, it is necessary to construct an analytic continuation of the hypergeometric function beyond the boundary of this circle, to the entire complex plane. One of the ways to continue analytically is to use the Euler integral representation of the form

$$
{}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b-c)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt, \tag{1.34}
$$
$$
0 < \operatorname{Re} b < \operatorname{Re} c, \qquad |\arg(1-z)| < \pi,
$$

in which the right side is defined under the specified conditions, ensuring the convergence of the integral.

An important property of the hypergeometric function is that many special and elementary functions can be obtained from it with certain values of the parameters and the transformation of an independent argument.

Examples for elementary functions are

$$
(1+x)^n = {}_2F_1(-n, \beta, \beta; -x), \qquad \frac{1}{x}\ln(1+x) = {}_2F_1(1, 1, 2; -x),
$$
$$
e^x = \lim_{n\to\infty} {}_2F_1\left(1, n, 1; \frac{x}{n}\right),
$$
$$
\cos x = \lim_{\alpha,\ \beta\to\infty} {}_2F_1\left(\alpha, \beta, \frac{1}{2}; -\frac{x^2}{4\alpha\beta}\right),
$$
$$
\cosh x = \lim_{\alpha,\ \beta\to\infty} {}_2F_1\left(\alpha, \beta, \frac{1}{2}; \frac{x^2}{4\alpha\beta}\right).
$$

The first-kind Bessel function and the Gauss hypergeometric function are related by the formula

$$
J_\nu(z) = \lim_{\alpha,\ \beta\to\infty} \left[\frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_2F_1\left(\alpha, \beta, \nu+1; -\frac{z^2}{4\alpha\beta}\right) \right].
$$

The degenerate Kummer hypergeometric function ${}_1F_1(a; b; z)$ is

$$
{}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!},
$$

associated with the Gauss hypergeometric function by the limiting relation

$$
{}_1F_1(a; c; z) = \lim_{b\to\infty} {}_2F_1\left(a, b; c; \frac{z}{b}\right).
$$

The degenerate Tricomi hypergeometric function $\Psi(a;b;z)$ is determined by the equality

$$\Psi(a;b;z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)}\,{}_1F_1(a;b;z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b}\,{}_1F_1(a+1-b;2-b;z).$$

Next we present Whittaker functions which appear in kernels of integral transforms connected with the fractional Bessel integral.

Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are special solutions of Whittaker's equation

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4-\mu^2}{z^2}\right) w = 0.$$

They are modified forms of Kummer's confluent hypergeometric functions and were introduced by Edmund Taylor Whittaker by

$$M_{\kappa,\mu}(z) = \exp(-z/2)\, z^{\mu+\frac{1}{2}} M\left(\mu-\kappa+\frac{1}{2}, 1+2\mu; z\right),$$
$$W_{\kappa,\mu}(z) = \exp(-z/2)\, z^{\mu+\frac{1}{2}} U\left(\mu-\kappa+\frac{1}{2}, 1+2\mu; z\right), \tag{1.35}$$

where

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} = {}_1F_1(a;b;z)$$

and

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)} M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a+1-b, 2-b, z)$$

are Kummer's functions.

The Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ are the same as those with opposite values of μ; in other words, considered as a function of μ at fixed κ and z, they are even functions. When κ and z are real, the functions give real values for real and imaginary values of μ.

A *generalized hypergeometric function* is defined as a power series

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

The functions of the form ${}_0F_1(;a;z)$ are called confluent hypergeometric limit functions and are closely related to the Bessel functions J_α and I_α. The relationships

are

$$J_\alpha(x) = \frac{(\frac{x}{2})^\alpha}{\Gamma(\alpha+1)} {}_0F_1\left(;\alpha+1;-\tfrac{x^2}{4}\right),$$
$$I_\alpha(x) = \frac{(\frac{x}{2})^\alpha}{\Gamma(\alpha+1)} {}_0F_1\left(;\alpha+1;\tfrac{x^2}{4}\right)$$

or

$${}_0F_1\left(;\alpha+1;-\tfrac{x^2}{4}\right) = j_\alpha(x), \qquad {}_0F_1\left(;\alpha+1;\tfrac{x^2}{4}\right) = i_\alpha(x).$$

We also need the function ${}_1F_2(;a;z)$. It is known (see [456]) that for $\alpha > 0$, $\xi \geq 0$, $t > 0$,

$$\int_0^t \left(t^2-u^2\right)^{\alpha-1} u^{1-\gamma} J_\gamma(u\xi)dt = \frac{\xi^\gamma t^{2\alpha}}{2^{\gamma+1}\alpha\Gamma(\gamma+1)} \times {}_1F_2\left(1;\alpha+1,\gamma+1;-\frac{t^2\xi^2}{4}\right)$$

and for $\gamma < 2$, $\alpha > 0$, $\xi \geq 0$, $t > 0$,

$$\int_0^t \left(t^2-u^2\right)^{\alpha-1} u^{1-\gamma} I_\gamma(u\xi)dt = \frac{\xi^\gamma t^{2\alpha}}{2^{\gamma+1}\alpha\Gamma(\gamma+1)} \times {}_1F_2\left(1;\alpha+1,\gamma+1;\frac{t^2\xi^2}{4}\right).$$

The *Appell hypergeometric function* $F_4(a,b,c_1,c_2;x,y)$ (see [456], p. 658) for $|x|^{1/2}+|y|^{1/2}<1$ has the form

$$F_4(a,b,c_1,c_2;x,y) = \sum_{m,n=0}^\infty \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n\, m!\, n!} x^m y^n. \tag{1.36}$$

For $|x|^{1/2}+|y|^{1/2} \geq 1$, function $F_4(a,b;c_1,c_2;x,y)$ is understood as an analytical continuation, which is determined by the formulas from [130].

Struve functions (of any order) can be expressed in terms of the generalized hypergeometric function ${}_1F_2$:

$$\mathbf{H}_\alpha(z) = \frac{z^{\alpha+1}}{2^\alpha\sqrt{\pi}\,\Gamma\left(\alpha+\frac{3}{2}\right)} {}_1F_2\left(1,\tfrac{3}{2},\alpha+\tfrac{3}{2},-\tfrac{z^2}{4}\right).$$

The Lauricella function ([500], p. 33) is

$$\begin{aligned}&\mathbf{F}_{\gamma}^{(n)}(a_1,...,a_n,b_1,...,b_n;c;z_1,...,z_n)\\&=\sum_{m_1,...,m_n}^{\infty}\frac{(a_1)_{m_1}...(a_n)_{m_n}(b_1)_{m_1}...(b_n)_{m_n}}{(c)_{m_1+...+m_n}}\frac{z_1^{m_1}}{m_1!}...\frac{z_n^{m_n}}{m_n!},\\&\max\{|z_1|,...,|z_n|\}<1.\end{aligned} \tag{1.37}$$

The *Fox–Wright function* ${}_p\Psi_q(z)$ is defined for $z\in\mathbb{C}$, $a_l,b_j\in\mathbb{C}$, $\alpha_l,\beta_j\in\mathbb{R}$, $l=1,...,p$, $j=1,...,q$ by the series

$${}_p\Psi_q(z)={}_p\Psi_q\left[\begin{matrix}(a_l,\alpha_l)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|z\right]=\sum_{k=0}^{\infty}\frac{\prod\limits_{l=1}^{p}\Gamma(a_l+\alpha_l k)}{\prod\limits_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{z^k}{k!}. \tag{1.38}$$

If the condition

$$\sum_{j=1}^{q}\beta_j-\sum_{l=1}^{p}\alpha_l>-1$$

is satisfied, the series in (1.38) is convergent for any $z\in\mathbb{C}$. Let

$$\delta=\prod_{l=1}^{p}|\alpha_l|^{-\alpha_l}\prod_{j=1}^{q}|\beta_j|^{\beta_j},$$

$$\mu=\sum_{j=1}^{q}b_j-\sum_{l=1}^{p}a_l+\frac{p-q}{2}.$$

If

$$\sum_{j=1}^{q}\beta_j-\sum_{l=1}^{p}\alpha_l=-1,$$

then the series in (1.38) is absolutely convergent for $|z|<\delta$ and for $|z|=\delta$ and $\mathrm{Re}\,\mu>\frac{1}{2}$.

The *Mittag-Leffler function* $E_{\alpha,\beta}(z)$ is the entire function of order $1/\alpha$ defined by the following series when the real part of α is strictly positive:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad z\in\mathbb{C},\ \alpha,\beta\in\mathbb{C},\ \mathrm{Re}\,\alpha>0,\ \mathrm{Re}\,\beta>0. \tag{1.39}$$

Function (1.39) was introduced by Gesta Mittag-Leffler in 1903 for $\alpha=1$ and by A. Wiman in 1905 in the general case. The first applications of these functions by

Mittag-Leffler and Wiman were applications in complex analysis (nontrivial examples of entire functions with noninteger orders of growth and generalized summation methods). In the USSR, these functions became mainly known after the publication of the famous monograph by M. M. Dzhrbashyan [98] (see also his later monograph [106]). The most famous application of the Mittag-Leffler functions in the theory of integro-differential equations and fractional calculus is the fact that through them the resolvent of the Riemann–Liouville fractional integral is explicitly expressed in accordance with the famous Hille–Tamarkin–Dzhrbashyan formula [494]. In view of the numerous applications to the solution of fractional differential equations, this function was deservedly named in [202] "*Royal function of fractional calculus*."

The derivative of the Mittag-Leffler function is calculated by the formula

$$E'_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta}(z)}{dz} = \sum_{k=0}^{\infty} \frac{(1+k)z^k}{\Gamma(\beta+\alpha(1+k))}.$$

Note that

$$\begin{aligned}
&E_{\alpha,\beta}(0) = 1,\\
&E_{0,1}(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z},\\
&E_{1,1}(z) = e^z, \qquad E_{1,2}(z) = \frac{e^z-1}{z}, \qquad E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}.
\end{aligned}$$

Using the Fox–Wright function (1.38) we can write

$$E_{\alpha,\beta}(z) = {}_1\Psi_1\left[\left.\begin{matrix}(1,1)\\(\beta,\alpha)\end{matrix}\right| z\right]. \tag{1.40}$$

A general definition of the Meijer G-function is given by the following line integral in the complex plane (see [20], p. 206):

$$\begin{aligned}
&G^{m,n}_{p,q}\left(\left.\begin{matrix}a_1,\dots,a_p\\b_1,\dots,b_q\end{matrix}\right| z\right)\\
&\quad = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}\, z^s\, ds.
\end{aligned} \tag{1.41}$$

The general Legendre equation reads

$$(1-x^2)\,y'' - 2xy' + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-x^2}\right] y = 0,$$

where the numbers λ and μ may be complex. This differential equation has two linearly independent solutions, which can both be expressed in terms of the hypergeometric function ${}_2F_1$:

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left[\frac{1+z}{1-z}\right]^{\mu/2} {}_2F_1\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right), \tag{1.42}$$

for $|1-z| < 2$,

$$Q_\lambda^\mu(z) = \frac{\sqrt{\pi}\,\Gamma(\lambda+\mu+1)}{2^{\lambda+1}\Gamma(\lambda+3/2)}\frac{e^{i\mu\pi}(z^2-1)^{\mu/2}}{z^{\lambda+\mu+1}} {}_2F_1\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2}; \lambda+\frac{3}{2}; \frac{1}{z^2}\right), \tag{1.43}$$

for $|z| > 1$.

Functions P_λ^μ and Q_λ^μ are generally known as Legendre functions of the first and second kind of noninteger degree, with the additional qualifier "associated" if μ is nonzero.

We will use also Legendre functions with $z = x$, where $-1 < x < 1$ (see [19]). The interval $(-1, 1)$ is "the cut." If μ is even integer, then the values of $P_\lambda^\mu(z)$ on both sides of the cut are equal and in this case it is sufficient to take the branch cut along the real axis from -1 to $-\infty$. In all other cases $P_\lambda^\mu(x-i0)$ and $P_\lambda^\mu(x+i0)$ are different (here $f(x \pm i0)$ means $\lim\limits_{\varepsilon\to 0} f(x \pm i\varepsilon)$, $\varepsilon > 0$). In order to avoid ambiguity it is usual to introduce slightly modified Legendre functions. These will be denoted by $\mathbb{P}_\nu^\mu(z)$ and $\mathbb{Q}_\nu^\mu(z)$:

$$\mathbb{P}_\nu^\mu(x) = \frac{1}{2}\left[e^{i\frac{\mu\pi}{2}} P_\nu^\mu(x+i0) + e^{-i\frac{\mu\pi}{2}} P_\nu^\mu(x-i0)\right], \tag{1.44}$$

$$\mathbb{Q}_\nu^\mu(x) = \frac{1}{2}e^{-i\mu\pi}\left[e^{-i\frac{\mu\pi}{2}} Q_\nu^\mu(x+i0) + e^{i\frac{\mu\pi}{2}} Q_\nu^\mu(x-i0)\right]. \tag{1.45}$$

1.1.4 Polynomials

Gegenbauer polynomials or ultraspherical polynomials $C_n^{(\alpha)}(x)$ are orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight function $(1-x^2)^{\alpha-\frac{1}{2}}$ that can be defined by the recurrence relation

$$C_0^\alpha(x) = 1, \qquad C_1^\alpha(x) = 2\alpha x,$$
$$C_n^\alpha(x) = \frac{1}{n}[2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-2)C_{n-2}^\alpha(x)].$$

The next decomposition is valid:

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n.$$

Gegenbauer polynomials are particular solutions of the Gegenbauer differential equation

$$(1-x^2)y''-(2\alpha+1)xy'+n(n+2\alpha)y=0.$$

When $\alpha=\frac{1}{2}$, this equation reduces to the Legendre equation, and the Gegenbauer polynomials reduce to the Legendre polynomials. When $\alpha=1$, the equation reduces to the Chebyshev differential equation, and the Gegenbauer polynomials reduce to the Chebyshev polynomials of the second kind.

They are given as Gauss hypergeometric functions in certain cases where the series is in fact finite (see [2], p. 561):

$$\begin{aligned} C_n^{(\alpha)}(z) &= \frac{(2\alpha)_n}{n!}\,{}_2F_1\left(-n,2\alpha+n;\alpha+\frac{1}{2};\frac{1-z}{2}\right)\\ &= \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!(n-2k)!}(2z)^{n-2k}. \end{aligned}$$

The Gegenbauer polynomial can also be represented by the Rodrigues formula

$$C_n^{(\alpha)}(x)=\frac{(-1)^n}{2^n n!}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(n+2\alpha)}{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})}(1-x^2)^{-\alpha+1/2}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\alpha-1/2}\right].$$

1.2 Functional spaces

1.2.1 Orthant $\mathbb{R}^n_+$, C^m_{ev}, S_{ev}, and L^γ_p spaces

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space,

$$\begin{aligned} &\mathbb{R}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n,\ \ x_1>0,\ldots,x_n>0\},\\ &\overline{\mathbb{R}}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n,\ \ x_1\geq 0,\ldots,x_n\geq 0\}, \end{aligned}$$

$\gamma=(\gamma_1,...,\gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, $i=1,...,n$, and $|\gamma|=\gamma_1+\ldots+\gamma_n$.

The part of the sphere of radius r with center at the origin belonging to $\mathbb{R}^n_+$ we will denote $S^+_r(n)$:

$$S^+_r(n)=\{x\in\overline{\mathbb{R}}^n_+:|x|=r\}\cup\{x\in\overline{\mathbb{R}}^n_+:x_i=0,|x|\leq r,i=1,...,n\}.$$

Let Ω be a finite or infinite open set in $\mathbb{R}^n$ symmetric with respect to each hyperplane $x_i=0$, $i=1,...,n$, $\Omega_+=\Omega\cap\mathbb{R}^n_+$, and $\overline{\Omega}_+=\Omega\cap\overline{\mathbb{R}}^n_+$, where $\overline{\mathbb{R}}^n_+=\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n, x_1\geq 0,\ldots,x_n\geq 0\}$. We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all derivatives of these functions with respect to x_i

for any $i = 1, ..., n$ are continuous up to x_i=0. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\right|_{x_i=0} = 0$ for all nonnegative integers $k \leq \frac{m-1}{2}$ (see [610] and [242], p. 21). In the following we will denote $C^m_{ev}(\overline{\mathbb{R}}^n_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+) = \bigcap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m and $C^\infty_{ev}(\overline{\mathbb{R}}_+) = C^\infty_{ev}$.

As the space of basic functions we will use the subspace of the space of rapidly decreasing functions:

$$S_{ev} = \left\{ f \in C^\infty_{ev} : \sup_{x \in \mathbb{R}^n_+} \left| x^\alpha D^\beta f(x) \right| < \infty \quad \forall \alpha, \beta \in \mathbb{Z}^n_+ \right\},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $\beta = (\beta_1, ..., \beta_n)$, $\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n$ are integer nonnegative numbers, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, $D^\beta = D^{\beta_1}_{x_1} ... D^{\beta_n}_{x_n}$, $D_{x_j} = \frac{\partial}{\partial x_j}$.

Let $L^\gamma_p(\mathbb{R}^n_+) = L^\gamma_p$, $1 \leq p < \infty$, be the space of all measurable in $\mathbb{R}^n_+$ functions even with respect to each variable x_i, $i = 1, ..., n$, such that

$$\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma dx < \infty,$$

where

$$x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p \geq 1$, the L^γ_p-norm of f is defined by

$$||f||_{L^\gamma_p(\mathbb{R}^n_+)} = ||f||_{p,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} |f(x)|^p x^\gamma dx \right)^{1/p}.$$

It is known (see [242]) that L^γ_p is a Banach space.

1.2.2 Weighted measure, space L^γ_∞, and definition of weak $(p,q)_\gamma$ type operators

The weighted measure of Ω_+ is denoted by $\mathrm{mes}_\gamma(\Omega_+)$ and is defined by the formula

$$\mathrm{mes}_\gamma(\Omega_+) = \int\limits_{\Omega_+} x^\gamma dx.$$

Let us consider the space with positive weighted measure mes_γ. For a scalar valued measurable function f that takes finite values almost everywhere we define

$$\mu_\gamma(f,t) = \mathrm{mes}_\gamma\{x \in \mathbb{R}^n_+ : |f(x)| > t\} = \int\limits_{\{x:\,|f(x)|>t\}^+} x^\gamma dx,$$

where $\{x{:}|f(x)|{>}t\}^+{=}\{x{\in}\mathbb{R}^n_+{:}|f(x)|{>}t\}$. We will call the function $\mu_\gamma = \mu_\gamma(f,t)$ a *weighted distribution function* of $|f(x)|$.

Statement 1. *For any function $f \in L^\gamma_p(\mathbb{R}^+_n)$ the following equality is correct:*

$$||f||_{L^\gamma_p} = \left(p \int\limits_0^\infty t^{p-1}\mu_\gamma(f,t)dt \right)^{1/p}. \tag{1.46}$$

Proof. Let us first suppose that the function f is continuous in $\mathbb{R}^+_n$ and has a limited support Ω^+. This area is divided into parts as follows. Let

$$m = t_0 < t_1 < t_2 < \ldots < t_\ell = M,$$

where m and M are the largest and the smallest value of the function $|f|$ on the $\overline{\Omega^+}$. We introduce the following partition of this area:

$$\Omega^+ = \bigcup_{i=1}^{\ell} \Omega^+_i = \bigcup_{i=1}^{\ell} \{x : \ t_{i-1} < |f(x)| < t_i\}^+ .$$

It is easy to see that the weight measure of a subset Ω^+_i is represented as a difference of the weight distribution functions at the points t_i and t_{i+1}:

$$\mathrm{mes}_\gamma\{\Omega^+_i\} = \mu_\gamma(f;t_i) - \mu_\gamma(f;t_{i+1}).$$

Then

$$\int\limits_{\mathbb{R}^+_{\mathbb{N}}} |f(x)|^p x^\gamma \, dx = \int\limits_{\Omega^+} |f(x)|^p x^\gamma \, dx$$

$$= \lim_{\lambda\to 0} \sum_{i=1}^{\ell} |f(\xi_i)|^p \,\mathrm{mes}_\gamma\, \Omega_i = \lim_{\lambda\to 0} \sum_{i=1}^{\ell} |f(\xi_i)|^p \left[\mu_\gamma(f;t_{i+1}) - \mu_\gamma(f;t_i) \right]$$

$$= -\lim_{\lambda\to 0} \sum_{i=1}^{\ell} t_i^p \left[\mu_\gamma(f;t_i) - \mu_\gamma(f;t_{i+1}) \right]$$

$$= -\int\limits_m^M t^p \, d_t\mu_\gamma(f,t)dt = -\int\limits_0^\infty t^p \, d_t\mu_\gamma(f,t)dt,$$

where λ is the maximum partition size and $\xi_i \in \Omega_i^+$ is the midpoint of the i-th partition and we took into account that $\mu_\gamma(f;t)$ for the continuous function f is a differentiable function. Now equality (1.46) is obtained by integration by parts.

Now let $f \in L_p^\gamma(\mathbb{R}_n^+)$. In this case an infinitely differentiable function f_ε exists such that

$$\lim_{\varepsilon\to 0}\|f - f_\varepsilon\|_{L_p^\gamma} = 0.$$

It remains to write equality (1.46) for the function f_ε and passing to the limit at $\varepsilon \to 0$ taking into account that both functions $\mu_\gamma(f_\varepsilon;t)$ and $\mu_\gamma(f;t)$ are monotone and that

$$\lim_{\varepsilon\to 0}\mu_\gamma(f_\varepsilon;t) = \mu_\gamma(f;t).$$

The proof is complete. □

The space $L_\infty^\gamma(\mathbb{R}_+^n)=L_\infty^\gamma$ is the space of all measurable in $\mathbb{R}_+^n$ functions even with respect to each variable x_i, $i = 1, ..., n$, for which the norm

$$||f||_{L_\infty^\gamma(\mathbb{R}_+^n)} = ||f||_{\infty,\gamma} = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^n}{}_\gamma |f(x)| = \inf_{a\in\mathbb{R}}\{\mu_\gamma(f,a) = 0\}$$

is finite.

Statement 2. *Norms of spaces L_p^γ and L_∞^γ related by equality*

$$||f||_{\infty,\gamma} = \lim_{p\to\infty}||f||_{p,\gamma}, \qquad f \in L_\infty^\gamma. \tag{1.47}$$

Proof. If $||f||_{\infty,\gamma} = 0$, then equality (1.47) is obvious.

Let $0<||f||_{\infty,\gamma}<\infty$. We introduce the notation $S_f^\gamma = \operatorname*{ess\,sup}_{x\in\mathbb{R}_+^n}{}_\gamma |f(x)| = ||f||_{\infty,\gamma}$.

We have

$$||f||_{p,\gamma} = \left(\int\limits_{\mathbb{R}_+^n}|f(x)|^p x^\gamma dx\right)^{1/2} \le (S_f^\gamma)^{1/2}\left(\int\limits_{\mathbb{R}_+^n}|f(x)|^{p/2} x^\gamma dx\right)^{1/p}.$$

Then

$$\begin{aligned}\overline{\lim_{p\to\infty}}||f||_{p,\gamma} &\le (S_f^\gamma)^{1/2}\overline{\lim_{p\to\infty}}\left(\int\limits_{\mathbb{R}_+^n}|f(x)|^{p/2} x^\gamma dx\right)^{1/p} \\ &= (S_f^\gamma)^{1/2}\left[\overline{\lim_{p\to\infty}}\left(\int\limits_{\mathbb{R}_+^n}|f(x)|^p x^\gamma dx\right)^{1/p}\right]^{1/2} \\ &= (S_f^\gamma)^{1/2}\overline{\lim_{p\to\infty}}||f||_{p,\gamma}^{1/2},\end{aligned}$$

so we get

$$\overline{\lim_{p\to\infty}}||f||_{p,\gamma} \leq S_f^\gamma. \tag{1.48}$$

From $S_f^\gamma = \operatorname{ess\,sup}_{\gamma,\, x\in\mathbb{R}_+^n} |f(x)| = \inf_{a\in\mathbb{R}}\{\mu_\gamma(f,a)=0\}$ it follows that for all $\varepsilon \in (0, S_f^\gamma]$ the set $E \subset \mathbb{R}_+^n$ such that $\text{mes}_\gamma E < \infty$ and

$$|f(x)| > S_f^\gamma - \varepsilon, \quad \forall x \in E,$$

exists. We obtain

$$\left[\int\limits_E (S_f^\gamma - \varepsilon)^p x^\gamma dx\right]^{1/p} < \left[\int\limits_E |f(x)|^p x^\gamma dx\right]^{1/p} \leq ||f||_{p,\gamma},$$

which implies that

$$(S_f^\gamma - \varepsilon)(\text{mes}_\gamma E)^{1/p} \leq ||f||_{p,\gamma}$$

and

$$\underline{\lim}_{p\to\infty} ||f||_{p,\gamma} \geq S_f^\gamma - \varepsilon$$

or, due to the arbitrariness of ε,

$$\underline{\lim}_{p\to\infty} ||f||_{p,\gamma} \geq S_f^\gamma. \tag{1.49}$$

From (1.48) and (1.49) it follows that $\lim\limits_{p\to\infty} ||f||_{p,\gamma} = S_f^\gamma$. □

Note that for $f \in L_\infty$, statement 2 is well known (see [419]).

A linear operator A is of *strong type* $(p,q)_\gamma$, $1\leq p\leq\infty$, $1\leq q\leq\infty$, if it is defined from L_p^γ to L_q^γ and the following inequality is valid:

$$||Af||_{q,\gamma} \leq h||f||_{p,\gamma}, \quad \forall f \in L_p^\gamma, \tag{1.50}$$

where constant h does not depend on f.

We say that a linear operator A is an operator of *weak type* $(p,q)_\gamma$ $(1 \leq p \leq \infty$, $1 \leq q < \infty)$ if

$$\mu_\gamma(Af,\lambda) \leq \left(\frac{h||f||_{p,\gamma}}{\lambda}\right)^q, \quad \forall f \in L_p^\gamma, \tag{1.51}$$

where h does not depend on f and λ, $\lambda > 0$.

If $q=\infty$, then a linear operator A is an operator of weak type $(p,q)_\gamma$ when it has strong type $(p,q)_\gamma$.

1.2.3 Space of weighted generalized functions S'_{ev}, absolutely continuous functions, and unitary operators

For $1 \leq p \leq \infty$, $L^\gamma_{p,loc}(\mathbb{R}^n_+) = L^\gamma_{p,loc}$ is the set of functions $u(x)$ defined almost everywhere in $\mathbb{R}^n_+$ such that $uf \in L^\gamma_p$ for any $f \in S_{ev}$.

Definition 2. *Let $\overset{\circ}{C}{}^k_{ev}(X)$ be the space of all functions $u \in C^k_{ev}(X)$ with compact support.*

Definition 3. *Let $L^1_{loc,\gamma}(X)$, $X \subset \mathbb{R}^n_+$, be the space of all functions integrable with the weight x^γ on compact subsets in X:*

$$f \in L^1_{loc,\gamma}(X) \quad \Leftrightarrow \quad \int\limits_X |f(x)|x^\gamma dx < \infty.$$

Definition 4. *The* ***space of weighted generalized functions*** *$S'_{ev}(\mathbb{R}^n_+) = S'_{ev}$ is a class of continuous linear functionals that map a set of test functions $f \in S_{ev}$ into the set of real numbers. Each function $u(x) \in L^\gamma_{1,loc}$ will be identified with the functional $u \in S'_{ev}(\mathbb{R}^n_+) = S'_{ev}$ acting according to the formula*

$$(u, f)_\gamma = \int\limits_{\mathbb{R}^n_+} u(x)\, f(x)\, x^\gamma\, dx, \qquad f \in S_{ev}. \tag{1.52}$$

Generalized functions $u \in S'_{ev}$ acting by formula (1.52) will be called ***regular weighted generalized functions****. All other generalized functions $u \in S'_{ev}$ will be called* ***singular weighted generalized functions****.*

As we have seen, a singular weighted generalized function cannot be identified with any locally integrable function. The simplest example of a singular weighted generalized function is the weighted delta function.

The weighted delta function $\delta_\gamma \in S'_{ev}$ is defined by the equality (by analogy with [177], p. 247)

$$(\delta_\gamma, \varphi)_\gamma = \varphi(0), \quad \varphi(x) \in S_{ev}.$$

The fact that this generalized function is weighted is explained as follows. Let

$$\omega_\varepsilon(x) = \begin{cases} lrC_\varepsilon\, e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}} & |x| \leq \varepsilon, \\ 0 & |x| > \varepsilon, \end{cases}$$

where C_ε is selected such that

$$\int\limits_{\mathbb{R}^n_+} \omega_\varepsilon(x) x^\gamma dx = 1.$$

Since

$$\lim_{\varepsilon\to+0}\int\limits_{\mathbb{R}^n_+}\omega_\varepsilon(x)\varphi(x)x^\gamma\,dx=\varphi(0),\qquad \varphi\in S_{ev},$$

we have

$$(\omega_\varepsilon(x),\varphi(x))_\gamma\to(\delta_\gamma(x),\varphi(x))_\gamma,\qquad \varepsilon\to+0,\qquad \varphi\in S_{ev}.$$

Considering that for convenience we will write

$$(\delta_\gamma,\varphi)_\gamma=\int\limits_{\mathbb{R}^n_+}\delta_\gamma(x)\varphi(x)x^\gamma\,dx=\varphi(0)$$

and understand it in the sense of the limit of delta-shaped sequences.

Following [494] we give the definition of the space $AC(\Omega)$ of absolutely continuous functions. Let $\Omega=[a,b]$.

Definition 5. *A function $f(x)$ is called absolutely continuous on an interval Ω if for any $\varepsilon>0$ there exists a $\delta>0$ such that for any finite set of pairwise n nonintersecting intervals $[a_k,b_k]\subset\Omega$, $k=1,2,...,n$, such that for $\sum\limits_{k=1}^{n}(b_k-a_k)<\delta$, the inequality $\sum\limits_{k=1}^{n}|f(b_k)-f(a_k)|<\varepsilon$ holds. The space of these functions is denoted by $AC(\Omega)$.*

The space $AC(\Omega)$ coincides with the space of primitives of Lebesgue summable functions (see [266], p. 338):

$$f(x)\in AC(\Omega)\Leftrightarrow f(x)=\int\limits_a^x\varphi(t)dt+c,\qquad \int\limits_a^b|\varphi(t)|dt<\infty.$$

Definition 6. *Let us denote by $AC^n(\Omega)$, where $n=1,2,...$, the space of functions $f(x)$ which have continuous derivatives up to order $n-1$ on Ω with $f^{(n-1)}(x)\in AC(\Omega)$.*

It is clear that $AC^1(\Omega)=AC(\Omega)$.

The space $AC^n(\Omega)$ consists of those and only those functions $f(x)$ which are represented in the form

$$f(x)=\frac{1}{(n-1)!}\int\limits_a^x(x-t)^{n-1}\varphi(t)dt+\sum_{k=0}^{n-1}c_k(x-a)^k,$$

where $\varphi(t)\in L_1(a,b)$, c_k are arbitrary constants.

A *unitary operator* is a surjective bounded operator on a Hilbert space preserving the inner product. Unitary operators are usually taken as operating on a Hilbert space, but the same notion serves to define the concept of isomorphism between Hilbert spaces.

1.2.4 Mixed case

In this subsection we give a summary of the basic notations, terminology, and results connected with the case when the Bessel operator acts on all variables except the first one.

Suppose that $\mathbb{R}^{n+1}$ is the $(n+1)$-dimensional Euclidean space,

$$\mathbb{R}^{n+1}_{+}=\{(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{R}^{n+1},\ \ x_1>0,\ldots,x_n>0\},$$

$\gamma=(\gamma_1,...,\gamma_n)$ is a multi-index consisting of positive fixed real numbers γ_i, $i=1,...,n$, and $|\gamma|=\gamma_1+\ldots+\gamma_n$. Let Θ be a finite or infinite open set in $\mathbb{R}^{n+1}$ symmetric with respect to each hyperplane $x_i=0$, $i=1,...,n$, and $\Theta_+=\Theta\cap\mathbb{R}^{n+1}_+$ and $\overline{\Theta}_+=\Theta\cap\overline{\mathbb{R}}^{n+1}_+$, where $\overline{\mathbb{R}}^{n+1}_+=\{(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{R}^{n+1}, x_1\geq 0,\ldots,x_n\geq 0\}$. We deal with the class $C^m(\Theta_+)$ consisting of m times differentiable on Θ_+ functions and denote by $C^m(\overline{\Theta}_+)$ the subset of functions from $C^m(\Theta_+)$ such that all existing derivatives of these functions with respect to x_i for any $i=1,...,n$ are continuous up to $x_i=0$ and all existing derivatives with respect to t are continuous for $t\in\mathbb{R}$. Class $\mathfrak{C}^m_{ev}(\overline{\Theta}_+)$ consists of all functions from $C^m(\overline{\Theta}_+)$ such that $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{x=0}=0$ for all nonnegative integers $k\leq\frac{m-1}{2}$ and for $i=1,...,n$ (see [610] and [242], p. 21). In the following we will denote $\mathfrak{C}^m_{ev}(\overline{\mathbb{R}}^{n+1}_+)$ by $\mathfrak{C}^m_{ev}$. We set

$$\mathfrak{C}^\infty_{ev}(\overline{\Theta}_+)=\bigcap\mathfrak{C}^m_{ev}(\overline{\Theta}_+)$$

with intersection taken for all finite m. Let $\mathfrak{C}^\infty_{ev}(\overline{\mathbb{R}}^{n+1}_+)=\mathfrak{C}^\infty_{ev}$. Assume that $\overset{\circ}{\mathfrak{C}}{}^\infty_{ev}(\overline{\Theta}_+)$ is the space of all functions $f\in\mathfrak{C}^\infty_{ev}(\overline{\Theta}_+)$ with a compact support. We will use the notation $\overset{\circ}{\mathfrak{C}}{}^\infty_{ev}(\overline{\Theta}_+)=\mathfrak{D}_+(\overline{\Theta}_+)$.

Let $\mathcal{L}^\gamma_p(\Theta_+)$, $1\leq p<\infty$, be the space of all measurable in Θ_+ functions such that

$$\int\limits_{\Theta_+}|f(t,x)|^p x^\gamma\,dtdx<\infty,\qquad x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p\geq 1$, the $\mathcal{L}^\gamma_p(\Theta_+)$-norm of f is defined by

$$||f||_{\mathcal{L}^\gamma_p(\Theta_+)}=\left(\int\limits_{\Theta_+}|f(t,x)|^p x^\gamma\,dtdx\right)^{1/p}.$$

The weighted measure of Θ_+ is denoted by $\mathrm{mesm}_\gamma(\Theta_+)$ and is defined by the formula

$$\mathrm{mesm}_\gamma(\Theta_+)=\int\limits_{\Theta_+}x^\gamma\,dtdx.$$

For every measurable function $f(x)$ defined on $\mathbb{R}^{n+1}_+$ we consider

$$\mathfrak{M}_\gamma(f,\sigma) = \mathrm{mesm}_\gamma\{(t,x)\in\mathbb{R}^{n+1}_+ : |f(t,x)|>\sigma\} = \int\limits_{\{(t,x):\ |f(t,x)|>\sigma\}^+} x^\gamma\,dtdx,$$

where $\{(t,x):|f(t,x)|>\sigma\}^+=\{(t,x)\in\mathbb{R}^{n+1}_+:|f(t,x)|>\sigma\}$.

Let a space $\mathcal{L}^\gamma_\infty(\Theta_+)$ be defined as a set of measurable on Θ_+ functions $f(t,x)$ such that

$$||f||_{\mathcal{L}^\gamma_\infty(\Theta_+)} = \underset{(t,x)\in\Theta_+}{\mathrm{ess\,sup}_\gamma}|f(t,x)| = \inf_{\sigma\in\Theta_+}\{\mathfrak{M}_\gamma(f,\sigma)=0\}<\infty.$$

For $1\le p\le\infty$, $\mathcal{L}^\gamma_{p,loc}(\Theta_+)$ is the set of functions u defined almost everywhere in Θ_+ such that $uf\in\mathcal{L}^\gamma_p(\Theta_+)$ for any $f\in\mathfrak{D}_+(\overline{\Theta}_+)$.

Let us define $\mathfrak{D}'_+(\overline{\Theta}_+)$ as a set of continuous linear functionals on $\overline{\Theta}_+$. Each function $u\in\mathcal{L}^\gamma_{1,loc}(\Theta_+)$ will be identified with the functional $u\in\mathfrak{D}'_+(\overline{\Theta}_+)$ acting according to the formula

$$(u,f)_\gamma = \int\limits_{\Theta_+} u(t,x)\,f(t,x)\,x^\gamma\,dtdx,\qquad f\in\mathfrak{D}_+(\overline{\Theta}_+). \tag{1.53}$$

Functionals $u\in\mathfrak{D}'_+(\overline{\Theta}_+)$ acting by formula (1.53) will be called *mixed regular weighted functionals*. All other continuous linear functionals $u\in\mathfrak{D}'_+(\overline{\Theta}_+)$ will be called *mixed singular weighted functionals*.

The generalized function δ_γ is defined by the equality (by analogy with [242], p. 12)

$$(\delta_\gamma,\varphi)_\gamma=\varphi(0),\quad \varphi\in\mathfrak{D}_+(\overline{\Theta}_+).$$

As the space of basic functions we will use the subspace of rapidly decreasing functions:

$$\mathfrak{S}_{ev}(\mathbb{R}^{n+1}_+) = \left\{ f\in\mathfrak{C}^\infty_{ev}:\ \sup_{(t,x)\in\mathbb{R}^{n+1}_+}\left|t^{\alpha_0}x^\alpha D^\beta f(t,x)\right|<\infty\right\},$$

where $\alpha=(\alpha_1,...,\alpha_n)$, $\beta=(\beta_0,\beta_1,...,\beta_n)$, $\alpha_0,\alpha_1,...,\alpha_n,\beta_0,\beta_1,...,\beta_n$ are arbitrary integer nonnegative numbers, and $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$, $D^\beta=D_t^{\beta_0}D_{x_1}^{\beta_1}...D_{x_n}^{\beta_n}$, $D_t=\frac{\partial}{\partial t}$, $D_{x_j}=\frac{\partial}{\partial x_j}$, $j=1,...,n$. In the same way as $\mathfrak{D}'_+$ we introduce the space $\mathfrak{S}'_{ev}$. In fact we identify S'_{ev} with a subspace of $\mathfrak{D}'_+$ since $\mathfrak{D}_+$ is dense in $\mathfrak{S}_{ev}$.

1.3 Integral transforms and Lizorkin–Samko space

1.3.1 One-dimensional integral transforms with Bessel functions in the kernels and Mellin transform

In this subsection, following [180], we consider some one-dimensional integral transforms which we will use later.

Definition 7. *The one-dimensional* ***Fourier transform*** *of an integrable function* $f: \mathbb{R} \to \mathbb{C}$ *is*

$$F[f](\xi) = \int\limits_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx,$$

for any real number ξ.

Under suitable conditions, f *is determined by* $F[f]$ *via the inverse transform:*

$$f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F[f](\xi)\, e^{ix\xi}\, d\xi,$$

for any real number x.

In $F[f]$, instead of the kernel $e^{-ix\xi}$, sometimes $e^{ix\xi}$, $e^{-2\pi ix\xi}$, or $(2\pi)^{-1/2}e^{\pm ix\xi}$ is chosen, as in certain instances these kernels are more convenient.

Theorem 1. *[180] Let* $f \in L_1$ *be piecewise smooth in each interval* $[a,b] \subset \mathbb{R}$. *Then we have for every* $x_0 \in \mathbb{R}$

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F[f](\xi)\, e^{ix_0\xi}\, d\xi = f(x_0)$$

if f *is continuous at* x_0, *and we have*

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F[f](\xi)\, e^{ix_0\xi}\, d\xi = \frac{f(x_0+0) - f(x_0-0)}{2}$$

if f *is discontinuous at* x_0, *and the integral in this case has to be understood in the sense of Cauchy's principal value.*

Let $f, g \in L_1 \cap L_2$. Then the **Parseval formula** follows

$$\int\limits_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx = \int\limits_{-\infty}^{\infty} F[f](\xi)\overline{F[g](\xi)}\, d\xi,$$

where the bar denotes complex conjugation.

The **Plancherel theorem**, which follows from the above, states that

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(f)(\xi)|^2\, d\xi.$$

Definition 8. *The **Laplace transform** of a function $f(t)$, defined for all real numbers $t > 0$, is the function $F(s)$, which is a unilateral transform defined by*

$$\mathcal{L}[f](s) = F(s) = \int_0^{\infty} f(t) e^{-st}\, dt, \tag{1.54}$$

where s is a complex number frequency parameter $s = \sigma + i\omega$, with real numbers σ and ω.

Let E_a, $a \in \mathbb{R}$, be the space of functions $f : \mathbb{R} \to \mathbb{C}$, $f \in L_1^{loc}(\mathbb{R})$, such that $\int\limits_0^{\infty} |f(t)| e^{-at} dt < \infty$ and $f(t)$ vanishes if $t < 0$.

Let $f \in E_a$. Then the Laplace integral (1.54) is absolutely and uniformly convergent on $\bar{H}_a = \{p : p \in \mathbb{C}, \operatorname{Re} p \geq a\}$. The Laplace transform of function $f \in E_a$ is bounded on $\bar{H}_a$ and it is an analytic function on $H_a = \{p : p \in \mathbb{C}, \operatorname{Re} p > a\}$.

Let $f \in E_a$ be smooth on every interval $(a, b) \in \mathbb{R}_+$. Then in points t of continuity the complex inversion formula

$$\mathcal{L}^{-1}[F](t) = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{ts} ds, \qquad c > a,$$

holds.

The Laplace transform of the Mittag-Leffler function multiplied by the power function is (see [241], p. 47, formula (1.9.13), where $\rho = 1$)

$$\mathcal{L}[x^{\beta-1} E_{\alpha,\beta}(\lambda x^{\alpha})](s) = \frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}. \tag{1.55}$$

Definition 9. *The one-dimensional **Hankel transform** of a function $f \in L_1^{\gamma}(\mathbb{R}_+^1)$ is expressed as*

$$F_{\gamma}[f](\xi) = F_{\gamma}[f(x)](\xi) = f(\xi) = \int_0^{\infty} f(x)\, j_{\frac{\gamma-1}{2}}(x\xi) x^{\gamma}\, dx, \tag{1.56}$$

where $\gamma > 0$ and the symbol j_{ν} is used for the normalized Bessel function of the first kind (1.19).

Let $f \in L_1^\gamma(\mathbb{R}_+)$ be of bounded variation in a neighborhood of a point x of continuity of f. Then for $\gamma > 0$ the inversion formula

$$F_\gamma^{-1}[\widehat{f}(\xi)](x) = f(x) = \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)} \int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\widehat{f}(\xi)\xi^\gamma\, d\xi \tag{1.57}$$

holds.

Definition 10. *For functions f the integral transform involving the normalized modified Bessel function of the second kind $k_{\frac{\gamma-1}{2}}$, $\gamma \geq 1$, as kernel is the* ***Meijer transform*** *defined by*

$$\mathcal{K}_\gamma[f](\xi) = F(\xi) = \int\limits_0^\infty k_{\frac{\gamma-1}{2}}(x\xi)\, f(x)x^\gamma\, dx. \tag{1.58}$$

Let $f \in L_1^{loc}(\mathbb{R}_+)$ and $f(t) = o\left(t^{\beta-\frac{\gamma}{2}}\right)$ as $t \to +0$, where $\beta > \frac{\gamma}{2} - 2$ if $\gamma > 1$ and $\beta > -1$ if $\gamma = 1$. Furthermore, let $f(t) = 0(e^{at})$ as $t \to +\infty$. Then its Meijer exists a.e. for $\mathrm{Re}\,\xi > a$ (see [180], p. 94).

Since $k_{-\frac{1}{2}}(z) = e^{-z}$,

$$\mathcal{K}_0[f](\xi) = F(\xi) = \int\limits_0^\infty e^{-x\xi}\, f(x)\, dx = \mathcal{L}[f](\xi)$$

is a Laplace transform (1.54).

Let S be the space of rapidly decreasing functions on $(0, \infty)$,

$$S = \left\{ f \in C^\infty(0,\infty) : \sup_{x\in(0,\infty)} \left| x^\alpha D^\beta f(x) \right| < \infty \quad \forall \alpha, \beta \in Z_+ \right\},$$

and $f \in S$. We also will use the following transforms with Bessel functions in the kernel:

$$H_\nu[f](x) = \int\limits_0^\infty (xt)^{\frac{1}{2}} J_\nu(xt) f(t) dt, \qquad \nu \geq -\frac{1}{2}, \tag{1.59}$$

and

$$\mathcal{Y}_\nu[f](x) = \int\limits_0^\infty (xt)^{\frac{1}{2}} Y_\nu(xt) f(t) dt, \tag{1.60}$$

where J_ν is the Bessel function of the first kind of order ν and Y_ν is the Bessel function of the second kind of order ν.

Transforms H_ν and $\mathcal{Y}_\nu$ are well studied in [477,478,166]. In these papers, their boundedness in $\mathcal{L}_{p,\mu}$-spaces have been completely examined. The $\mathcal{L}_{p,\mu}$-space is the space of functions (or, more precisely, equivalence classes) such that their Lebesgue integral $\int\limits_0^\infty |x^\mu f(x)|^p \frac{dx}{x}$ is finite. The $\mathcal{L}_{p,\mu}$-norm is defined by

$$||f||_{p,\mu} = \left(|x^\mu f(x)|^p \frac{dx}{x} \right)^{\frac{1}{2}}. \tag{1.61}$$

Here $1 \leq p < \infty$ and μ is any number.

Definition 11. *The* ***Mellin transform*** *of a function $f : \mathbb{R}_+ \to \mathbb{C}$ is the function f^* defined by*

$$f^*(s) = \mathcal{M}[f](s) = \int\limits_0^\infty x^{s-1} f(x) dx,$$

where $s = \sigma + i\tau \in \mathbb{C}$, provided that the integral exists.

As space of originals we choose the space P_a^b, $-\infty < a < b < \infty$, which is the linear space of $\mathbb{R}_+ \to \mathbb{C}$ functions such that $x^{s-1} f(x) \in L_1(\mathbb{R}_+)$ for every $s \in \{p \in \mathbb{C} : a \leq \operatorname{Re} p \leq b\}$.

If additionally $f^*(c + i\tau) \in L_1(\mathbb{R})$ with respect to τ, then the complex inversion formula holds:

$$\mathcal{M}^{-1}[\varphi](x) = f(x) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} x^{-s} \varphi(s)\, ds.$$

In fact, the gamma function $\Gamma(z)$ corresponds to the Mellin transform of the negative exponential function:

$$\Gamma(z) = \mathcal{M}[e^{-x}](z).$$

If an operator A is acting in the images of the Mellin transform as a multiplication to a function

$$\mathcal{M}[Af](s) = m_A \mathcal{M}[f](s), \tag{1.62}$$

then we name m_A a *multiplier of operator* A.

The *Mellin convolution* $(f * g)_M(y)$ of two functions f and g is given by

$$\int\limits_0^\infty f(x) g\left(\frac{y}{x}\right) \frac{dx}{x}. \tag{1.63}$$

We have

$$\mathcal{M}\left[\int_0^\infty K\left(\frac{x}{t}\right) f(t)\frac{dt}{t}\right](s) = \mathcal{M}[K](s)\,\mathcal{M}[f](s), \tag{1.64}$$

so $\mathcal{M}[K](s)$ is the multiplier for the Mellin convolution (see [361]).

1.3.2 Properties of composition of integral transforms with Bessel functions in the kernel

The next lemma, which allows to reduce the question of the boundedness of some operator in L_2 to study its multiplier, is well known (see [114,98,565]). We give this lemma here.

Lemma 1. *(1) Let for operator A equality (1.62) be true. Then, in order for expansion of A to the bounded operator in L_2 to be allowed, it is necessary and sufficient that*

$$\sup_{\xi\in\mathbb{R}}\left|m_A\left(i\xi+\frac{1}{2}\right)\right| = M_1 < \infty. \tag{1.65}$$

Thus we have $||A||_{L_2} = M_1$.

(2) In order for expansion of the inverse operator A^{-1} to the bounded operator in L_2 to be allowed, it is necessary and sufficient that

$$\inf_{\xi\in\mathbb{R}}\left|m_A\left(i\xi+\frac{1}{2}\right)\right| = m_1 < \infty. \tag{1.66}$$

Thus we have $||A^{-1}||_{L_2} = \frac{1}{m_1}$.

(3) Let operators A and A^{-1} be defined and bounded in L_2. Then in order for A and A^{-1} to be unitary it is necessary and sufficient that an equality

$$\left|m_A\left(i\xi+\frac{1}{2}\right)\right| = 1 \tag{1.67}$$

is true for almost all $\xi\in\mathbb{R}$.

Here, we give some more known results from [477,478,166] which we will use later.

Lemma 2. *(1) The integral transform H_ν bijectively maps the space $\mathcal{L}_{2,\mu}$ into itself when $\frac{1}{2}\le\mu<\nu+\frac{3}{2}$.*

(2) Let $f\in\mathcal{L}_{2,\mu}$, $\frac{1}{2}\le\mu<\frac{3}{2}-\nu$. Then for $\operatorname{Re}s=\mu$ we have

$$\mathcal{M}[H_\nu f](s) = m_\nu(s)\,\mathcal{M}[f](1-s), \tag{1.68}$$

where

$$m_\nu(s) = 2^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{2s+2\nu+1}{4}\right)}{\Gamma\left(\frac{2s-2\nu+3}{4}\right)}. \tag{1.69}$$

Lemma 3. *(1) The integral transform $\mathcal{Y}_\nu$ bijectively maps the space $\mathcal{L}_{2,\mu}$ into itself when $\frac{1}{2} \le \mu < \frac{3}{2} - |\nu|$, except for the case $\mu = \frac{1}{2} - \nu$. For $\mu = \frac{1}{2} - \nu$ we have $\mathcal{Y}_\nu[\mathcal{L}_{2,\frac{1}{2}-\nu}] = H_\nu[\mathcal{L}_{2,\frac{1}{2}-\nu}]$, and $\mathcal{Y}_\nu = C_{1\nu} H_\nu + C_{2\nu} H_{-\nu}$ when $-\frac{1}{2} < \nu < 0$ and $H_\nu[\mathcal{L}_{2,\frac{1}{2}-\nu}] = H_{-\nu}[\mathcal{L}_{2,\frac{1}{2}-\nu}] = \mathcal{L}_{2,\frac{1}{2}-\nu}$. It is obvious that $\mathcal{Y}_\nu[\mathcal{L}_{2,\frac{1}{2}-\nu}] \subset \mathcal{L}_{2,\frac{1}{2}-\nu}$. For $\mathcal{Y}_0$ it is proved in [478].*

(2) Let $f \in \mathcal{L}_{2,\mu}$, $\frac{1}{2} \le \mu < \nu + \frac{3}{2}$. Then for $\operatorname{Re} s = \mu$ *we have*

$$\mathcal{M}[\mathcal{Y}_\nu f](s) = -m_\nu(s) \operatorname{ctg}\left(s + \frac{1}{2} - |\nu|\right) \frac{\pi}{2} \mathcal{M}[f](1-s), \tag{1.70}$$

where $m_\nu(s) = 2^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{2s+2\nu+1}{4}\right)}{\Gamma\left(\frac{2s-2\nu+3}{4}\right)}$.

Now let us study the composition of operators H_ν and $\mathcal{Y}_\nu$.

Theorem 2. *Let $f \in \mathcal{L}_{2,\mu}$, $\frac{1}{2} \le \mu < \frac{3}{2} - \nu$. Then:*

1) Operator $H_\nu \mathcal{Y}_\nu$ acts in Mellin images according to (1.62) *with the multiplier*

$$m(s) = -\operatorname{tg}\left(\frac{2s+2\nu-1}{4}\right)\pi, \qquad \operatorname{Re} s = \mu. \tag{1.71}$$

2) The formula

$$H_\nu[\mathcal{Y}_\nu f](x) = \frac{2}{\pi} \int_0^\infty \frac{t f(t)}{t^2 - x^2} \left(\frac{x}{t}\right)^{\nu+\frac{1}{2}} dt \tag{1.72}$$

is valid.

Proof. 1) According to [478,166], if $f \in \mathcal{L}_{2,\mu}$, then $\mathcal{Y}_\nu f \in \mathcal{L}_{2,\mu}$ for $\frac{1}{2} \le \mu < \frac{1}{2} - |\nu|$ and we can apply formula (1.70) to $\mathcal{Y}_\nu f$:

$$\mathcal{M}[H_\nu \mathcal{Y}_\nu f](s) = m_\nu(s) \mathcal{M}[\mathcal{Y}_\nu f](1-s).$$

Then using (1.68) we get

$$\mathcal{M}[H_\nu \mathcal{Y}_\nu f](s) = -m_\nu(s)\, m_\nu(1-s) \operatorname{ctg}\left(\left(s + \frac{1}{2} - \nu\right)\frac{\pi}{2}\right) \mathcal{M}[f](s).$$

To prove (1.71) it remains to substitute

$$m_\nu(s) = 2^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{2s+2\nu+1}{4}\right)}{\Gamma\left(\frac{2s-2\nu+3}{4}\right)}, \qquad m_\nu(1-s) = 2^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{3-2s+2\nu}{4}\right)}{\Gamma\left(\frac{5-2s-2\nu}{4}\right)}$$

(see (1.69)) and to apply formulas (1.5) and (1.6).

Formula (1.71) can also be obtained directly. We denote $\mathcal{Y}_\nu[f](s)=g(s)$ and change the variables in (1.59) by $t=\frac{1}{y}$. We obtain

$$H_\nu[g](x)=\int\limits_0^\infty \mathfrak{K}_\nu\left(\frac{x}{y}\right)\frac{1}{y}\,g\left(\frac{1}{y}\right)\frac{dy}{y}, \tag{1.73}$$

where $\mathfrak{K}_\nu(x)=x^{\frac{1}{2}}\,J_\nu(x)$.

Then using formula (1.64) we get

$$\mathcal{M}H_\nu[g](s)=\mathcal{M}[\mathfrak{K}_\nu](s)\,\mathcal{M}\left[\frac{1}{y}\,g\left(\frac{1}{y}\right)\right]=\mathcal{M}[\mathfrak{K}_\nu](s)\,\mathcal{M}[g(y)](1-s).$$

We used formulas (3) and (4) from [19], p. 268, for obtaining the last expression. It is clear that

$$g(y)=\mathcal{Y}_\nu[f](y)=\int\limits_0^\infty \mathfrak{Y}_\nu\left(\frac{y}{p}\right)\frac{1}{p}\,f\left(\frac{1}{p}\right)\frac{dp}{p},$$

where $\mathfrak{Y}_\nu(x)=x^{\frac{1}{2}}\,Y_\nu(x)$. Then

$$\mathcal{M}[g](1-s)=\mathcal{M}[\mathcal{K}_\nu](1-s),$$
$$\mathcal{M}\left[\frac{1}{p}\,f\left(\frac{1}{p}\right)\right](1-s)=\mathcal{M}[\mathcal{K}_\nu](1-s)\,\mathcal{M}[f](s).$$

So

$$\mathcal{M}[H_\nu\,\mathcal{Y}_\nu\,f](s)=\mathcal{M}[K_1](s)\,\mathcal{M}[K_2](1-s)\,\mathcal{M}[f](s).$$

Substitution of values $\mathcal{M}[K_1](s)$ and $\mathcal{M}[K_2](1-s)$ from [19] into this expression gives (1.71).

2) Denoting $m(s)=\mathcal{M}[K](s)$ we can write

$$\mathcal{M}[H_\nu\,\mathcal{Y}_\nu\,f](s)=\mathcal{M}[K](s)\,\mathcal{M}[f](s).$$

Using formula (18) from [19], p. 302, we get

$$K(x)=\frac{2}{\pi}\,\frac{x^{\nu+\frac{1}{2}}}{1-x^2}.$$

Applying (1.64) we obtain (1.72). □

It is easy to see that for $\nu = \mp\frac{1}{2}$ the operator $H_\nu \mathcal{Y}_\nu$ is equal to Hilbert transform on semiaxes:

$$H_{\frac{1}{2}}[\mathcal{Y}_{\frac{1}{2}}](x) = \frac{2}{\pi}\int\limits_0^\infty \frac{xf(t)}{t^2 - x^2}dt.$$

It is easy to explain this fact. We know that (see formulas (14) and (15) in [21], p. 90)

$$J_{\frac{1}{2}}(x) = Y_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin(x),$$

$$Y_{\frac{1}{2}}(x) = -J_{-\frac{1}{2}}(x) = -\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos(x).$$

Then for $\nu = \frac{1}{2}$ operators H_ν and $\mathcal{Y}_\nu$ are cosine- and sine-Fourier transforms. Accordingly, for $\nu = -\frac{1}{2}$ operators H_ν and $\mathcal{Y}_\nu$ are sine- and cosine-Fourier transforms. Superposition of such transform operators is a Hilbert transform on semiaxes.

Theorem 3. *Let $f \in \mathcal{L}_{2,\mu}$, $\frac{1}{2} \le \mu < \frac{3}{2} - |\nu|$. Then:*

1) Operator $\mathcal{Y}_\nu H_\nu$ acts in Mellin images by formula (1.62) *with multiplier*

$$m(s) = -\operatorname{ctg}\left[\left(\frac{2s - 2\nu + 1}{4}\right)n\right], \qquad \operatorname{Re} s = \mu. \tag{1.74}$$

2) The formula

$$\mathcal{Y}_\nu[H_\nu f](x) = \frac{2}{\pi}\int\limits_0^\infty \left(\frac{x}{t}\right)^{\frac{1}{2}-\nu} \frac{tf(t)}{x^2 - t^2}\,dt \tag{1.75}$$

is valid.

The proof of this theorem resembles the proof of Theorem 2.

From (1.72) it is clear that $\mathcal{Y}_\nu H_\nu = -H_{-\nu}\mathcal{Y}_{-\nu}$ and we can consider only one of these compositions.

Next we consider norms of H_ν, $\mathcal{Y}_\nu$ and their compositions in $L_2(0,\infty)$. For estimation of norms of H_ν and $\mathcal{Y}_\nu$ we use (1.70) and (1.68), which we write in the following form:

$$\mathcal{M}[H_\nu f](s) = m_\nu(s)\,\mathcal{M}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right](s), \tag{1.76}$$

$$\mathcal{M}[\mathcal{Y}_\nu f](s) = -m_\nu(s)\,\cos\left(s + \frac{1}{2} - \nu\right)\frac{\pi}{2}\,\mathcal{M}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right](s). \tag{1.77}$$

To obtain (1.76) and (1.77) we used formulas (3) and (4) from [19], p. 268.

It is obvious that for operators H_ν and $\mathcal{Y}_\nu$ representation (1.62) does not hold; hence we cannot apply Lemma 1 directly. We introduce auxiliary operators $\hat{H}_\nu$ and $\hat{\mathcal{Y}}_\nu$ according to the formulas

$$\hat{H}_\nu[f](x) = \int_0^\infty J_\nu\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)^{\frac{1}{2}} f(t)\frac{dt}{t}, \tag{1.78}$$

$$\hat{\mathcal{Y}}_\nu[f](x) = \int_0^\infty Y_\nu\left(\frac{x}{t}\right)\left(\frac{x}{t}\right)^{\frac{1}{2}} f(t)\frac{dt}{t}. \tag{1.79}$$

Obviously (see (1.73)), for any function $f \in L_2$ we have

$$H_\nu f = \hat{H}_\nu \hat{f}, \qquad \mathcal{Y}_\nu f = \hat{\mathcal{Y}}_\nu \hat{f},$$

where

$$\hat{f}(x) = \frac{1}{x} f\left(\frac{1}{x}\right),$$

and

$$\mathcal{M}[H_\nu f](s) = \mathcal{M}[\hat{H}_\nu \hat{f}](s) = m_\nu(s)\mathcal{M}[\hat{f}](s),$$

$$\mathcal{M}[\mathcal{Y}_\nu f](s) = \mathcal{M}[\hat{\mathcal{Y}}_\nu \hat{f}](s) = m_\nu(s)\operatorname{ctg}\left(s+\frac{1}{2}-\nu\right)\frac{\pi}{2}\mathcal{M}[\hat{f}](s).$$

Thus for (1.78) and (1.79) the representation (1.62) holds with multipliers of operators H_ν and $\mathcal{Y}_\nu$, accordingly. Let show that

$$\left\|\frac{1}{x} f\left(\frac{1}{x}\right)\right\|_{L_2} = \|f\|_{L_2}.$$

Indeed

$$\int_0^\infty \left|\frac{1}{x} f\left(\frac{1}{x}\right)\right|^2 dx = -\int_\infty^0 |tf(t)|^2\frac{dt}{t^2} = \int_0^\infty |f(t)|^2 dt.$$

We have

$$||H_\nu f||_{L_2} = \sup_{f\in L_2}\frac{||H_\nu f||_{L_2}}{\|f\|_{L_2}} = \sup_{f\in L_2}\frac{||\hat{H}_\nu \hat{f}||_{L_2}}{\|\hat{f}\|_{L_2}} = ||\hat{H}_\nu||_{L_2}.$$

Similarly we obtain $||\mathcal{Y}_\nu||_{L_2} = ||\hat{\mathcal{Y}}_\nu||_{L_2}$. Now we can use Lemma 1 to prove the next two theorems.

Theorem 4. *For $\nu > -1$ operator H_ν is unitary in L_2.*

Proof. The space L_2 is obtained from $\mathcal{L}_{2,\mu}$ when $\mu = \frac{1}{2}$; therefore the multiplier of operator H_ν for $\nu > \frac{1}{2}$ is defined on the line $\operatorname{Re} s = \frac{1}{2}$. Let us write its values on this line:

$$\left| m\left(i\xi + \frac{1}{2}\right)\right| = \left|\frac{\Gamma\left(\frac{1+\nu-i\xi}{2}\right)}{\Gamma\left(\frac{1-\nu-i\xi}{2}\right)}\right|.$$

Considering that $|z| = |\bar{z}|$ and $\bar{\Gamma}(z) = \Gamma(\bar{z})$ we obtain

$$\left| m\left(i\xi + \frac{1}{2}\right)\right| = 1$$

for any $\xi \in \mathbb{R}$. That means unitarity of $\hat{H}_\nu$ and, consequently, H_ν by Lemma 1. $\square$

Remark 1. *It is easy to see that for complex ν the operator H_ν is not unitary in L_2. Indeed, let $\nu = \lambda + i\mu$, $\mu \neq 0$. Then*

$$\left| m\left(i\xi + \frac{1}{2}\right)\right| = \left|\frac{\Gamma\left(\frac{\lambda+1+i(\mu+\xi)}{2}\right)}{\Gamma\left(\frac{\lambda+1+i(\mu-\xi)}{2}\right)}\right|.$$

It is obvious that the modules of imaginary parts of the arguments of gamma functions are equal here if and only if $\xi = 0$ and, consequently, equality (1.67) is not true for almost all $\xi \in \mathbb{R}$.

Theorem 5. *For $\nu \in (-1, 1)$ the operator $\mathcal{Y}_\nu$ is bounded in L_2 and*

$$||\mathcal{Y}_\nu||_{L_2} = \begin{cases} 1 & -\frac{1}{2} \le \nu \le \frac{1}{2}, \\ \left|\operatorname{tg}\left(\frac{n\nu}{n}\right)\right| & \nu \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right). \end{cases}$$

Proof. Let us write the multiplier of operator $\mathcal{Y}_\nu$ on the line $\operatorname{Re} s = \frac{1}{2}$:

$$\left| m\left(i\xi + \frac{1}{2}\right)\right| = \left|\frac{\Gamma\left(\frac{1+\nu+i\xi}{2}\right)\Gamma\left(\frac{1-\nu+i\xi}{2}\right)}{\Gamma\left(\frac{1-\nu+i\xi}{2}\right)\Gamma\left(\frac{\nu-i\xi}{2}\right)}\right| = \left|\operatorname{tg}\left(\frac{i\xi-\nu}{2}\right)n\right|.$$

Here we use formulas (1.5), (1.6), and $|\Gamma(z)| = |\Gamma(\bar{z})|$. Since for $k \in \mathbb{Z}$

$$\lim_{z \to \frac{\pi}{2} + \pi k} \operatorname{tg}(z) = \infty$$

we require that $\nu \neq 2k+1$. For $-1<\nu<1$ this requirement is obviously satisfied. Because (see [20])

$$\lim_{\xi \to \infty} \left| m\left(i\xi + \frac{1}{2}\right)\right| = 1,$$

$\mathcal{Y}_\nu$ is bounded in L_2. Using

$$\operatorname{tg}(z) = (-i)\frac{e^{2iz}-1}{e^{2iz}+1},$$

we obtain

$$\left|\operatorname{tg}\left(\frac{i\xi-\nu}{2}\right)\pi\right|^2 = f_\nu(\xi),$$

where

$$f_\nu(\xi) = \frac{e^{-2i\xi} - 2e^{-n\xi}\cos(\nu\pi)+1}{e^{-2i\xi} + 2e^{-n\xi}\cos(\nu\pi)+1}.$$

We have

$$\sup_{\xi\in\mathbb{R}} f_\nu(\xi) = f_\nu(0) = \left|\operatorname{tg}\left(\frac{\nu}{2}\pi\right)\right|^2$$

for $\nu \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ and

$$\sup_{\xi\in\mathbb{R}} f_\nu(\xi) = f_\nu(\infty) = 1$$

for $-\frac{1}{2} \le \nu \le \frac{1}{2}$. Therefore, noting that $||\mathcal{Y}_\nu||_{L_2} = ||\hat{\mathcal{Y}}_\nu||_{L_2}$ we obtain the statement of the theorem by Lemma 1. $\square$

Theorem 6. *Operators $H_\nu\mathcal{Y}_\nu$ and $\mathcal{Y}_\nu H_\nu$ are bounded in L_2 for $-1 < \nu < 1$. The formula for norms*

$$||H_\nu\mathcal{Y}_\nu||_{L_2} = ||\mathcal{Y}_\nu H_\nu||_{L_2} = \begin{cases} 1 & -\frac{1}{2} \le \nu \le \frac{1}{2}, \\ \left|\operatorname{tg}\left(\frac{n\nu}{n}\right)\right| & \nu \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$$

is valid.

Proof. The multiplier of $\mathcal{Y}_\nu H_\nu$ on a line $\operatorname{Re} s = \frac{1}{2}$ is

$$m\left(i\xi + \frac{1}{2}\right) = -\operatorname{ctg}\left(\frac{i\xi-\nu+1}{2}\right)\pi = \operatorname{tg}\left(\frac{i\xi-\nu}{2}\right)\pi.$$

We have

$$\sup_{\xi\in\mathbb{R}}\left|\operatorname{tg}\left(\frac{i\xi-\nu}{2}\pi\right)\right| = ||\mathcal{Y}_\nu H_\nu||_{L_2}.$$

It remains to note that $||\mathcal{Y}_\nu H_\nu||_{L_2}$ is an even function of ν and is defined on symmetric intervals. Finally, since $H_\nu\mathcal{Y}_\nu = -\mathcal{Y}_{-\nu}H_{-\nu}$ we have

$$||H_\nu\mathcal{Y}_\nu||_{L_2} = ||\mathcal{Y}_\nu H_\nu||_{L_2}. \qquad \square$$

Let us consider weighted Lebesgue spaces $\mathcal{L}_{2,\mu}$. Since the boundedness of operator A acting from $\mathcal{L}_{2,\mu}$ to $\mathcal{L}_{2,\mu}$ is equal to the boundedness of operator $B = x^{\mu-\frac{1}{2}}Ax^{\frac{1}{2}-\mu}$ acting from L_2 to L_2, all statements of Lemma 1 are preserved for the weighted case with changing $i\xi + \frac{1}{2}$ to $i\xi + \mu$. However, as was noted earlier, we cannot apply Lemma 1 directly to operators H_ν and $\mathcal{Y}_\nu$; that is why we will use $\hat{H}_\nu$ and $\hat{\mathcal{Y}}_\nu$ again. It is easy to see that

$$\left\| \frac{1}{x} f\left(\frac{1}{x}\right) \right\|_{\mathcal{L}_{2,\mu}} = \|f\|_{\mathcal{L}_{2,\mu}}$$

and

$$||H_\nu||_{\mathcal{L}_{2,\mu}\to\mathcal{L}_{2,1-\mu}}$$
$$= \sup_{f\in\mathcal{L}_{2,\mu}} \frac{||H_\nu f||_{\mathcal{L}_{2,1-\mu}}}{\|f\|_{\mathcal{L}_{2,\mu}}} = \sup_{f\in\mathcal{L}_{2,\mu}} \frac{||\hat{H}_\nu \hat{f}||_{\mathcal{L}_{2,1-\mu}}}{\|\hat{f}\|_{\mathcal{L}_{2,1-\mu}}} = ||\hat{H}_\nu||_{\mathcal{L}_{2,1-\mu}}.$$

Similarly, we find that

$$||\mathcal{Y}_\nu||_{\mathcal{L}_{2,\mu}\to\mathcal{L}_{2,1-\mu}} = ||\hat{\mathcal{Y}}_\nu||_{\mathcal{L}_{2,1-\mu}}.$$

It follows that if in Lemma 1 we change $i\xi + \frac{1}{2}$ to $i\xi + 1 - \mu$, it is possible to use this lemma for operators H_ν and $\mathcal{Y}_\nu$ in spaces $\mathcal{L}_{2,\mu}$.

First we prove the auxiliary lemma.

Lemma 4. *Let $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq \beta$. Then the relation*

$$\sup_{\xi\in\mathbb{R}} \left| \frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)} \right|$$
$$= \begin{cases} \infty & \alpha > \beta, \\ \infty & \alpha = -n,\ \beta \neq -m,\ n, m \in \mathbb{N}_0, \\ q,\ \textit{where}\ \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)}\right| \leq q < \infty & \textit{in other cases} \end{cases}$$

is valid.

Proof. (1) Let $\alpha > \beta$. In virtue of the asymptotic formula (see [20], p. 62, formula (4))

$$\frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)} \sim (i\xi)^{\alpha-\beta}, \qquad |\xi| \to \infty,$$

we obtain

$$\lim_{|\xi|\to\infty} \left| \frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)} \right| = \begin{cases} \infty & \alpha > \beta, \\ 0 & \alpha < \beta \end{cases} \tag{1.80}$$

and the first statement in curly brackets is proved.

(2) Let $\alpha = -n$, $\beta \neq -m$, $n, m \in \mathbb{N}_0$. In this case the numerator of the considered relation has a pole at $\xi = 0$ and at the same time the denominator at this point is finite.

(3) Let $|\alpha| \leq \beta$, $\alpha \neq -n$, $n \in \mathbb{N}_0$. Using the formula (see [430])

$$\left|\frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)}\right| = \left|\frac{\Gamma(\alpha)}{\Gamma(\beta)}\right| \prod_{s=0}^{\infty} \frac{1+\left(\frac{\xi}{(\beta+s)}\right)^2}{1+\left(\frac{\xi}{(\alpha+s)}\right)^2},$$

we obtain $\frac{1+\left(\frac{\xi}{(\beta+s)}\right)^2}{1+\left(\frac{\xi}{(\alpha+s)}\right)^2} < 1$ for $(\alpha-\beta)(\alpha+\beta+s) \leq 0$. Therefore for $\alpha < \beta$ this is true for any s if $\alpha+\beta \geq 0$. These two conditions are equal to $|\alpha| \leq \beta$ since by the condition of the lemma $\alpha \neq \beta$. So for $|\alpha| \leq \beta$ the supremum is at $\xi = 0$.

(4) Let $\alpha < \beta$, $\alpha + \beta < 0$. From (1.80) it follows that $\sup\limits_{\xi \in \mathbb{R}} \left|\frac{\Gamma(i\xi+\alpha)}{\Gamma(i\xi+\beta)}\right|$ is finite for $\alpha \neq -n$, $n \in \mathbb{N}_0$. If $\alpha = -n$, $\beta = -m$, $n, m \in \mathbb{N}_0$, then using formula (11) from [20], p. 61, we get

$$\left|\frac{\Gamma(-n)}{\Gamma(-m)}\right| = \frac{m!}{n!},$$

and thus the lemma is completely proved. □

Theorem 7. *The operator H_ν acting from $\mathcal{L}_{2,\mu}$ to $\mathcal{L}_{2,1-\mu}$ is unbounded under any of the following conditions:*

(1) $\mu - \nu = 2n + \frac{3}{2}$, $n \in \mathbb{N}_0$,

(2) $\mu < \frac{1}{2}$.

In other cases it is bounded and for its norm the formula

$$||H_\nu||_{\mathcal{L}_{2,\mu}\to\mathcal{L}_{2,1-\mu}} = 2^{\frac{1}{2}} \left|\frac{\Gamma\left(\frac{2\nu-2\mu+3}{4}\right)}{\Gamma\left(\frac{2\nu+2\mu+3}{4}\right)}\right|$$

is valid.

Proof. We can write the module of the multiplier of H_ν on the line $\mathrm{Re}\, s = 1-\mu$:

$$|m(i\xi+1-\mu)| = 2^{\frac{1}{2}-\mu} \left|\frac{\Gamma\left(\frac{2\nu+i2\xi-2\mu+3}{4}\right)}{\Gamma\left(\frac{2\nu-i2\xi+2\mu+1}{4}\right)}\right| = 2^{\frac{1}{2}-\mu} \left|\frac{\Gamma\left(\frac{2\nu+i2\xi-2\mu+3}{4}\right)}{\Gamma\left(\frac{2\nu+i2\xi+2\mu+1}{4}\right)}\right|.$$

Now setting $\alpha = \frac{2\nu-2\mu+3}{4}$, $\beta = \frac{2\nu+2\mu+3}{4}$ and applying Lemma 4 we obtain the statement of the theorem. □

Theorem 8. *The operator $\mathcal{Y}_\nu$ acting from $\mathcal{L}_{2,\mu}$ to $\mathcal{L}_{2,1-\mu}$ is unbounded under any of the following conditions:*

(1) $\mu < \frac{1}{2}$,

(2) $\mu \pm \nu = 2n + \frac{3}{2}$, $n \in \mathbb{N}_0$.

In other cases it is bounded. If at least one of the following conditions:

(3) $\mu \geq 1$ *and* $\nu \geq -\frac{1}{2}$ *and* $\mu + \nu \leq 2$,

(4) $\mu \leq 2$ *and* $\nu\frac{1}{2}$ *and* $\mu + \nu \geq 1$,

is satisfied, then for the $\mathcal{Y}_\nu$-norm the formula

$$||\mathcal{Y}_\nu||_{\mathcal{L}_{2,\mu}\to\mathcal{L}_{2,1-\mu}} = 2^{\frac{1}{2}-\mu}\left|\frac{\Gamma\left(\frac{2\nu-2\mu+3}{4}\right)\Gamma\left(\frac{-2\nu-2\mu+3}{4}\right)}{\Gamma\left(\frac{-2\nu+2\mu+5}{4}\right)\Gamma\left(\frac{2\nu+2\mu-1}{4}\right)}\right|$$

is valid.

Proof. Let us consider the multiplier module of $\mathcal{Y}_\nu$ on the line $\mathrm{Re}\, s = 1 - \mu$:

$$|m(i\xi + 1 - \mu)| = 2^{\frac{1}{2}-\mu}\left|\frac{\Gamma\left(\frac{i2\xi+2\nu-2\mu+3}{4}\right)\Gamma\left(\frac{i2\xi-2\nu-2\mu+3}{4}\right)}{\Gamma\left(\frac{i2\xi-2\nu-2\mu+5}{4}\right)\Gamma\left(\frac{i2\xi+2\nu+2\mu-1}{4}\right)}\right|.$$

Now we set $\alpha = \frac{2\nu-2\mu+3}{4}$, $\beta = \frac{-2\nu-2\mu+5}{4}$, $\theta = \frac{-2\nu-2\mu+3}{4}$, $\gamma = \frac{2\nu+2\mu-1}{4}$. Note that if each function $f(x)$ and $g(x)$ has a maximum at x_0, then the function $p(x)=f(x)g(x)$ has a maximum at the same point. Now applying Lemma 4 to relations

$$\left|\frac{\Gamma\left(\frac{1\xi}{2}+\alpha\right)}{\Gamma\left(\frac{1\xi}{2}+\beta\right)}\right|, \left|\frac{\Gamma\left(\frac{1\xi}{2}+\theta\right)}{\Gamma\left(\frac{1\xi}{2}+\gamma\right)}\right|, \left|\frac{\Gamma\left(\frac{1\xi}{2}+\alpha\right)}{\Gamma\left(\frac{1\xi}{2}+\gamma\right)}\right|, \left|\frac{\Gamma\left(\frac{1\xi}{2}+\theta\right)}{\Gamma\left(\frac{1\xi}{2}+\beta\right)}\right|,$$

we obtain the statement of the theorem. □

Turning to the consideration of the superposition of H_ν and $\mathcal{Y}_\nu$ we should note that the representation (1.62) is valid for them and we can apply Lemma 1 directly.

Theorem 9. *The operator $H_\nu\mathcal{Y}_\nu$ is unbounded below in $\mathcal{L}_{2,\mu}$ if $\mu \pm \nu = 2n + \frac{3}{2}$, $n \in \mathbb{Z}$. In other cases it is bounded and for its norm the formula*

$$||H_\nu\mathcal{Y}_\nu||_{\mathcal{L}_{2,\mu}} = \begin{cases} 1 & \mu+\nu \in (2n, 2n+1),\ n \in \mathbb{Z}, \\ \left|\mathrm{tg}\left(\frac{2\mu+2\nu-1}{4}\right)n\right| & \mu+\nu \in \left(2n-1, 2n-\frac{1}{2}\right) \cup \left(2n-\frac{1}{2}, 2n\right),\ n \in \mathbb{Z}, \end{cases}$$

is valid. For $\mu + \nu = n$, $n \in \mathbb{Z}$, and only for them operator $H_\nu\mathcal{Y}_\nu$ is unitary in $\mathcal{L}_{2,\mu}$.

Proof. By formula (1.71) for $s = i\xi + \mu$ we can write out the multiplier module of $H_\nu\mathcal{Y}_\nu$:

$$|m(i\xi + \mu)| = \left|\mathrm{tg}\left(\frac{i2\xi + 2\mu + 2\nu - 1}{4}\right)n\right|.$$

Putting $\frac{2\mu+2\nu-1}{4}=-\frac{\gamma}{2}$ we get function $f_\gamma(\xi)$, which we studied in the proof of Theorem 6. From the properties of this function and from Lemma 1 we obtain the statement of the theorem. $\square$

Corollary 1. *Since $\mathcal{Y}_\nu H_\nu=-H_{-\nu}\mathcal{Y}_{-\nu}$ we have an analogous theorem for $\mathcal{Y}_\nu H_\nu$ by replacing ν with $-\nu$.*

Similarly to what was done in the proof of Theorem 7, by formula (1.70) for $f\in\mathcal{L}_{2\mu}$ we obtain

$$\mathcal{M}[H_\nu H_\gamma f](s)=m_\nu(s)m_\gamma(1-s)\mathcal{M}[f](s)=m(s)\mathcal{M}[f](s),$$

$$\frac{1}{2}\le\mu<\min\left(\frac{3}{2}+\nu,\frac{3}{2}+\gamma\right). \tag{1.81}$$

Here

$$m(s)=\frac{\Gamma\left(\frac{2s+2\nu+1}{4}\right)\Gamma\left(\frac{-2s+2\gamma+3}{4}\right)}{\Gamma\left(\frac{-2s+2\nu+3}{4}\right)\Gamma\left(\frac{2s+2\gamma+1}{4}\right)},\qquad \operatorname{Re}s=\mu.$$

It is obvious that $\left|m\left(i\xi+\frac{1}{2}\right)\right|=1$ for all $\xi\in\mathbb{R}$. So the operator $H_\nu H_\gamma$ cannot be presented as Mellin convolution with a kernel from L_2. One way or another equality must be fulfilled (see [608], p. 53), i.e.,

$$||K||_{L_2}=\frac{1}{2\pi}\int\limits_0^\infty\left|m\left(i\xi+\frac{1}{2}\right)\right|^2d\xi.$$

We are looking for $H_\nu H_\gamma$ in the form suggested by the lemma from [6], p. 78.

Lemma 5. *To each bounded linear operator T in $L_2(0,\infty)$ corresponds a function $P(x,t)$ belonging to $L_2(0,\infty)$ by t for each $x\in(0,\infty)$ and having the following property: For all $f(t)\in L_2(0,\infty)$, almost everywhere on x,*

$$T[f](x)=\frac{d}{dx}\int\limits_0^\infty P(x,t)f(t)dt.$$

Therefore, let

$$H_\nu[H_\gamma f](x)=\frac{d}{dx}\int\limits_0^\infty K\left(\frac{x}{y}\right)f(y)dy=\frac{d}{dx}\int\limits_0^\infty K\left(\frac{x}{y}\right)[yf(y)]\frac{dy}{y}. \tag{1.82}$$

Then

$$\mathcal{M}[H_\nu H_\gamma f](s) = \mathcal{M}\left[\frac{d}{dx} K * (yf)\right](s)$$
$$= (1-s)\mathcal{M}[K * (yf)](1-s) = (1-s)\mathcal{M}[K](1-s)M[f](s).$$

Comparing with (1.81) we obtain

$$m(s) = (1-s)\mathcal{M}[K](1-s),$$

whence

$$\mathcal{M}[K](s) = \frac{1}{2}\frac{\Gamma\left(\frac{2s+2\nu+3}{4}\right)\Gamma\left(\frac{-2s+2\mu+1}{4}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{-2s+2\nu+1}{4}\right)\Gamma\left(\frac{2s+2\mu+3}{4}\right)\Gamma\left(\frac{s}{2}+1\right)}.$$

Applying formulas (9) on p. 728 from [457] and (5) on p. 268 from [430], we get

$$K(x) = -G^{2,1}_{3,3}\left(\begin{matrix}\frac{s-2\gamma}{4}, 1, \frac{s+2\gamma}{4}\\ \frac{s+2\nu}{4}, 0, \frac{s-2\nu}{4}\end{matrix}\,\middle|\, x^2\right),$$

where $G^{2,1}_{3,3}$ is the Meijer G-function (1.41).

The substitution in (1.82) gives the final expression for $H_\nu H_\gamma$:

$$H_\nu[H_\gamma f](x) = -\frac{d}{dx}\int\limits_0^\infty G^{2,1}_{3,3}\left(\begin{matrix}\frac{s-2\gamma}{4}, 1, \frac{s+2\gamma}{4}\\ \frac{s+2\nu}{4}, 0, \frac{s-2\nu}{4}\end{matrix}\,\middle|\,\left(\frac{x}{y}\right)^2\right) f(y)dy.$$

Remark 2. *Similarly, one can obtain the formula for $\mathcal{Y}_\nu\mathcal{Y}_\gamma$ and $H_\nu\mathcal{Y}_\gamma$ but they are cumbersome.*

1.3.3 Multi-dimensional integral transforms

Definition 12. *The* ***multi-dimensional Hankel transform*** *of a function $f \in L_1^\gamma(\mathbb{R}^n_+)$ is expressed as*

$$\mathbf{F}_\gamma[f](\xi) = \mathbf{F}_\gamma[f(x)](\xi) = \widehat{f}(\xi) = \int\limits_{\mathbb{R}^n_+} f(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx,$$

where

$$\mathbf{j}_\gamma(x;\xi) = \prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i), \qquad \gamma_1 > 0, ..., \gamma_n > 0,$$

and the symbol j_ν is used for the normalized Bessel function of the first kind (1.19).

Let $f \in L_1^\gamma(\mathbb{R}_+)$ be of bounded variation in a neighborhood of a point x of continuity of f. Then for $\gamma > 0$ the inversion formula

$$\mathbf{F}_\gamma^{-1}[\widehat{f}(\xi)](x) = f(x) = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n} \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}_+^n} \mathbf{j}_\gamma(x,\xi)\widehat{f}(\xi)\xi^\gamma \, d\xi$$

holds.

Definition 13. *The* ***multi-dimensional Fourier–Bessel transform*** *of a function $f \in \mathcal{L}_1^\gamma(\mathbb{R}_+^{n+1})$ is*

$$\mathcal{F}_\gamma[f](\tau,\xi) = \widehat{f}(\tau,\xi) = \int\limits_{\mathbb{R}_+^{n+1}} f(t,x)\, e^{-it\tau}\, \mathbf{j}_\gamma(x;\xi) x^\gamma \, dt dx,$$

where

$$\mathbf{j}_\gamma(x;\xi) = \prod_{i=1}^{n} j_{\frac{\gamma_i-1}{2}}(x_i\xi_i), \qquad \gamma_1 > 0, \ldots, \gamma_n > 0.$$

If $g \in S'_{ev}$, then the equality

$$(\mathbf{F}_\gamma g, \varphi)_\gamma = (g, \mathbf{F}_\gamma \varphi)_\gamma, \qquad \varphi \in S_{ev}, \tag{1.83}$$

defines the Hankel transform of functional $g \in S'_{ev}$.

In [491] the space Ψ_V consisting of functions vanishing on a given closed set V of measure zero was considered. The Lizorkin–Samko space Φ_V is dual to Φ_V in the sense of Fourier transforms. We introduce the space Ψ_V^γ of functions S_{ev} vanishing with all their derivatives on a given closed set V:

$$\Psi_V^\gamma = \{\psi \in S_{ev}(\mathbb{R}_+^n) : (D^k\psi)(x) = 0, x \in V, |k| = 0, 1, 2, \ldots\}.$$

Space Ψ_V^γ is dual to Φ_V^γ in the sense of Hankel transforms,

$$\Phi_V^\gamma = \{\varphi : \mathbf{F}_\gamma\varphi \in \Psi_V^\gamma\}. \tag{1.84}$$

For the mixed case let $\widetilde{\Psi}_{\gamma,V}$ denote the following class of functions:

$$\widetilde{\Psi}_{\gamma,V} = \{\psi \in \mathfrak{S}_{ev}(\mathbb{R}_+^{n+1}) : (D^k\psi)(x) = 0, x \in V, |k| = 0, 1, 2, \ldots\}$$

and

$$\widetilde{\Phi}_{\gamma,V} = \{\varphi : \mathcal{F}_\gamma\varphi \in \widetilde{\Psi}_{\gamma,V}\}.$$

We will consider weighted generalized functions over Φ and Ψ. The Hankel transform of $f \in \Phi'$ is

$$(\mathbf{F}_\gamma f, \psi)_\gamma = (f, \mathbf{F}_\gamma \psi)_\gamma, \;\; \psi \in \Psi. \tag{1.85}$$

If $g \in \Psi'$, then

$$(\mathbf{F}_\gamma g, \varphi)_\gamma = (g, \mathbf{F}_\gamma \varphi)_\gamma, \quad \varphi \in \Phi. \tag{1.86}$$

Definition 14. *[163] Let f be a function on $\mathbb{R}^n$, integrable on each hyperplane in $\mathbb{R}^n$. Let $\mathbb{P}^n$ denote the space of all hyperplanes in $\mathbb{R}^n$. The* ***Radon transform*** *of $f(x) : \mathbb{R}^n \to \mathbb{R}$ is defined as the function $\mathcal{R}f : S^{n-1} \times \mathbb{R} \to \mathbb{R}$ on $\mathbb{P}^n$ given by*

$$\mathcal{R}f(\omega, s) = \int\limits_{x\cdot\omega=s} f(x)dm(x),$$

where dm is the Euclidean measure on the hyperplane $x \cdot \omega = s$.

Along with the transformation $f \to \mathcal{R}f$ we consider also the dual transform $\mathcal{R}^*\varphi$ which associates with a continuous function φ on $\mathbb{R}^n$ the function $\mathcal{R}^*\varphi$ on $\mathbb{R}^n$ given by

$$\mathcal{R}^*\varphi(x) = \int\limits_{x\in\xi} \varphi(\xi)d\mu(\xi),$$

where $\xi = \{x \cdot \omega = s\}$ is the hyperplane incident with the point $x \in \mathbb{R}^n$ and $d\mu$ is the measure on the compact set $\{\xi \in \mathbb{P}^n : x \in \xi\}$ which is invariant under the group of rotations around x and for which the measure of the whole set is 1.

1.4 Basic facts and formulas

1.4.1 Kipriyanov's classification of second order linear partial differential equations

We will deal with the **singular Bessel differential operator** B_γ (see, for example, [242], p. 5):

$$(B_\gamma)_t = \frac{\partial^2}{\partial t^2} + \frac{\gamma}{t}\frac{\partial}{\partial t} = \frac{1}{t^\gamma}\frac{\partial}{\partial t}t^\gamma\frac{\partial}{\partial t}, \qquad t > 0, \qquad \gamma \in \mathbb{R}. \tag{1.87}$$

The Russian mathematician I. A. Kipriyanov introduced convenient classifications for linear differential operators and linear partial differential equations with operator (1.87).

Let $a_k > 0$, $k = 1, ..., n$. In accordance with I. A. Kipriyanov's terminology the operator

$$\sum_{k=1}^n a_k (B_{\gamma_k})_{x_k} = \sum_{k=1}^n a_k \left(\frac{\partial^2}{\partial x_k^2} + \frac{\gamma_k}{x_k}\frac{\partial}{\partial x_k}\right) = \sum_{k=1}^n a_k \frac{1}{x_k^{\gamma_k}}\frac{\partial}{\partial x_k}x_k^{\gamma_k}\frac{\partial}{\partial x_k}$$

is classified as B**-elliptic operator**, the operator

$$a_1 \frac{\partial}{\partial x_1} - \sum_{k=2}^{n} a_k (B_{\gamma_i})_{x_i}$$

is classified as B**-parabolic operator**, the operator

$$a_1 (B_{\gamma_1})_{x_1} - \sum_{k=2}^{n} a_k (B_{\gamma_k})_{x_k}$$

is classified as B**-hyperbolic operator**, and

$$\sum_{k=1}^{p} a_k (B_{\gamma_k})_{x_k} - \sum_{k=p+1}^{n} a_k (B_{\gamma_k})_{x_k}, \qquad 1 < p < n,$$

is classified as B**-ultrahyperbolic operator**.

We will use notations

$$\triangle_\gamma = (\triangle_\gamma)_x = \sum_{k=1}^{n} (B_{\gamma_k})_{x_k} \tag{1.88}$$

and

$$\square_\gamma = (\square_\gamma)_x = (B_{\gamma_1})_{x_1} - \sum_{i=2}^{n} (B_{\gamma_i})_{x_i}. \tag{1.89}$$

For $\triangle_\gamma$ the term **Laplace–Bessel operator** is used.

Let $u = u(x) = u(x_1, \ldots, x_n)$, $f = f(x) = f(x_1, \ldots, x_n)$. B-elliptic linear differential equations have the form

$$\sum_{k=1}^{n} a_k (B_{\nu_k})_{x_k} u = f. \tag{1.90}$$

Generalized axisymmetric potential theory (GASPT) for particular cases of (1.90) was studied by A. Weinstein [592,594,599,598]. Also Eq. (1.90) and related ones were considered by L. D. Kudryavtsev [303], P. I. Lizorkin and S. M. Nikol'skii [334], I. A. Kipriyanov [242], V. V. Katrakhov [226], and others.

B-parabolic linear differential equations

$$a_1 \frac{\partial u}{\partial x_1} - \sum_{k=2}^{n} a_k (B_{\nu_k})_{x_k} u = f \tag{1.91}$$

were studied by Ya. I. Zhitomirskii [610,609], M. I. Matiichuk [364,363], and A. B. Muravnik [394–397].

B-hyperbolic linear differential equations have the form

$$a_1 \frac{\partial^2 u}{\partial x_1^2} - \sum_{k=2}^{n} a_k (B_{\nu_k})_{x_k} u = f. \tag{1.92}$$

The study of this class of equations was begun in the works of L. Euler [128], S. D. Poisson [447], and J. G. Darboux [77] and continued in the works of R. Carroll and R. Showatler [56], A. Weinstein [599,595,593,596], D. Fox [147], I. A. Kipriyanov and L. A. Ivanov [246], S. A. Tersenov [564], etc.

In [349,350,354] the B-ultrahyperbolic linear differential equation

$$\sum_{i=1}^{p} a_k (B_{\gamma_i})_{x_i} u - \sum_{i=p+1}^{n} a_k (B_{\gamma_i})_{x_i} u = 0 \tag{1.93}$$

was considered.

For equations of fractional order, a similar classification is adopted.

Let $\alpha \in \mathbb{R}$, $\alpha > 0$. B-elliptic linear differential equations of fractional order have the form

$$\left(\sum_{k=1}^{n} a_k (B_{\nu_k})_{x_k} \right)^{\alpha} u = f,$$

B-parabolic linear differential equations of fractional order have the form

$$\left(a_1 \frac{\partial}{\partial x_1} - \sum_{k=2}^{n} a_k (B_{\nu_k})_{x_k} \right)^{\alpha} u = f,$$

and B-hyperbolic linear differential equations of fractional order have the forms

$$\left(a_1 \frac{\partial^2}{\partial x_1^2} - \sum_{k=2}^{n} a_k (B_{\nu_k})_{x_k} \right)^{\alpha} u = f \quad \text{and} \quad \left((B_\gamma)_t - \sum_{k=1}^{n} a_k (B_{\nu_k})_{x_k} \right)^{\alpha} u = f.$$

Kipriyanov (see [242]) also introduced the *B-polyharmonic of order p* function. It is the function $u = u(x) = u(x_1, ..., x_n)$ such that

$$\Delta_\gamma^p u = 0, \tag{1.94}$$

where Δ_γ is operator (1.88). The B-polyharmonic of order 1 function will be called *B-harmonic*.

The multi-dimensional Hankel transform $\mathbf{F}_\gamma$ given in Definition 12 acts to Δ_γ as the Fourier transform acts to the Laplace operator. This is proved in the next Lemma.

Lemma 6. *Let $u \in S_{ev}$. Then*

$$\mathbf{F}_\gamma [\Delta_\gamma f](\xi) = -|\xi|^2 \mathbf{F}_\gamma [f](\xi). \tag{1.95}$$

Proof. We have

$$\mathbf{F}_\gamma[\Delta_\gamma f](\xi)=\int\limits_{\mathbb{R}^n_+}[\Delta_\gamma f(x)]\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx$$
$$=\sum_{i=1}^n\int\limits_{\mathbb{R}^n_+}\left[\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}f(x)\right]\mathbf{j}_\gamma(x;\xi)x^\gamma dx.$$

Integrating by parts by variable x_i and using formula (1.23), we obtain

$$\mathbf{F}_\gamma[\Delta_\gamma f](\xi)=\sum_{i=1}^n\int\limits_{\mathbb{R}^n_+}f(x)\left[\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial}{\partial x_i}\mathbf{j}_\gamma(x;\xi)\right]x^\gamma dx$$
$$=\sum_{i=1}^n(-\xi_i^2)\int\limits_{\mathbb{R}^n_+}f(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx$$
$$=-|\xi|^2\int\limits_{\mathbb{R}^n_+}f(x)\,\mathbf{j}_\gamma(x;\xi)x^\gamma dx=-|\xi|^2\mathbf{F}_\gamma[f](\xi).$$

□

We will also need the formula

$$\left(\frac{d}{2x\,dx}\right)^n x^{2\mu+2n}=\frac{\Gamma(\mu+n+1)}{\Gamma(\mu+1)}x^{2\mu}. \tag{1.96}$$

1.4.2 Divergence theorem and Green's second identity for B-elliptic and B-hyperbolic operators

Here we give the generalization of the divergence theorem to the case of weighted divergence and derive Green's second identity for Δ_γ and $\Box_\gamma$.

Suppose that $\vec{e}=(e_1,...,e_n)$ is an orthonormal basis in $\mathbb{R}^n$,

$$\nabla'_\gamma=\left(\frac{1}{x_1^{\gamma_1}}\frac{\partial}{\partial x_1},...,\frac{1}{x_n^{\gamma_n}}\frac{\partial}{\partial x_n}\right)$$

is the first weighted nabla operator,

$$\vec{F}=\vec{F}(x)=(F_1(x),...,F_n(x))$$

is a vector field, and

$$(\nabla'_\gamma\cdot\vec{F})=\frac{1}{x_1^{\gamma_1}}\frac{\partial F_1}{\partial x_1}+...+\frac{1}{x_i^{\gamma_n}}\frac{\partial F_n}{\partial x_n}$$

is the weighted divergence.

In $\overline{\mathbb{R}}^n_+$ let us consider a domain G^+ bounded by a piecewise smooth surface $S^+ \in \overline{\mathbb{R}}^n_+$. Thus, a surface can be represented as a union $S^+ = \bigcup\limits_{k=1}^{q} S_k^+$ of a finite number of its parts S_k^+ without common internal points. Let there be for each interior point a neighborhood within which the surface S_k^+ is represented by parametric equations of the form

$$x_i = \chi_i(y_1, ..., y_{n-1}), \qquad i = 1, ..., n,$$

where $\chi_i(y)$, $y = (y_1, ..., y_{n-1})$, has continuous first derivatives and the rank of the Jacobi matrix $\left\| \frac{\partial(\chi_1,...,\chi_n)}{\partial(y_1,...,y_{n-1})} \right\|$ is equal to $n-1$. The vector

$$\vec{N} = \begin{Vmatrix} e_1 & ... & e_n \\ \frac{\partial \chi_1(y)}{\partial y_1} & ... & \frac{\partial \chi_n(y)}{\partial y_1} \\ ... & ... & ... \\ \frac{\partial \chi_1(y)}{\partial y_{n-1}} & ... & \frac{\partial \chi_n(y)}{\partial y_{n-1}} \end{Vmatrix}$$

is normal to the surface S^+ in each point $y \in S^+$ with the exception of the junction points of surfaces S_k^+, $k = 1, .., q$, where it is not defined unambiguously and will not be considered. The vector

$$\vec{\nu} = \frac{\vec{N}}{|\vec{N}|}$$

is determined to within sign. Of the two possible directions $\vec{\nu}$, we choose the external with respect to the domain G^+. Such a vector will be called the unit normal vector to the surface S^+ at the point y. We denote by η_i the angle which forms a vector $\vec{\nu}$ with an axis x_j. Then

$$\vec{\nu} = e_1 \cos\eta_1 + ... + e_n \cos\eta_n.$$

Theorem 10. *Let G^+ be the domain in $\overline{\mathbb{R}}^n_+$ such that each line perpendicular to the plane $x_i = 0$, $i = 1, ..., n$, either does not cross G^+ or has one common segment with G^+ (maybe degenerating to a point) of the form*

$$\alpha_i(x') \le x_i \le \beta_i(x'), \qquad x' = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n), \qquad i = 1, ..., n.$$

If $\vec{g} = (g_1(x), ..., g_n(x))$ is a continuously differentiable in G^+ vector field and $\vec{F} = (F_1(x), ..., F_n(x))$, $F_1(x) = x_1^{\gamma_1} g_1(x), ..., F_n(x) = x_n^{\gamma_n} g_n(x)$, then the following formula is valid:

$$\int\limits_{G^+} (\nabla'_\gamma \cdot \vec{F})\, x^\gamma\, dx = \int\limits_{S^+} (\vec{g} \cdot \vec{\nu})\; x^\gamma\, dS, \tag{1.97}$$

where $\vec{\nu}$ is the external unit normal vector S^+.

Proof. Let $i=1,...,n$ be fixed. If the part of the surface S^+ defined by the equation $x_i=\beta_i(x')$ is denoted by S_u^+ and the part of the surface S^+ defined by the equation $x_i=\alpha_i(x')$ is denoted by S_d^+, then

$$(\vec{\nu},e_i)=\begin{cases} -\dfrac{1}{\sqrt{1+\left(\frac{\partial\alpha_i}{\partial x_1}\right)^2+...+(\frac{\partial\alpha_i}{\partial x_{i-1}})^2+\left(\frac{\partial\alpha_i}{\partial x_{i+1}}\right)^2+...+\left(\frac{\partial\alpha_i}{\partial x_n}\right)^2}} & x\in S_d^+,\\ \dfrac{1}{\sqrt{1+\left(\frac{\partial\beta_i}{\partial x_1}\right)^2+...+\left(\frac{\partial\beta_i}{\partial x_{i-1}}\right)^2+\left(\frac{\partial\beta_i}{\partial x_{i+1}}\right)^2+...+\left(\frac{\partial\beta_i}{\partial x_n}\right)^2}} & x\in S_u^+.\end{cases}$$

We have

$$\int\limits_{G^+}(\nabla'_\gamma\cdot\vec{F})\,x^\gamma\,dx=\sum_{i=1}^n\int\limits_{G^+}\frac{1}{x_i^{\gamma_i}}\frac{\partial F_i}{\partial x_i}\,x^\gamma\,dx.$$

Let us consider

$$\int\limits_{G^+}\frac{1}{x_i^{\gamma_i}}\frac{\partial F_i}{\partial x_i}\,x^\gamma\,dx$$
$$=\int\limits_Q x_1^{\gamma_1}...x_{i-1}^{\gamma_{i-1}}x_{i+1}^{\gamma_{i+1}}...x_n^{\gamma_n}\,dx_1...dx_{i-1}dx_{i+1}...dx_n\int\limits_{\alpha_i(x')}^{\beta_i(x')}\frac{\partial F_i}{\partial x_i}dx_i,$$

where Q is a projection of G^+ to $x_i=0$. Integrating by x_i we obtain

$$\int\limits_{G^+}\frac{1}{x_i^{\gamma_i}}\frac{\partial F_i}{\partial x_i}\,x^\gamma\,dx$$
$$=\int\limits_Q F_i(x)|_{x_i=\alpha_i(x')}^{x_i=\beta_i(x')}x_1^{\gamma_1}...x_{i-1}^{\gamma_{i-1}}x_{i+1}^{\gamma_{i+1}}...x_n^{\gamma_n}\,dx_1...dx_{i-1}dx_{i+1}...dx_n.$$

Let $(x')^{\gamma'}=x_1^{\gamma_1}...x_{i-1}^{\gamma_{i-1}}x_{i+1}^{\gamma_{i+1}}...x_n^{\gamma_n}$, $dx'=dx_1...dx_{i-1}dx_{i+1}...dx_n$. Then

$$\int\limits_{G^+}\frac{1}{x_i^{\gamma_i}}\frac{\partial F_i}{\partial x_i}\,x^\gamma\,dx$$
$$=\int\limits_Q F_i(x_1,...,x_{i-1},\beta_i(x'),x_{i+1},...,x_n)(x')^{\gamma'}dx'$$
$$-\int\limits_Q F_i(x_1,...,x_{i-1},\alpha_i(x'),x_{i+1},...,x_n)(x')^{\gamma'}dx'$$
$$=\int\limits_Q F_i(x_1,...,x_{i-1},\beta_i(x'),x_{i+1},...,x_n)(\vec{\nu},e_i)$$

$$\times\sqrt{1+\left(\frac{\partial\beta_i}{\partial x_1}\right)^2+...+\left(\frac{\partial\beta_i}{\partial x_{i-1}}\right)^2+\left(\frac{\partial\beta_i}{\partial x_{i+1}}\right)^2+...+\left(\frac{\partial\beta_i}{\partial x_n}\right)^2}(x')^{\gamma'}dx'$$
$$+\int\limits_Q F_i(x_1,...,x_{i-1},\alpha_i(x'),x_{i+1},...,x_n)(\vec{\nu},e_i)$$
$$\times\sqrt{1+\left(\frac{\partial\alpha_i}{\partial x_1}\right)^2+...+\left(\frac{\partial\alpha_i}{\partial x_{i-1}}\right)^2+\left(\frac{\partial\alpha_i}{\partial x_{i+1}}\right)^2+...+\left(\frac{\partial\alpha_i}{\partial x_n}\right)^2}(x')^{\gamma'}dx'$$
$$=\int\limits_{S_u^+} F_i(x)(\vec{\nu},e_i)(x')^{\gamma'}dS_u+\int\limits_{S_d^+} F_i(x)(\vec{\nu},e_i)(x')^{\gamma'}dS_d$$
$$=\int\limits_{S_u^+} g_i(x)(\vec{\nu},e_i)x^{\gamma}dS_u+\int\limits_{S_d^+} g_i(x)(\vec{\nu},e_i)x^{\gamma}dS_d$$
$$=\int\limits_{S^+} g_i(x)\cos\eta_i\, x^{\gamma}dS.$$

Then

$$\int\limits_{G^+}(\nabla'_\gamma\cdot\vec{F})\,x^{\gamma}dx=\sum_{i=1}^n\int\limits_{S^+} g_i(x)\cos\eta_i\,x^{\gamma}dS=\int\limits_{S^+}(\vec{g}\cdot\vec{\nu})\;x^{\gamma}\,dS. \qquad \square$$

Remark 3. *Suppose that a domain $G^+\in\overline{\mathbb{R}}^n_+$ is a union of domains $G_1^+,...,G_m^+$ without common internal points. Let each G_j^+ be the domain in $\overline{\mathbb{R}}^n_+$ such that each line perpendicular to the plane $x_i=0$, $i=1,...,n$, either does not cross G_j^+ or has one common segment with G_j^+ (maybe degenerating to a point) of the form*

$$\alpha_i^j(x')\le x_i\le\beta_i^j(x'),\qquad x'{=}(x_1,...,x_{i-1},x_{i+1},...,x_n),\qquad i=1,...,n,$$

and $\vec{F}=(F_1(x),...,F_n(x))$, $F_1(x)=x_1^{\gamma_1}g_1(x)$, ..., $F_n(x)=x_n^{\gamma_n}g_n(x)$, $\vec{g}=(g_1(x),...,g_n(x))$, is a continuously differentiable in G^+ vector field. Then the following formula is valid:

$$\int\limits_{G^+}(\nabla'_\gamma\cdot\vec{F})\,x^{\gamma}dx=\int\limits_{S^+}(\vec{g}\cdot\vec{\nu})\;x^{\gamma}\,dS, \tag{1.98}$$

where $S^+\in\overline{\mathbb{R}}^n_+$ is a piecewise smooth surface boundary and $\vec{\nu}$ is the external unit normal vector S^+.

Theorem 11. *Let G^+ satisfy the conditions in Remark 3. If φ,ψ are twice continuously differentiable functions defined on G^+, such that*

$$\left.\frac{\partial\varphi}{\partial x_i}\right|_{x_i=0}=0,\qquad\left.\frac{\partial\psi}{\partial x_i}\right|_{x_i=0}=0,\qquad i=1,...,n,$$

then Green's second identity for the Laplace–Bessel operator has the form

$$\int\limits_{G^+} (\psi\,\Delta_\gamma\varphi - \varphi\Delta_\gamma\psi)\, x^\gamma\, dx = \int\limits_{S^+} \left(\psi\frac{\partial\varphi}{\partial\vec{\nu}} - \varphi\frac{\partial\psi}{\partial\vec{\nu}}\right) x^\gamma\, dS. \tag{1.99}$$

Proof. If φ, ψ are twice continuously differentiable functions defined on a neighborhood of $B_R^+(n)$, such that

$$\left.\frac{\partial\varphi}{\partial x_i}\right|_{x_i=0} = 0, \qquad \left.\frac{\partial\psi}{\partial x_i}\right|_{x_i=0} = 0, \qquad i = 1, ..., n,$$

one may choose

$$\begin{aligned}
\vec{F} &= \psi\,\nabla''_\gamma\varphi - \varphi\nabla''_\gamma\psi \\
&= \left(\psi\cdot x_1^{\gamma_1}\frac{\partial\varphi}{\partial x_1} - \varphi\cdot x_1^{\gamma_1}\frac{\partial\psi}{\partial x_1}, ..., \psi\cdot x_n^{\gamma_n}\frac{\partial\varphi}{\partial x_n} - \varphi\cdot x_n^{\gamma_n}\frac{\partial\psi}{\partial x_n}\right) \\
&= \left(x_1^{\gamma_1}\left(\psi\frac{\partial\varphi}{\partial x_1} - \varphi\frac{\partial\psi}{\partial x_1}\right), ..., x_n^{\gamma_n}\left(\psi\frac{\partial\varphi}{\partial x_n} - \varphi\frac{\partial\psi}{\partial x_n}\right)\right)
\end{aligned}$$

to obtain Green's second identity for the Laplace–Bessel operator. In this case

$$\vec{g} = \left(\psi\frac{\partial\varphi}{\partial x_1} - \varphi\frac{\partial\psi}{\partial x_1}, ..., \psi\frac{\partial\varphi}{\partial x_n} - \varphi\frac{\partial\psi}{\partial x_n}\right)$$

is a continuously differentiable vector field defined on a neighborhood of $B_R^+(n)$,

$$\begin{aligned}
(\nabla'_\gamma\cdot\vec{F}) &= (\nabla'_\gamma\cdot(\psi\,\nabla''_\gamma\varphi - \varphi\nabla''_\gamma\psi)) \\
&= \sum_{i=1}^n\left(\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}\left(\psi\cdot x_i^{\gamma_i}\frac{\partial\varphi}{\partial x_i}\right) - \frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}\left(\varphi\cdot x_i^{\gamma_i}\frac{\partial\psi}{\partial x_i}\right)\right) \\
&= \sum_{i=1}^n\left(\frac{1}{x_i^{\gamma_i}}\frac{\partial\psi}{\partial x_i}\cdot x_i^{\gamma_i}\frac{\partial\varphi}{\partial x_i} + \psi\cdot\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial\varphi}{\partial x_i}\right. \\
&\qquad \left. - \frac{1}{x_i^{\gamma_i}}\frac{\partial\varphi}{\partial x_i}\cdot x_i^{\gamma_i}\frac{\partial\psi}{\partial x_i} - \varphi\cdot\frac{1}{x_i^{\gamma_i}}\frac{\partial}{\partial x_i}x_i^{\gamma_i}\frac{\partial\psi}{\partial x_i}\right) \\
&= \sum_{i=1}^n\left(\psi\,B_{\gamma_i}\varphi - \varphi B_{\gamma_i}\psi\right) = \psi\,\Delta_\gamma\varphi - \varphi\Delta_\gamma\psi,
\end{aligned}$$

$$\begin{aligned}
(\vec{g}\cdot\vec{\nu}) &= \left(\psi\frac{\partial\varphi}{\partial x_1}\cos\eta_1 + ... + \psi\frac{\partial\varphi}{\partial x_n}\cos\eta_n\right) \\
&\quad - \left(\varphi\frac{\partial\psi}{\partial x_1}\cos\eta_1 + ... + \varphi\frac{\partial\psi}{\partial x_n}\cos\eta_n\right) = \psi\frac{\partial\varphi}{\partial\vec{\nu}} - \varphi\frac{\partial\psi}{\partial\vec{\nu}},
\end{aligned}$$

and applying (1.98) we obtain (1.99). □

Let

$$\Diamond_\gamma = \left(x_1^{\gamma_1} \frac{\partial}{\partial x_1}, -x_2^{\gamma_{p+1}} \frac{\partial}{\partial x_2}, ..., -x_n^{\gamma_n} \frac{\partial}{\partial x_n} \right).$$

Then

$$(\nabla'_\gamma \cdot \Diamond_\gamma) = \Box_\gamma.$$

Theorem 12. *Let G^+ satisfy the conditions in Remark 3. If φ, ψ are twice continuously differentiable functions defined on G^+, such that*

$$\left.\frac{\partial \varphi}{\partial x_i}\right|_{x_i=0} = 0, \qquad \left.\frac{\partial \psi}{\partial x_i}\right|_{x_i=0} = 0, \qquad i = 1, ..., n,$$

then Green's second identity for the B-ultrahyperbolic operator has the form

$$\int\limits_{G^+} (\psi \Box_\gamma \varphi - \varphi \Box_\gamma \psi)\, x^\gamma \, dx = \int\limits_{S^+} \left(\psi \frac{\partial \varphi}{\partial \vec{\tau}} - \varphi \frac{\partial \psi}{\partial \vec{\tau}} \right) x^\gamma \, dS, \tag{1.100}$$

where $\vec{\tau} = (\cos\eta_1, -\cos\eta_2, ..., -\cos\eta_n)$.

Proof. Let

$$\begin{aligned} \vec{F} &= \psi \Diamond_\gamma \varphi - \varphi \Diamond_\gamma \psi \\ &= \left(x_1^{\gamma_1} \left(\psi \frac{\partial \varphi}{\partial x_1} - \varphi \frac{\partial \psi}{\partial x_1} \right), -x_2^{\gamma_2} \left(\psi \frac{\partial \varphi}{\partial x_2} - \varphi \frac{\partial \psi}{\partial x_2} \right), ..., \right. \\ &\qquad \left. -x_n^{\gamma_n} \left(\psi \frac{\partial \varphi}{\partial x_n} - \varphi \frac{\partial \psi}{\partial x_n} \right) \right), \end{aligned}$$

so that

$$\begin{aligned} &\vec{g} = \left(\psi \frac{\partial \varphi}{\partial x_1} - \varphi \frac{\partial \psi}{\partial x_1}, -\left(\psi \frac{\partial \varphi}{\partial x_2} - \varphi \frac{\partial \psi}{\partial x_2} \right), ..., -\left(\psi \frac{\partial \varphi}{\partial x_n} - \varphi \frac{\partial \psi}{\partial x_n} \right) \right), \\ &(\nabla'_\gamma \cdot \vec{F}) = \psi \Box_\gamma \varphi - \varphi \Box_\gamma \psi, \\ &(\vec{g} \cdot \vec{\nu}) = \left(\psi \frac{\partial \varphi}{\partial x_1} \cos\eta_1 - \varphi \frac{\partial \psi}{\partial x_1} \cos\eta_1 \right) - \sum_{i=2}^n \left(\psi \frac{\partial \varphi}{\partial x_i} \cos\eta_i - \varphi \frac{\partial \psi}{\partial x_i} \cos\eta_i \right) \\ &\qquad = \psi \frac{\partial \varphi}{\partial \vec{\tau}} - \varphi \frac{\partial \psi}{\partial \vec{\tau}}, \end{aligned}$$

where $\vec{\tau} = (\cos\eta_1, -\cos\eta_2, ..., -\cos\eta_n)$. Then applying (1.98) we obtain (1.100). $\Box$

As a corollary of Theorem 12 we obtain the formula of integral with weight $x^\gamma = x_1^{\gamma_1} ... x_n^{\gamma_n}$, $\gamma_i > 0, i{=}1, ..., n$, when the region of integration is a part of a ball belonging to the orthant $\mathbb{R}^n_+$.

We will denote the part of a ball $|x| \leq r$, $|x| = \sqrt{x_1^2 + ... + x_n^2}$ belonging to $\mathbb{R}^n_+$ by $B_r^+(n)$. The boundary of $B_r^+(n)$ denoted by $S_r^+(n)$ consists of a part of a sphere $\{x \in \mathbb{R}^n_+ : |x|=r\}$ and of parts of coordinate hyperplanes $x_i=0$, $i=1,\dots,n$, such that $|x^i| \leq r$.

Corollary 2. *For $w \in C^2_{ev}(B_R^+(n))$ the following formula is valid:*

$$\int\limits_{B_R^+(n)} (\Delta_\gamma w(x))\, x^\gamma\, dx = \int\limits_{S_R^+(n)} \left(\frac{\partial w(x)}{\partial \vec{\nu}}\right) x^\gamma\, dS, \tag{1.101}$$

where $\vec{\nu}$ is external normal to the $S_R^+(n)$.

Now let us consider integration by $S_r^+(n)$ with a weight of the form

$$\int\limits_{S_r^+(n)} u(x) x^\gamma dS_r, \qquad x^\gamma = \prod_{i=1}^n x_i^{\gamma_i},$$

where dS_r is a surface element of $S_r^+(n)$. It is easy to see that the formulas

$$\int\limits_{B_r^+(n)} u(x) x^\gamma dx = r^{n+|\gamma|} \int\limits_{B_1^+(n)} u(rx) x^\gamma dx \tag{1.102}$$

and

$$\int\limits_{S_r^+(n)} u(x) x^\gamma dS_r = r^{n+|\gamma|-1} \int\limits_{S_1^+(n)} u(rx) x^\gamma dS, \tag{1.103}$$

where dS is a surface element of $S_1^+(n)$, are valid.

Let a function $f(x)$ be integrable by $B_r^+(n)$ and a function $g(t)$ be continuous of variable t, $t \in [0, \infty)$. For integration by $B_r^+(n)$ with weight x^γ formulas

$$\int\limits_{B_r^+(n)} g(|x|) f(x)\, x^\gamma\, dx = \int\limits_0^r g(\lambda) \lambda^{n+|\gamma|-1} d\lambda \int\limits_{S_1^+(n)} f(\lambda x) x^\gamma d\omega, \tag{1.104}$$

$$\int\limits_{S_1^+(n)} f(rx) x^\gamma dS = r^{1-n-|\gamma|} \frac{d}{dr} \int\limits_{B_r^+(n)} f(z) z^\gamma dz \tag{1.105}$$

are valid (see [349]).

Let us find $\int\limits_{S_1^+(n)} x^\gamma dS$ taking into account the formula (see [73]). We have

$$\int\limits_{y_1+...+y_n\leq 1, y_1\geq 0,...,y_n\geq 0} \!\!\!\dots\!\!\! \int y_1^{\alpha_1}...y_n^{\alpha_n} dy_1...dy_n = \frac{\Gamma(\alpha_1+1)...\Gamma(\alpha_n+1)}{\Gamma(\alpha_1+...+\alpha_n+n+1)}, \tag{1.106}$$

where $\alpha_1, ..., \alpha_n \in \mathbb{R}$. Using new coordinates $y_1 = x_1^2, ..., y_n = x_n^2$ we rewrite (1.106) in the form

$$\int\limits_{B_1^+(n)} x_1^{2\alpha_1+1}...x_n^{2\alpha_n+1} dx_1...dx_n = \frac{\Gamma(\alpha_1+1)...\Gamma(\alpha_n+1)}{2^n\Gamma(\alpha_1+...+\alpha_n+n+1)},$$

or putting $2\alpha_i + 1 = \gamma_i$, $i = 1, ..., n$,

$$\int\limits_{B_1^+(n)} x^\gamma dx = \frac{\Gamma\left(\frac{\gamma_1+1}{2}\right)...\Gamma\left(\frac{\gamma_n+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|}{2}+1\right)}.$$

Then from (1.102) and (1.105), for $u = 1$, applying the formula $\Gamma(z+1) = z\Gamma(z)$, putting $r = 1$, and denoting the resulting integral by $|S_1^+(n)|_\gamma$, we obtain

$$|S_1^+(n)|_\gamma = \int\limits_{S_1^+(n)} x^\gamma dS = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}. \tag{1.107}$$

1.4.3 Tricomi equation

The Tricomi equation is a second order partial differential equation of mixed elliptic-hyperbolic type for $u(x; y)$ with the following form:

$$u_{xx} + xu_{yy} = 0.$$

It was first analyzed in the work by Francesco Giacomo Tricomi (1923) on the well-posedness of a boundary value problem. The equation is hyperbolic in the half-plane $x < 0$, is elliptic in the half-plane $x > 0$, and degenerates on the line $x = 0$. Its characteristic equation is

$$dy^2 + xdx^2 = 0,$$

whose solutions are

$$y \pm \frac{2}{3}(-x)^{\frac{3}{2}} = C$$

for any constant C, which are real for $x < 0$. The characteristics comprise two families of semicubical parabolas lying in the half-plane $x < 0$, with cusps on the line $x = 0$. This is of hyperbolic degeneracy, for which the two characteristic families coincide, perpendicularly to the line $x = 0$.

For $\pm x > 0$, set $\tau = \frac{2}{3}(\pm x)^{\frac{3}{2}}$. Then the Tricomi equation becomes the classical elliptic or hyperbolic Euler–Poisson–Darboux equation:

$$u_{\tau\tau} \pm u_{yy} + \frac{\beta}{\tau} u_\tau = 0.$$

The index $\beta = \frac{1}{3}$ determines the singularity of solutions near $\tau = 0$, equivalently, $x = 0$.

Many important problems in fluid mechanics and differential geometry can be reduced to corresponding problems for the Tricomi equation, particularly transonic flow problems and isometric embedding problems. The Tricomi equation is a prototype of the generalized Tricomi equation:

$$u_{xx} + K(x) u_{yy} = 0.$$

For a steady-state transonic flow in $\mathbb{R}^2$, $u(x; y)$ is the stream function of the flow, $K(x)$ and x are functions of the velocity, which are positive at subsonic and negative at supersonic speeds, and y is the angle of inclination of the velocity. The solutions $u(x; y)$ also serve as entropy generators for entropy pairs of the potential flow system for the velocity. For the isometric embedding problem of two-dimensional Riemannian manifolds into $\mathbb{R}^3$, the function $K(x)$ has the same sign as the Gaussian curvature.

A closely related partial differential equation is the Keldysh equation

$$x u_{xx} + u_{yy} = 0.$$

It is hyperbolic when $x < 0$, is elliptic when $x > 0$, and degenerates on the line $x = 0$. Its characteristics are

$$y \pm \frac{1}{2}(-x)^{\frac{1}{2}} = C$$

for any constant C, which are real for $x < 0$. The two characteristic families are (quadratic) parabolas lying in the half-plane $x < 0$ and coincide tangentially to the degenerate line $x = 0$, which is of parabolic degeneracy. For $\pm x > 0$, the Keldysh equation becomes the elliptic or hyperbolic Euler–Poisson–Darboux equation with index $\beta = -\frac{1}{4}$ by setting $\tau = \frac{1}{2}(\pm x)\frac{1}{2}$. Many important problems in continuum mechanics can also be reduced to corresponding problems for the Keldysh equation, particularly shock reflection–diffraction problems in gas dynamics.

1.4.4 Abstract Euler–Poisson–Darboux equation

The abstract Euler–Poisson–Darboux equation has the form

$$Au = (B_\gamma)_t \, u, \qquad u = u(x, t; \gamma), \tag{1.108}$$

where A is a linear operator acting only by variable $x = (x_1, ..., x_n)$.

Lemma 7. *Let $u^k = u(x,t;k)$ be a solution to Eq. (1.108). The two recurrent formulas*

$$u^k = t^{1-k}u^{2-k}, \tag{1.109}$$

$$u_t^k = tu^{k+2} \tag{1.110}$$

are valid.

Proof. Let us show (1.109). Putting $w = t^{k-1}v$, $v = u^k$, we obtain

$$\begin{aligned}
w_t &= (k-1)t^{k-2}v + t^{k-1}v_t = \frac{k-1}{t}\,w + t^{k-1}v_t,\\
w_{tt} &= (k-1)(k-2)t^{k-3}v + (k-1)t^{k-2}v_t + (k-1)t^{k-2}v_t + t^{k-1}v_{tt}\\
&= \frac{(k-1)(k-2)}{t^2}\,w + 2(k-1)t^{k-2}v_t + t^{k-1}v_{tt},\\
\frac{2-k}{t}\,w_t &= -\frac{(k-1)(k-2)}{t^2}\,w + (2-k)t^{k-2}v_t,\\
w_{tt} + \frac{2-k}{t}\,w_t &= 2(k-1)t^{k-2}v_t + t^{k-1}v_{tt} + (2-k)t^{k-2}v_t\\
&= t^{k-1}\left(v_{tt} + \frac{k}{t}\,v_t\right)
\end{aligned}$$

or

$$w_{tt} + \frac{2-k}{t}\,w_t = t^{k-1}\left(v_{tt} + \frac{k}{t}\,v_t\right). \tag{1.111}$$

If $w = t^{k-1}v$ satisfies the equation

$$Aw = w_{tt} + \frac{2-k}{t}\,w_t,$$

then using (1.111) we get

$$t^{k-1}\,Av = t^{k-1}\left(v_{tt} + \frac{k}{t}\,v_t\right),$$

which means that v satisfies the equation

$$Av = v_{tt} + \frac{k}{t}\,v_t.$$

Denoting $w = u^{2-k}$ we obtain (1.109).

Let us prove now (1.110). Putting $tw = v_t$, $v = u^k$, we can write

$$w_t = -\frac{1}{t^2}\,v_t + \frac{1}{t}\,v_{tt},$$

$$w_{tt} = \frac{2}{t^3}\, v_t - \frac{2}{t^2}\, v_{tt} + \frac{1}{t}\, v_{ttt},$$

and

$$\frac{k+2}{t}\, w_t = -\frac{k+2}{t^3}\, v_t + \frac{k+2}{t^2}\, v_{tt}.$$

We have

$$\begin{aligned} w_{tt} + \frac{k+2}{t}\, w_t &= \frac{2}{t^3}\, v_t - \frac{2}{t^2}\, v_{tt} + \frac{1}{t}\, v_{ttt} - \frac{k+2}{t^3}\, v_t + \frac{k+2}{t^2}\, v_{tt} \\ &= \frac{1}{t}\, v_{ttt} - \frac{k}{t^3}\, v_t + \frac{k}{t^2}\, v_{tt} = \frac{1}{t}\left(v_{ttt} - \frac{k}{t^2}\, v_t + \frac{k}{t}\, v_{tt}\right) \\ &= \frac{1}{t}\frac{\partial}{\partial t}\left(v_{tt} + \frac{k}{t}\, v_t\right) \end{aligned}$$

or

$$w_{tt} + \frac{k+2}{t}\, w_t = \frac{1}{t}\frac{\partial}{\partial t}\left(v_{tt} + \frac{k}{t}\, v_t\right). \tag{1.112}$$

If $w = \frac{1}{t}\, v_t$ satisfies the equation

$$Aw = w_{tt} + \frac{k+2}{t}\, w_t,$$

then using (1.112) we obtain

$$\frac{1}{t}\frac{\partial}{\partial t} Av = \frac{1}{t}\frac{\partial}{\partial t}\left(v_{tt} + \frac{k}{t}\, v_t\right),$$

which means that v satisfies the equation

$$Av = v_{tt} + \frac{k}{t} v_t.$$

Denoting $w = u^{k+2}$, $v = u^k$, we get (1.110). □

For $A = \Delta_\gamma$ formula (1.109) is proved in [147] and formula (1.110) is proved in [16]. Both formulas are present in Weinsteins' article [595], but in the case when in Eq. (1.108) for each of variables x_i, $i = 1, ..., n$, the second derivative acts. These recurring formulas will allow using the solution u^k to (1.108) to get a solution to the same equation, but with the parameters $k + 2$ and $2 - k$, respectively.

So, if the function $u(x, t; k)$ is a solution of the abstract Euler–Poisson–Darboux equation $Au = (B_k)_t\, u$, where A is a linear operator acting only by $x = (x_1, ..., x_n)$, then the function $t^{1-k} u(x, t; 2 - k)$ also is a solution to this equation.

Basics of fractional calculus and fractional order differential equations

2

2.1 Short history of fractional calculus and fractional order differential equations

Operators of fractional integro-differentiation play an important role in many modern fields of mathematics. For special function theory its importance is reflected in the title of the well-known paper [253] "***All special functions are fractional integrals of elementary functions!***" (But there is a remark of Professor A. A. Kilbas – all but Fox functions.)

In this section we list essential one- and multi-dimensional fractional operators and include some historical and priority information.

2.1.1 One-dimensional fractional derivatives and integrals

Euler's introduction in 1729 of the gamma function (1.1) allowed to expand the concept of factorial to the case of a fractional value of the argument. This, in turn, allowed Euler to note that the concept of the n-th order derivative of the power function x^p acquired meaning for a nonintegral n. Namely, let $n, p \in \mathbb{N}$ and $p \geq n$. It is well known that

$$(x^p)' = px^{p-1}, \qquad (x^p)'' = p(p-1)x^{p-2}, \dots,$$
$$(x^p)^{(n)} = p(p-1)\dots(p-n+1)x^{p-n}$$

or

$$(x^p)^{(n)} = \frac{p!}{(p-n)!}x^{p-n}. \tag{2.1}$$

Expression (2.1) can also be meaningful for noninteger n and p. Namely, by virtue of the well-known equality for the gamma function

$$\Gamma(p+1) = p(p-1)\dots(p-n+1)\Gamma(p-n+1),$$

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formula (2.1) can be written as

$$\frac{d^n}{dx^n}x^p = \frac{\Gamma(p+1)}{\Gamma(p-n+1)}x^{p-n}$$

and can be used for all real n.

Further, we note that in 1823, Liouville formally expanded the formula for the derivative of the integer order of the exponential $\frac{d^n}{dx^n}e^{bx}$, $b \in \mathbb{R}$, to the derivative of the exponential of arbitrary order $\frac{d^\alpha}{dx^\alpha}e^{bx}$. Specifically,

$$\frac{d^\alpha e^{bx}}{dx^\alpha} = b^\alpha e^{bx}. \tag{2.2}$$

Based on formula (2.2) we can formally write the derivative of order $\alpha \in \mathbb{R}$ of an arbitrary function f represented by the series

$$f(x) = \sum_{k=0}^{\infty} c_k e^{b_k x}.$$

Therefore,

$$\frac{d^\alpha f(x)}{dx^\alpha} = \sum_{k=0}^{\infty} c_k b_k^\alpha e^{b_k x}.$$

The limitations of this definition are related not only to the convergence or divergence of the series but mainly to the fact that not every function can be represented as a series in exponentials (in modern terms as Dirichlet series).

In 1847 B. Riemann proposed the definition of a fractional integral which is now used as standard. His definition was based on a generalization of the formula for an n-fold integral of the form

$$\underbrace{\int\limits_a^x dx \ldots \int\limits_a^x dx \int\limits_a^x}_{n} f(x)dx = \frac{1}{(n-1)!}\int\limits_a^x (x-t)^{n-1} f(t)dt, \tag{2.3}$$

$$x \in [a,b], \qquad a, b \in \mathbb{R}.$$

Generalizing formula (2.3) to the case of real $n = \alpha > 0$ one can obtain

$$(I_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int\limits_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a.$$

Liouville in 1832 introduced a fractional integral operator in the form close to $I_{a+}^\alpha f$. Currently, I_{a+}^α is called the **left-sided fractional Riemann–Liouville integral** of order $\alpha > 0$.

It is known that the derivative of the integral with a variable upper limit of the continuous function equals the integrand in which the integration variable is replaced by the upper limit:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Thus, the differentiation operator $\frac{d}{dx}$ can be interpreted as the left inverse to the integration operator $\int_a^x$. If we can interchange the integration and differentiation operators, then we obtain

$$\int_a^x \left(\frac{d}{dt} f(t)\right) dt = f(x) - f(a)$$

and $\int_a^x$ will no longer be inverse for $\frac{d}{dx}$ for all functions.

So we can see that in order to obtain the Riemann–Liouville left-sided fractional derivative we should solve an integral equation

$$\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt = g(x).$$

This equation was already solved by Abel in 1823 in connection with the tautochrone problem, and the solution has the form

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}},$$

where $n = [\alpha] + 1$, $\alpha > 0$. The expression $(D_{a+}^{\alpha} f)(x)$ is now called the **left-sided fractional Riemann–Liouville derivative**. Similarly the right-sided fractional Riemann–Liouville integral and derivative have been introduced, in which an integration is taken from x to b.

Historically it were really Liouville and Riemann who considered the above forms of fractional derivatives and integrals. Riemann considered them on a finite segment and Liouville on the half-axes. But these considerations were not strict; Liouville suggested that every function is represented by the series in exponentials, and Riemann used for the derivation consciously divergent integrals. The first strict approach was developed in fact by A. V. Letnikov [498], and he was also the first researcher who applied fractional integrals for transmutation [273,498]. Important generalizations of Abel's equation were studied among others by N. Ya. Sonine [554].

The resulting definition of the fractional derivative is not very similar to the definition of the ordinary derivative as a limit of the form

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}, \qquad h = \triangle x. \tag{2.4}$$

However, A. Grünwald in 1867 and A. V. Letnikov in 1868 proposed the construction of fractional differentiation of the form

$$(\mathcal{D}^{\alpha}_{a+} f)(x) = \lim_{N\to\infty} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \sum_{k=0}^{N-1} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} f\,(x-kh)\,,\ N \in \mathbb{N},\ \alpha < N-1,$$
$$h = \frac{x-a}{N},$$

which is a natural generalization of formula (2.4) that is also convenient for computer approximations.

The fractional Grünwald–Letnikov integral for $\alpha > 0$ has the form

$$(\mathcal{I}^{\alpha}_{a+} f)(x) = \lim_{N\to\infty} \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{N-1} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)} f\,(x-kh)\,, \quad h = \frac{x-a}{N}. \tag{2.5}$$

It is known [494] that for an integrable on $[a,b]$ function $f(x)$ the limit (2.5) exists almost for all x and (2.5) coincides with the left-sided fractional Riemann–Liouville integral

$$\lim_{N\to\infty} \frac{h^{\alpha}}{\Gamma(\alpha)} \sum_{k=0}^{N-1} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)} f\,(x-kh) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}},$$
$$h = \frac{x-a}{N}.$$

Riemann–Liouville and Grünwald–Letnikov integro-differential operators were adapted for x from the finite segment.

It is important to note that an idea on which a definition of the Grünwald–Letnikov derivative is based, namely, the usage of finite differences, was used again after many years for a generalization of Sobolev spaces of fractional orders – Besov spaces [28].

Next many new forms of fractional integrals and derivatives appeared in particular adapted for x from the axes or semiaxes. In 1917 Weil defined fractional integration suitable for periodic functions:

$$I^{(\alpha)}_{\pm} f \sim \sum_{k=-\infty}^{\infty} (\pm ik)^{-\alpha} f_k e^{ikx}, \qquad f_k = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{-ikt} f(t)dt, \qquad f_0 = 0,$$

and showed that $I_{\pm}^{(\alpha)}$ for $0 < \alpha < 1$ may be written in the form

$$(I_{+}^{(\alpha)} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^{x} (x-t)^{\alpha-1} f(t)dt,$$

$$(I_{-}^{(\alpha)} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{x}^{\infty} (t-x)^{\alpha-1} f(t)dt.$$

We should also mention the form of fractional differentiation which was presented by Marchaud in 1927 and has the form

$$(\mathbf{D}^{\alpha} f)(x) = c \int\limits_{0}^{\infty} \frac{(\Delta_t^l f)(x)}{t^{\alpha+1}} dt, \qquad \alpha > 0,$$

where Δ_t^l is the forward finite difference:

$$(\Delta_t^l f)(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f\bigl(x + (l-k)t\bigr).$$

Also backward finite difference may be used,

$$(\nabla_t^l f)(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x - kt),$$

and central finite difference,

$$(\delta_t^l f)(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f\left(x + \left(\frac{l}{2} - k\right) t\right).$$

To use finite differences is one of the ways of regularizing the integral in $\mathbf{D}^{\alpha}$.

As a conclusion let us mention that important results concerning fractional operators were obtained by different mathematical schools. Let us mention just some of them.

In Minsk, Belarus, the founder of a well-known school on fractional calculus and its applications to fractional differential equations was Anatoly Kilbas; his ideas are now developed by his disciples S. Ragosin, A. Koroleva, A. Grin'ko, O. Skoromnik, A. Shlapakov, and others.

In Sofia, Bulgaria, important results were obtained by I. Dimovski, V. Kiryakova, J. Paneva-Konovska, and others.

The town Nal'chik, Russia, is sometimes called the "Mekka" of fractional calculus due to a well-known school of fractional calculus founded and developed by A. M. Nahushev. From his books [413–415] most mathematicians in the Soviet Union

and Russia first learned about and studied fractional calculus. He brought up many talented disciples now working on fractional calculus and its applications. Now this school is headed by Arsen Pskhu – also his disciple.

An important impact was made by the famous Voronezh mathematical school, originally headed by M. Krasnosel'skii and S. Krein. In their books on differential equations in Banach spaces fractional powers of operators were used for the general theory (cf. [294,301]).

Also let us mention works of mechanics of the Voronezh school which developed applications of fractional calculus. An essential impact was made by S. Meshkov, Yu. Rossikhin, and M. Shitikova. Now this work is continued at Voronezh Polytechnical (former Construction and Architecture) University. There, the International Scientific Center named after professor Yu. A. Rossikhin is headed by M. Shitikova.

Recently results on fractional calculus and applications were summed up in eight volumes of the "Handbook of Fractional Calculus with Applications" [161].

But we need also mention that nowadays many unprofessional and inappropriate generalizations of fractional operators have appeared which cannot be considered as "true" generalizations of fractional operators. Some of them do not obey the semigroup property, some are simply reduced to multiplication by a function, and some are similar to fractional operator's resolvents and are also not "true" generalizations. The project of Yu. Luchko [340] is devoted to critics of inappropriate generalizations of fractional operators (cf. also papers [211,432,433,556,563]). This concludes our brief review of the history of one-dimensional fractional integrals and derivatives. A detailed history of this issue is set forth, for example, in [494].

2.1.2 *Fractional derivatives in mechanics*

Methods of fractional calculus and fractional integro-differential equations are widely used in different applied sciences. Among them these methods turned out to be very effective in theory and numerically in mechanics, especially in viscoelasticity and construction mechanics. We want to stress that a remarkable impact in these branches was made by Soviet and Russian scientists. Starting from fundamental works of A. N. Gerasimov [179] and Yu. N. Rabotnov in 1947–1948, it was continued in works and books of Yu. N. Rabotnov [464,484] and many of his disciples. Based on works and researches of Rabotnov in the USSR period, there was a standard used for foundations of buildings, so we may say that every building during the period of the 1950s–1980s was constructed with the use of fractional calculus! An essential impact on the field was also made by D. Shermegor, M. Rosovskii, A. Rzhanitsyn, and others. And also a serious impact was made by the Voronezh school of mechanics, namely, by S. Meshkov [378], Yu. Rossikhin, and M. Shitikova.

For the real history of applications of fractional calculus in mechanics, cf. the survey papers of Yu. Rossikhin and M. Shitikova [482,483] and also [574].

2.1.3 Fractional powers of multi-dimensional operators

The most developed type of fractional multi-dimensional integrals are Riesz potentials, which generalize both the Newton potential to the fractional case and the Riemann–Liouville fractional integral to the multi-dimensional case.

Let us start from the classical Newton potential. If f is an integrable function with compact support, then the Newton potential of f is the convolution product (see [585])

$$V_N f(x) = \int\limits_{\mathbb{R}^n} v(x-y) f(y) dy,$$

where

$$v(x) = \begin{cases} \frac{1}{2\pi} \log|x| & n = 2, \\ \frac{1}{n(2-n)\omega_n} |x|^{2-n} & n \neq 2, \end{cases} \qquad \omega_n \text{ is a volume of unit ball } \mathbb{R}^n.$$

The Newton potential V_N of f is the solution to the Poisson equation

$$\Delta V_N = f, \qquad \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}, \qquad x = (x_1, ..., x_n) \in \mathbb{R}^n,$$

and therefore, it can be considered as a negative degree of the Laplace operator:

$$V_N f = \Delta^{-1} f.$$

The term *potential* is due to Green [204] (1828) and Gauss [176] (1840) (see [267]).

Along with the Newtonian potential, the wave potential of the function f has found wide applications (see [585]):

$$V_W f(x) = \int\limits_{\mathbb{R}^n} \varepsilon(x-y) f(y) dy,$$

where ε is a fundamental solution of the wave operator. For the wave potential V_W the following equality is true:

$$\Box V_W = f, \qquad \Box = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - ... - \frac{\partial^2}{\partial x_n^2},$$

and therefore, it can be considered as a negative degree of the D'Alembert operator: $V_W f = \Box^{-1} f$.

Marcel Riesz was a Hungarian mathematician who first introduced fractional powers of Laplace and D'Alembert operators. On November 1, 1933 M. Riesz presented the report "Integral of Riemann–Liouville, Potential, Waves" at the Physiographic Society of Lund, where he generalized the fractional Riemann–Liouville integral to the multi-dimensional case and also generalized Newtonian and wave potentials to the case of fractional power of a kernel. However, the first publication on this subject was

made by O. Frostman in 1935 [154], who was a PhD student of M. Riesz at Lund University. Frostman wrote that in addition to the Riemann–Liouville integrals of elliptical character studied in [154], in the speech in 1933 M. Riesz also considered integrals of hyperbolic and parabolic character.

On the International Mathematical Congress in Oslo in 1936 M. Riesz published four abstracts in different sections, two of which [472,473] were about the solution to the Cauchy problem for the wave equation. He found a solution to the equation

$$\Box u = f, \qquad u = u(x_1, x_2, ..., x_n), \qquad f = f(x_1, x_2, ..., x_n). \tag{2.6}$$

In order to solve (2.6) Riesz presented a potential

$$I^\alpha f(P) = \frac{1}{H_n(\alpha)} \int\limits_{D_S^P} f(Q) r_{PQ}^{\alpha-n} dQ, \tag{2.7}$$

where $r{=}r_{PQ}{=}\sqrt{(x_1-\xi_1)^2-(x_2-\xi_2)^2-...-(x_n-\xi_n)^2}$ is a Lorentz distance, $P{=}(x_1, x_2, ..., x_n)$, $Q{=}(\xi_1, \xi_2, ..., \xi_n)$, $D_S^P{=}\{Q \in \mathbb{R}^n{:}r_{PQ}^2 < 0, x_1-\xi_1 > 0\}$ is a retrograde light cone, α is a positive real number, and $H_n(\alpha) = \pi^{\frac{n-2}{2}} 2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha+2-n}{2}\right)$. The integral (2.7) converges for $\alpha > n-2$ and satisfies the relations $I^\alpha I^\beta = I^{\alpha+\beta}$ and $\Box I^{\alpha+2} = I^\alpha$. But in order to solve (2.6) using (2.7) is it necessary to extend analytically I^α with respect to α to the values $\alpha \leq n-2$ and show that $I^0 = I$ under the suitable regularity conditions. Riesz emphasized that the starting point of the analytical continuation of potentials is the concept of finite-part integrals presented by J. Hadamard in [157,158]. Hadamard's regularization reduces to dropping some divergent terms of a divergent integral and keeping the finite part. Riesz showed that this can be interpreted as taking the meromorphic continuation of a convergent integral. In [472] the Green formula for the potential (2.7) was presented in the following form:

$$\begin{aligned} I_\Box^\alpha f(P) = I_\Box^{\alpha+2} \Box f(P) - \frac{1}{H_m(\alpha+2)} \\ \times \int\limits_{S^P} \left(\frac{\partial f(Q)}{\partial \nu} r_{PQ}^{\alpha+2-n} - f(Q) \frac{\partial r_{PQ}^{\alpha+2-n}}{\partial \nu} \right) dS, \end{aligned} \tag{2.8}$$

where S^P denotes the portion of the surface cut by S from the cone, ν is the outer conormal to S^P, and S is a surface such that the generators of the retrograde cone belonging to the points P are considered cut only at a single point. M. Riesz presented the results of the abstracts [472,473] in a more detailed form in 1939 in the article [474]. In [151] Riesz potential was considered as a generalization of the Riemann–Liouville integral.

In 1949 Riesz published an extended paper [475] about two forms of potentials: with Euclidean and with Lorentz distances. Such potentials are now called the elliptic and hyperbolic **Riesz potentials** and have the following forms, respectively:

$$I_\Delta^\alpha f(P) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} f(Q) r^{\alpha-n} dQ$$

and (2.7). Here $\gamma_n(\alpha)=\frac{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$, $r=\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+...+(x_n-\xi_n)^2}$ is a Euclidean distance. In [475] it was shown that

$$\Delta I_\Delta^{\alpha+2}f(P)=-I_\Delta^\alpha f(P) \qquad \text{and} \qquad \Box I_\Box^{\alpha+2}f(P)=I_\Box^\alpha f(P),$$

proving the Green formula (2.8), and analytical continuations of I_Δ^α and $I_\Box^\alpha$ were constructed. M. Riesz considered the potential (2.7) not only over D_S^P but also over the cone $K_+^+=\{x: x_1^2\geq x_2^2+...+x_n^2, x_1\geq 0\}$.

As for the theory of elliptic Riesz potentials during the 1940s and 1950s, it has been the subject of many independent studies for a wide variety of generalizations (see [87,215]). Besides the studies of the Dirichlet problem, elliptic Riesz potentials were adapted to the study of the sign of the integral of energy and the principle of the maximum.

Popularization of the theory of distributions presented in books of L. Schwartz [495] and I. M. Gelfand and G. E. Shilov [177] had a significant effect on the further development of the Riesz potential theory. In [177,495] generalized functions generated by quadratic forms r^λ, $(P\pm i0)^\lambda$ $(x\in\mathbb{R}^n$, $r=|x|$, $P=x_1^2+...+x_p^2-x_{p+1}^2-...-x_n^2)$ and their Fourier transforms were studied. It turned out that it is convenient to present Riesz potentials I_Δ^α and $I_\Box^\alpha$ as convolutions with functions r^λ and $(P\pm i0)^\lambda$, respectively ($p=1$ for $I_\Box^\alpha$). In addition, the Riesz ultrahyperbolic potential immediately appeared as a convolution with $(P\pm i0)^\lambda$, $p>1$.

Some fundamental problems were not solved by M. Riesz; among them are boundedness estimates, finding spaces invariant with respect to potentials, and inversion of potentials I_Δ^α and $I_\Box^\alpha$.

Necessary and sufficient conditions for boundedness of I_Δ^α from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ $(0<\alpha<n$, $1<p<n/\alpha$, $1/g=1/p-\alpha/n)$ were given by Sobolev [544] in 1938.

As for the elliptic potential I_Δ^α the space invariant with respect to this potential is a Lizorkin space $\Phi=\{F\psi\in S(\mathbb{R}^n), \psi\in\Psi\}$, where $S(\mathbb{R}^n)$ is a Schwartz space, F is a Fourier transform, and $\Psi=\{\psi\in S(\mathbb{R}^n), (D^j\psi)(0)=0, |j|=0,1,2,...\}$. This fact was first noted by Semyanistyi [496] in 1960 and Lizorkin [331] in 1963.

The inversion of the elliptic Riesz potential was constructed by S. G. Samko in the form of a hypersingular integral [493] in 1976 and by the method of approximate operators [492] in 1998.

In [173] and [194] kernels of fractional powers which are the set of all positive powers of the operator generated by the Green function for the Laplace equation were studied. Riesz potentials with Euclidean distance was also studied in [15,162,241,431, 475,485,494,495,543,575].

Generalization of the Riesz hyperbolic potential or potential with Lorentz distance to the ultrahyperbolic case was considered by Nozaki [428] in 1964. I. A. Kipriyanov and L. A. Ivanov [244,245] in 1986–1987 introduced the following modification of the hyperbolic Riesz potential:

$$I_{\Box}^{\alpha}f(x)=\frac{1}{H_n(\alpha)}\int\limits_{D_x}f(y)\mathrm{sh}^{\alpha-n}(r_{xy})dz. \tag{2.9}$$

Such modification is connected with the Lorentz space. In [244,245] the Fourier transform was obtained, an analogue of the Hardy–Littlewood–Sobolev theorem on potential estimates was proved, and an application to the Cauchy problem for the Euler–Poisson–Darboux equation on the Lorentz space was given. Potential studied by I. A. Kipriyanov and L. A. Ivanov was used by S. Helgason [163] to represent a function through its orbital integrals in isotropic Lorentzian manifolds.

V. A. Nogin and E. V. Sukhinin [425,426] in 1992–1993 completely solved the problems of the boundedness and inversion of the Riesz hyperbolic and ultrahyperbolic Riesz potentials of the form

$$I_{\square}^{\alpha} f(x) = \frac{1}{H_n(\alpha)} \int\limits_{K_+^+} f(x-y)[y_1^2 - y_2^2 - \ldots - y_n^2]^{\frac{\alpha-n}{2}} dy, \qquad n-2 < \alpha < n,$$

where $f \in L_p(\mathbb{R}^n)$, $1 < p < \frac{n}{\alpha}$, and

$$I_{P\pm i0}^{\alpha} f(x) = C \int\limits_{\mathbb{R}^n} f(x-y)(P \pm i0)^{\frac{\alpha-n}{2}} dy, \qquad n-2 < \alpha < n,$$

where $P(y) = y_1^2 + \ldots + y_s^2 - y_{s+1}^2 - \ldots - y_n^2$, $C = \frac{e^{\pm\frac{n-s}{2}\pi i}}{\gamma_n(\alpha)}$, respectively. Boundedness was proved using the Marcinkiewicz interpolation theorem and inversion was obtained by the method of approximate operators. Another approach to inversion of Riesz ultra-hyperbolic was given in [58].

Another widely studied potential is the Bessel potential

$$G^{\alpha} f(x) = \int\limits_{\mathbb{R}^n} \mathcal{G}_{\alpha}(x-y) f(y) dy, \qquad \alpha > 0,$$

where

$$\mathcal{G}_{\alpha}(x) = \frac{2^{\frac{2-n-\alpha}{2}}}{\pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \frac{K_{\frac{n-\alpha}{2}}(|x|)}{|x|^{\frac{n-\alpha}{2}}},$$

which realizes the fractional powers of the operator $(I-\Delta)^{-\frac{\alpha}{2}}$. Here K_ν is the modified Bessel functions of the second kind (1.17). Such potential appeared in the papers of N. Aronzajn and K. T. Smith [10] in 1961 and Calderon [44] in 1961. The space of Bessel potentials is sometimes called the Liouville space of fractional smoothness α. This space is an extension of the Sobolev spaces $L_p^m(\mathbb{R}^n)$ to the case of fractional order α; that is why it is also called Sobolev space of fractional order. Results about the space of Bessel potentials were obtained by I. Stein [542] in 1961 in the case $0 < \alpha < 2$ and by Lizorkin [332] in 1970 in the general case. The inversion of Bessel potentials using hypersingular integrals was given by V. A. Nogin [422–424] in 1981–1985.

Necessary and sufficient conditions for the strong and weak boundedness of the Riesz potential on Orlicz spaces were given in [209].

The potential theory comes from mathematical physics. The most well-known areas of its application are electrostatic and gravitational theory, probability theory, scattering theory, and biological systems.

The first application of Riesz potentials was given by M. Riesz himself, and it was a solution of the Maxwell equations for the electromagnetic field (see [475], p. 146, and [152]). The Maxwell equations are fundamental equations of classical electrodynamics and optics. The equations completely describe all electromagnetic phenomena in an arbitrary environment and give a mathematical model for electric, optical, and radio technologies, such as power generation, electric motors, wireless communication, lenses, radar, etc. So the Riesz potential can be used for studying realistic single-particle energy levels.

An interesting fact was noted in [43]. Namely, in this paper it was shown that the elliptic Riesz potential can be interpreted as a transmutation operator. More precisely, the operator square root of the Laplacian was obtained from the harmonic extension problem to the upper half-space as the operator that maps the Dirichlet boundary condition to the Neumann condition. The same result but for hyperbolic Riesz potentials was obtained in [120].

Spaces of Riesz and Bessel potentials are used in connection with problems which arise in the theory of integral equations of the first kind with a potential type or oscillating kernel.

P. I. Lizorkin in [333] showed that the space of elliptic Riesz potentials is the functional completion of infinitely differentiable functions finite in $\mathbb{R}^n$ in $l_p^r(\mathbb{R}^n)$, where $l_p^r(\mathbb{R}^n)$ is the class of such functions that are Liouville derivatives of order r belonging to $L_p(\mathbb{R}^n)$.

In [195–201] optimal embedding of spaces of Bessel and Riesz type potentials are obtained.

S. Helgason (see [163], p. 137) solved the problem of determining a function by its orbital integrals over Lorentzian spheres. In other words he built the inversion of the generalized Radon transform in the isotropic Lorentz spaces. It turned out that the inverse operator to such a Radon transform would be the operator (2.9).

2.1.4 Differential equations of fractional order

Here we give brief overview of the results for differential equations of fractional order on a finite interval of the real axis following [241,494].

The differential equations of fractional order $\alpha \notin \mathbb{N}$ have the following general form:

$$F[x, y(x), D^{\alpha_1} y(x), ..., D^{\alpha_m} y(x)] = f(x). \tag{2.10}$$

Here $x = (x_1, ..., x_n)$ is a point of n-dimensional Euclidean space $\mathbb{R}^n$, F, f are given functions, and D^{α_k} are fractional differentiation operators of real $\alpha_k > 0$ or complex $\operatorname{Re}\alpha_{\mathrm{k}} > 0$ numbers, $k = 1, 2, ..., m$.

The most studied equation of fractional order is

$$(D_{a+}^{\alpha} y)(x) = f[x, y(x)], \qquad \operatorname{Re}\alpha > 0, \qquad x > a, \qquad a \in \mathbb{R},$$

where D^{α}_{a+} is the left-sided Riemann–Liouville fractional derivative on a segment (2.14) or on semiaxes (2.28) (in this case $a=0$). To this equation initial conditions

$$(D^{\alpha-k}_{a+}y)(a+)=b_k, \quad \operatorname{Re} k=1,...,n, \quad n=[\operatorname{Re}\alpha]+1, \quad x>a, \quad b_k\in\mathbb{C},$$

are added. The notation $(D^{\alpha-k}_{a+}y)(a+)$ means that the limit is taken at almost all points of the right-sided neighborhood $(a,a+\varepsilon)$ $(\varepsilon>0)$. Let us note that

$$(D^{\alpha-n}_{a+}y)(a+)=(I^{n-\alpha}_{a+}y)(a+), \qquad (D^{0}_{a+}y)(a+)=y(a),$$

where $I^{n-\alpha}_{a+}$ is the Riemann–Liouville fractional integration operator defined by (2.12) or (2.26) (in this case $a=0$), accordingly. Such problems are called *Cauchy type problems*.

When $0<\operatorname{Re}\alpha<1$ the weighted Cauchy type problem is

$$(D^{\alpha}_{a+}y)(x)=f[x,y(x)], \qquad \lim_{x\to a+0}(x-a)^{1-\alpha}y(x)=c, \qquad c\in\mathbb{C}.$$

In [17] it was shown that if $f(x)\in L(a,b)$, the Cauchy type problem for the linear differential equation

$$(D^{\alpha}_{a+}y)(x)-\lambda y(x)=f(x),\ (D^{\alpha-k}_{a+}y)(a+)=b_k,$$
$$k=1,...,n,\ n=[\operatorname{Re}\alpha]+1,\ b_k\in\mathbb{C},$$

has the unique solution $y(x)$ in some subspace of $L(a,b)$ given by

$$y(x)=\sum_{k=1}^{n}b_k x^{\alpha-k}E_{\alpha,\alpha-k+1}(\lambda(x-a)^{\alpha})$$
$$+\int_a^x (x-t)^{\alpha-1}E_{\alpha,\alpha}(\lambda(x-t)^{\alpha})f(t)dt,$$

where $E_{\alpha,\alpha}$ is the Mittag-Leffler function defined by (1.39). In particular for $0<\operatorname{Re}\alpha<1$ and $f(x)=0$ the function

$$y(x)=b_1x^{\alpha-1}E_{\alpha,\alpha}(\lambda(x-a)^{\alpha})$$

will be the solution to

$$(D^{\alpha}_{a+}y)(x)=\lambda y(x),\ (I^{1-\alpha}_{a+}y)(a+)=b_1,\ b_1\in\mathbb{C}.$$

Luchko and Gorenflo in [341] showed that the problem with the Gerasimov–Caputo derivative (2.30)

$$({}^{GC}D^{\alpha}_{0+}y)(x)-\lambda y(x)=f(x),\ x>0,\ \alpha>0,\ \lambda>0,$$
$$y^{(k)}(0)=b_k,\ k=1,...,n,\ n=[\operatorname{Re}\alpha]+1,\ b_k\in\mathbb{C},$$

has a unique solution of the form

$$y(x) = \sum_{k=1}^{n} b_k x^{\alpha-k} E_{\alpha,k}(\lambda x^{\alpha}) + \int_0^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(\lambda(x-t)^{\alpha}) f(t)dt.$$

2.2 Standard fractional order integro-differential operators

Fractional integro-differential operators are studied in [17,78,107,202,381,413,414,429,431,446,494].

2.2.1 *Riemann–Liouville fractional integrals and derivatives on a segment*

In this subsection we present definitions of the Riemann–Liouville fractional integrals and fractional derivatives on a finite segment of the real line. Also we give some of their properties in spaces of summable and continuous functions.

Definition 15. *Let* $0 < \alpha$, $f \in L_1(a,b)$, $a, b \in \mathbb{R}$. *Then integrals*

$$(I_{b-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \qquad x < b, \tag{2.11}$$

and

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \qquad x > a, \tag{2.12}$$

are called ***right-sided*** (2.11) ***and left-sided*** (2.12) ***Riemann–Liouville fractional integrals of the order*** α ***on a segment*** $[a,b]$.

Let $\alpha > 0$ *and not integer,* $n = [\alpha] + 1$. ***Right-sided and left-sided Riemann–Liouville fractional derivatives of the order*** α ***on a segment*** $[a,b]$ *for a function* $f \in L_1(a,b)$ *are introduced by the relations*

$$\begin{aligned}(D_{b-}^{\alpha} f)(x) &= \left(-\frac{d}{dx}\right)^n (I_{b-}^{n-\alpha} f)(x) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b \frac{f(t)dt}{(t-x)^{\alpha-n+1}},\end{aligned} \tag{2.13}$$

where $I_{b-}^{n-\alpha} f \in C^n(a,b)$ *and*

$$(D_{a+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n (I_{a+}^{n-\alpha} f)(x)$$
$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \tag{2.14}$$

where $I_{a+}^{n-\alpha} f \in C^n(a,b)$.

When $\alpha = n \in \mathbb{N}$ *for* $x \in [a,b]$, $f \in C^n(a,b)$,

$$(D_{b-}^{n} f)(x) = \left(-\frac{d}{dx}\right)^n f(x), \qquad (D_{a+}^{n} f)(x) = \left(\frac{d}{dx}\right)^n f(x).$$

Next, following [241,494] we present some properties of the Riemann–Liouville fractional integrals and fractional derivatives on the segment.

Theorem 13. *(see [241], p. 73) Let* $0 \le \alpha$ *and* $n = [\alpha]+1$. *If* $f \in AC^n[a,b]$, *then the fractional derivatives* D_{b-}^{α} *and* D_{a+}^{α} *exist almost everywhere on a segment* $[a,b]$ *and can be represented in the forms*

$$(D_{b-}^{\alpha} f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int\limits_x^b \frac{f^{(n)}(t)dt}{(t-x)^{\alpha-n+1}} + \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(1+k-\alpha)} (b-x)^{k-\alpha}$$

and

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_a^x \frac{f^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)} (x-a)^{k-\alpha},$$

respectively.

Theorem 14. *(see [241], p. 72) (1) The fractional Riemann–Liouville integrals* I_{b-}^{α} *and* I_{a+}^{α} *with* $\alpha > 0$ *are bounded in* $L_p(a.b)$, $1 \le p \le \infty$:

$$||I_{b-}^{\alpha} f||_p \le K||f||_p, \qquad ||I_{a+}^{\alpha} f||_p \le K||f||_p, \qquad K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

(2) If $0 < \alpha < 1$ *and* $1 < p < \frac{1}{\alpha}$, *then the operators* I_{b-}^{α} *and* I_{a+}^{α} *are bounded from* $L_p(0,\infty)$ *into* $L_q(0,\infty)$, *where* $q = \frac{p}{1-\alpha p}$.

The semigroup property of the fractional integration operators I_{b-}^{α} and I_{a+}^{α} is given by the following result. Often the semigroup property is called index law.

Theorem 15. *(see [494], Sections 2.3 and 2.5) If $\alpha > 0$ and $\beta > 0$, then equalities*

$$(I^{\alpha}_{b-} I^{\beta}_{b-} f)(x) = (I^{\alpha+\beta}_{b-} f)(x), \qquad (I^{\alpha}_{a+} I^{\beta}_{a+} f)(x) = (I^{\alpha+\beta}_{a+} f)(x)$$

are satisfied at almost every point $x \in [a,b]$ for $f \in L_p(a,b)$, $1 \le p \le \infty$. If $\alpha + \beta > 1$, then these relations hold at any point of $[a,b]$.

It is well known that the usual differentiation $\frac{d}{dx}$ and integration $\int\limits_a^x ...dt$ are mutually inverse operations if the differentiation is applied on the left, i.e., $\frac{d}{dx} \int\limits_a^x f(t)dt = f(x)$. However, generally speaking, $\int\limits_a^x f'(t)dt \neq f(x)$ (as the constant $-f(a)$ is added). Similarly $\left(\frac{d}{dx}\right)^n I^n_{a+} f = f$ but $I^n_{a+} f^{(n)}$ does not coincide with a function f and differs from it by a polynomial of order $(n-1)$. In the same way for fractional differentiation the equality $D^{\alpha}_{a+} I^{\alpha}_{a+} f = f$ is valid but $I^{\alpha}_{a+} D^{\alpha}_{a+} f$ does not coincide with a function $f(x)$ and differs from it by a sum of functions $(x-a)^{\alpha-k}$, $k = 1, 2, ..., [\alpha] - 1$.

Lemma 8. *(see [241], p. 74) If $\alpha > 0$ and $f \in L_p(a,b)$, $1 \le p \le \infty$, then the equalities*

$$(D^{\alpha}_{b-} I^{\beta}_{b-} f)(x) = f(x), \qquad (D^{\alpha}_{a+} I^{\beta}_{a+} f)(x) = f(x)$$

hold almost everywhere on $[a,b]$.

We need the following class of functions.

Definition 16. *(see [494], p. 43) Let $I^{\alpha}_{b-}(L_p)$, $\alpha > 0$, be the class of functions $f(x)$ such that*

$$f \in I^{\alpha}_{b-}(L_p), \quad \alpha > 0 \qquad \Leftrightarrow \qquad f = I^{\alpha}_{b-}\varphi, \quad \varphi \in L_p(a,b), \quad 1 \le p \le \infty.$$

Let $I^{\alpha}_{a+}(L_p)$, $\alpha > 0$, be the class of functions $f(x)$ such that

$$f \in I^{\alpha}_{a+}(L_p), \quad \alpha > 0 \qquad \Leftrightarrow \qquad f = I^{\alpha}_{a+}\varphi, \quad \varphi \in L_p(a,b), \quad 1 \le p \le \infty.$$

The next theorem gives the description of the classes $I^{\alpha}_{b-}(L_1)$ and $I^{\alpha}_{a+}(L_1)$.

Theorem 16. *(see [494], p. 43) (1) The necessary and sufficient conditions required for $f \in I^{\alpha}_{b-}(L_1)$, $\alpha > 0$, are*

$$f_{n-\alpha}(x) = I^{n-\alpha}_{b-} f \in AC^n([a,b]),$$

where $n = [\alpha] + 1$ and

$$f^{(k)}_{n-\alpha}(b) = 0, \quad k = 0, 1, 2, ..., n-1.$$

(2) The necessary and sufficient conditions required for $f \in I^{\alpha}_{a+}(L_1)$, $\alpha > 0$, are

$$f_{n-\alpha}(x) = I^{n-\alpha}_{a+} f \in AC^n([a,b]),$$

where $n=[\alpha]+1$ *and*

$$f_{n-\alpha}^{(k)}(a)=0, \quad k=0,1,2,...,n-1.$$

We note that the representability of a function f by fractional integral of the order α and the existence of a fractional derivative of f are two different things. The hypothesis "the fractional derivative exists almost everywhere and is summable" is not enough to produce satisfactory theory. Therefore we give the next definition.

Definition 17. *(see [494], p. 44) Let* $\alpha>0$, $n=[\alpha]+1$.

Function $f(x)\in L_1(a,b)$ *has a* ***summable fractional derivative*** $D_{b-}^{\alpha}f$ *if* $I_{b-}^{n-\alpha}f\in AC^n([a,b])$.

Function $f(x)\in L_1(a,b)$ *has a* ***summable fractional derivative*** $D_{a+}^{\alpha}f$ *if* $I_{a+}^{n-\alpha}f\in AC^n([a,b])$.

Remark 4. *(see [494], p. 44) If* $D_{b-}^{\alpha}f=\left(-\frac{d}{dx}\right)^n I_{b-}^{n-\alpha}f$ *exists in the usual sense, i.e.,* $I_{b-}^{n-\alpha}f$ *is differentiable by the order* n *at each point of* $[a,b]$, *then* f *has a summable fractional derivative* $D_{b-}^{\alpha}f$ *in the sense of Definition 17.*

If $D_{a+}^{\alpha}f=\left(\frac{d}{dx}\right)^n I_{a+}^{n-\alpha}f$ *exists in the usual sense, i.e.,* $I_{a+}^{n-\alpha}f$ *is differentiable by the order* n *at each point of* $[a,b]$, *then* f *has a summable fractional derivative* $D_{a+}^{\alpha}f$ *in the sense of Definition 17.*

The next theorems give conditions when fractional Riemann–Liouville integration and differentiation are used as reciprocal operations.

The following statement characterizes the composition of the fractional integration operator I_{b-}^{α} with the fractional differentiation operator D_{b-}^{α}.

Theorem 17. *(see [241], p. 75) Let* $\alpha>0$. *Then the equality*

$$(D_{b-}^{\alpha}I_{b-}^{\alpha}f)(x)=f(x) \tag{2.15}$$

is valid for any summable function $f(x)$.

The equality

$$(I_{b-}^{\alpha}D_{b-}^{\alpha}f)(x)=f(x) \tag{2.16}$$

is satisfied for

$$f(x)\in I_{a+}^{\alpha}(L_1). \tag{2.17}$$

If we assume that instead of (2.17) *a function* $f\in L_1(a,b)$ *has a summable fractional derivative* $D_{b-}^{\alpha}f$ *(in the sense of Definition 17), then* (2.15) *is not true in general and has to be replaced by the result*

$$(I_{b-}^{\alpha}D_{b-}^{\alpha}f)(x)=f(x)-\sum_{k=0}^{n-1}\frac{(-1)^{n-k-1}(b-x)^{\alpha-k-1}}{\Gamma(\alpha-k)}\,f_{n-\alpha}^{(n-k-1)}(b), \tag{2.18}$$

where $n = [\alpha] + 1$ *and* $f_{n-\alpha}(x) = I_{b-}^{n-\alpha} f$*. In particular, for* $0 < \alpha < 1$ *we have*

$$(I_{b-}^{\alpha} D_{b-}^{\alpha} f)(x) = f(x) - \frac{f_{1-\alpha}(b)}{\Gamma(\alpha)}(b - x)^{\alpha-1}. \tag{2.19}$$

Theorem 18. *(see [494], p. 44) Let* $\alpha > 0$*. Then the equality*

$$(D_{a+}^{\alpha} I_{a+}^{\alpha} f)(x) = f(x) \tag{2.20}$$

is valid for any summable function $f(x)$*.*

The equality

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x) \tag{2.21}$$

is satisfied for

$$f(x) \in I_{a+}^{\alpha}(L_1). \tag{2.22}$$

If we assume that instead of (2.22) *a function* $f \in L_1(a, b)$ *has a summable fractional derivative* $D_{a+}^{\alpha} f$ *(in the sense of Definition 17), then* (2.20) *is not true in general and is to be replaced by the result*

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x - a)^{\alpha-k-1}}{\Gamma(\alpha - k)} f_{n-\alpha}^{(n-k-1)}(a), \tag{2.23}$$

where $n = [\alpha] + 1$ *and* $f_{n-\alpha}(x) = I_{a+}^{n-\alpha} f$*. In particular, for* $0 < \alpha < 1$ *we have*

$$(I_{a+}^{\alpha} D_{a+}^{\alpha} f)(x) = f(x) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)}(x - a)^{\alpha-1}. \tag{2.24}$$

2.2.2 Riemann–Liouville fractional integrals and derivatives on a semiaxis

Definition 18. *Let* $0 < \alpha$, $f \in L_1(0, \infty)$*. Then integrals*

$$(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(t)}{(t - x)^{1-\alpha}} dt, \qquad x > 0, \tag{2.25}$$

and

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(t)}{(x - t)^{1-\alpha}} dt, \qquad x > 0, \tag{2.26}$$

are called ***right-sided*** (2.26) ***and left-sided*** (2.25) ***Riemann–Liouville fractional integrals of the order*** α ***on a semiaxis*** $(0, \infty)$.

Let $\alpha > 0$ and not integer, $n = [\alpha] + 1$. ***Right-sided and left-sided Riemann–Liouville fractional derivatives of the order*** α ***on a semiaxis*** $(0, \infty)$ *for function $f \in L_1(0, \infty)$ are introduced by the relations*

$$(D_-^\alpha f)(x) = \left(-\frac{d}{dx}\right)^n (I_-^{n-\alpha} f)(x)$$
$$= \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int\limits_x^\infty \frac{f(t)dt}{(t-x)^{\alpha-n+1}}, \tag{2.27}$$

where $I_-^{n-\alpha} f \in C^n(0, \infty)$ and

$$(D_{0+}^\alpha f)(x) = \left(\frac{d}{dx}\right)^n (I_{0+}^{n-\alpha} f)(x)$$
$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_0^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \tag{2.28}$$

where $I_{0+}^{n-\alpha} f \in C^n(0, \infty)$.

When $\alpha = n \in \mathbb{N}$ for $x \in (0, \infty)$, $f \in C^n(0, \infty)$,

$$(D_-^n f)(x) = \left(-\frac{d}{dx}\right)^n f(x), \qquad (D_{0+}^n f)(x) = \left(\frac{d}{dx}\right)^n f(x).$$

Theorem 19. *(see [241], p. 83) If $\alpha > 0$, $\beta > 0$, $p \geq 1$, $\alpha + \beta < 1/p$, and $f \in L_p(0, \infty)$, then the equalities*

$$(I_-^\alpha I_-^\beta f)(x) = (I_-^{\alpha+\beta} f)(x), \qquad (I_{0+}^\alpha I_{0+}^\beta f)(x) = (I_{0+}^{\alpha+\beta} f)(x)$$

hold.

Lemma 9. *(see [241], p. 83) If $\alpha > 0$ and $f \in L_1(0, \infty)$, then the equalities*

$$(D_-^\alpha I_-^\beta f)(x) = f(x), \qquad (D_{0+}^\alpha I_{0+}^\beta f)(x) = f(x)$$

hold.

Theorem 20. *(Hardy–Littlewood–Polya theorem, see [241], p. 82) Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $\alpha > 0$. Then the operators I_-^α and I_{0+}^α are bounded from $L_p(0, \infty)$ into $L_q(0, \infty)$ if and only if*

$$0 < \alpha < 1, \qquad 1 < p < \frac{1}{\alpha}, \qquad q = \frac{p}{1 - \alpha p}.$$

Theorem 21. *(see [241], p. 83) If $\alpha > 0$, then the relation*

$$\int\limits_0^\infty \varphi(x)(I_{0+}^\alpha \psi)(x)dx = \int\limits_0^\infty \psi(x)(I_-^\alpha \varphi)(x)dx$$

holds for functions $\varphi \in L_p(0,\infty)$ *and* $\psi \in L_q(0,\infty)$. *The relation*

$$\int_0^\infty f(x)(D_{0+}^\alpha g)(x)dx = \int_0^\infty g(x)(I_-^\alpha f)(x)dx$$

holds for functions $f \in I_-^\alpha(L_p(0,\infty))$ *and* $g \in I_{0+}^\alpha(L_q(0,\infty))$, *where* $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$.

2.2.3 Gerasimov–Caputo fractional derivatives

In 1948 (see [179], submitted in 1947) the Soviet mechanic A. N. Gerasimov introduced the fractional derivative of the form

$$({}^G D_{-,t}^\alpha u)(x,t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \frac{u_y(x,y)dy}{(t-y)^\alpha}, \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha < 1.$$

In the same work, A. N. Gerasimov studied two new problems in viscoelasticity theory. He reduced this problem to differential equations with partial fractional derivative. In the same way it is possible to introduce fractional derivatives of Gerasimov type for other fractional derivatives analogous to Riemann–Liouville ones (cf. the paper by the author [517]).

After 20 years the same construction with higher order derivatives was introduced by the Italian mechanic M. Caputo in 1967 in the paper [49] and studied in the monograph [50]. Therefore, in many books and papers, the fractional derivative of the type ${}^{GC}D_{-,t}^\alpha$ is called the Caputo derivative.

An obvious modification of the fractional derivative of Gerasimov to the case of a higher order leads to fractional derivatives and integrals of Gerasimov–Caputo.

Obvious generalization of Gerasimov's fractional derivative for higher order derivatives leads to Gerasimov–Caputo operators. The general form of the Gerasimov–Caputo derivative is ([179], [241], p. 97, formula (2.4.47))

$$({}^{GC} D_{a+}^\alpha g)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{g^{(n)}(t)dt}{(x-t)^{\alpha-n+1}}, \qquad n = [\alpha]+1, \qquad x \in (a,b), \tag{2.29}$$

where $\alpha > 0$, $\alpha \notin \mathbb{N}_0$, and $y \in C^n(a,b)$. For $\alpha = n = 0, 1, 2, ...$

$$({}^{GC} D_{a+}^n f)(x) = f^{(n)}(x).$$

For noninteger $\alpha > 0$,

$$({}^{GC} D_{0+}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)dt}{(x-t)^{\alpha+1-n}}, \qquad x \in [0,\infty), \tag{2.30}$$

is the left-sided Gerasimov–Caputo fractional derivative on semiaxes ([179], [241], p. 97, formula (2.4.47)) and for $\alpha = n = 0, 1, 2, ...$

$$({}^{GC}D^n_{0+} f)(x) = f^{(n)}(x).$$

The paper of A. N. Gerasimov is the first in mathematical literature in which fractional derivatives were used for studying viscoelastic materials. Let us mention the book of O. Novozhenova [427], who gathered many biographical facts and papers of A. N. Gerasimov, including his pioneering paper [179]. The priority of A. N. Gerasimov was first pointed out by A. Kilbas in his lectures [238]. Note that M. Caputo never insisted on his priority, as he worked in the field of mechanics; it was propagandized by his disciple F. Mainardi.

The importance of Gerasimov–Caputo fractional derivatives is illustrated by the simple example of its application to fractional differential equations. If we consider the fractional differential equation with Riemann–Liouville fractional derivative of the form

$$(D^\alpha_{0+} y)(x) = \lambda y(x), \qquad x > 0, \qquad 0 < \alpha \le 1, \qquad \lambda \in \mathbb{R},$$

we should add the initial conditions here,

$$(D^{\alpha-1}_{0+} y)(0+) = 1,$$

and the solution can be found explicitly and equals

$$y(x) = x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha), \tag{2.31}$$

via the Mittag-Leffler function

$$E_{\alpha,\alpha}(\lambda x^\alpha) = \sum_{n=0}^{\infty} \frac{(\lambda x^\alpha)^n}{\Gamma(\alpha n + \alpha)}$$

(cf. (1.39) for its definition). Note a singularity at zero in (2.31) when $\alpha < 1$. It is not possible to consider the classical Cauchy problem at zero initial point with Riemann–Liouville fractional derivative

$$(D^\alpha_{0+} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int\limits_0^x \frac{f(t)dt}{(x-t)^\alpha}, \qquad 0 < \alpha \le 1.$$

Of course we can use the conditions $(D^{\alpha-1}_{0+} y)(0+) = 1$ instead of classical initial conditions, but repeat once more – in this case the classical Cauchy problem is incorrect. Opposite to it the general solution to the fractional differential equation with Gerasimov–Caputo fractional derivative

$$({}^{GC}D^\alpha_{0+} y)(x) = \lambda y(x), \qquad x > 0, \qquad 0 < \alpha \le 1, \qquad \lambda \in \mathbb{R},$$

$$y(0+) = 1,$$

is also found explicitly and equals

$$y(x) = E_{\alpha,1}(\lambda x^{\alpha}), \tag{2.32}$$

via the Mittag-Leffler function

$$E_{\alpha,1}(\lambda x^{\alpha}) = \sum_{n=0}^{\infty} \frac{(\lambda x^{\alpha})^n}{\Gamma(\alpha n + 1)}.$$

Now it is bounded at zero so the classical Cauchy problem is correct.

As a conclusion we may formulate that different problems need different fractional derivatives to be involved; some of them need Riemann–Liouville fractional derivatives, some Gerasimov–Caputo fractional derivatives, and some other types of fractional operators.

2.2.4 Dzrbashian–Nersesyan fractional operators and sequential order fractional operators

Dzrbashian–Nersesyan fractional derivatives, associated with a sequence

$$\{\gamma_0, \gamma_1, \ldots, \gamma_m\}$$

of order σ, where $\sigma = \gamma_0 + \gamma_1 + ... + \gamma_m$, are defined by

$$D_{DN}^{\sigma} = D^{\gamma_0} D^{\gamma_1} \cdots D^{\gamma_m}, \tag{2.33}$$

where D^{γ_k} are fractional integrals and derivatives of Riemann–Liouville with some endpoint. These operators were introduced in [108–110] and then studied and applied in [102,105,111,112]. The original definitions demand $-1 \le \gamma_0 \le 0$, $0 \le \gamma_k \le 1$, $1 \le k \le m$, as in the above papers integro-differential equations under such conditions were studied for operators (2.33). But Dzrbashian–Nersesyan fractional operators may be defined and considered for any parameter γ_k if appropriate definitions of Riemann–Liouville operators are used. Riemann–Liouville, Gerasimov, and Gerasimov–Caputo fractional operators are special cases of Dzrbashian–Nersesyan fractional operators defined in the above-mentioned generalized sense.

Operators of Gerasimov, Gerasimov–Caputo, and Dzrbashian–Nersesyan were patterns for introducing in the book of Miller and Ross [381] more general *sequential operators of fractional integro-differentiation* for which compositions in definitions of the form (2.33) consist of *any fractional operators* (cf. a useful discussion in [446]).

2.3 Some more fractional order integro-differential operators

2.3.1 The Erdélyi–Kober operators

Definition 19. *Let $\alpha > 0$. The Erdélyi–Kober operators are defined by the following formulas:*

$$I^{\alpha}_{0+;\,2,\,y} f = \frac{2}{\Gamma(\alpha)} x^{-2(\alpha+y)} \int\limits_0^x (x^2 - t^2)^{\alpha-1} t^{2y+1} f(t)\, dt, \tag{2.34}$$

$$I^{\alpha}_{-;\,2,\,y} f = \frac{2}{\Gamma(\alpha)} x^{2y} \int\limits_x^{\infty} (t^2 - x^2)^{\alpha-1} t^{2(1-\alpha-y)-1} f(t)\, dt. \tag{2.35}$$

For $\alpha > -n$, $n \in \mathbb{N}$ they are defined by

$$I^{\alpha}_{0+;\,2,y} f = x^{-2(\alpha+y)} \left(\frac{d}{dx^2}\right)^n x^{2(\alpha+y+n)} I^{\alpha+n}_{0+;\,2,\,y} f, \tag{2.36}$$

$$I^{\alpha}_{-;\,2,y} f = x^{2y} \left(-\frac{d}{dx^2}\right)^n x^{2(\alpha-y)} I^{\alpha+n}_{-;\,2,\,y-n} f. \tag{2.37}$$

Let us denote that in the classical Russian version of the monograph [494] cases of integral limits 0 and ∞ were not considered. In the English version of this book [494] these limits are considered, but definitions are given with inaccuracies, and in particular cases they lead to complex values in integrals.

The Erdélyi–Kober operators are essential and important in transmutation theory because the most well-known transmutations of Sonine and Poisson are of this class; they are discussed in monographs [234,537], where a more historically exact term is used for them: Sonine–Poisson–Delsarte transmutations. Important properties of Erdélyi–Kober operators were studied in the monograph [234] and in papers of Yu. Luchko (cf. [340] and more references therein).

2.3.2 Fractional integrals and fractional derivatives of a function with respect to another function

Definition 20. *Let $\operatorname{Re}\alpha > 0$. The left- and right-sided fractional integrals of a function f with respect to another function g are*

$$I^{\alpha}_{0+,g} f = \frac{1}{\Gamma(\alpha)} \int\limits_0^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) dt, \tag{2.38}$$

$$I^{\alpha}_{-,g} f = \frac{1}{\Gamma(\alpha)} \int\limits_{x}^{\infty} (g(t) - g(x))^{\alpha-1} g'(t) f(t) dt. \tag{2.39}$$

Moreover, Riemann–Liouville fractional integrals on a semiaxis (2.26) and (2.25) are obtained by choosing $g(x) = x$ in (2.38) and (2.39), respectively. If we take in (2.38) and (2.39) the function $g(x) = x^2$ we obtain Erdélyi–Kober operators (2.34) and (2.35); if we take $g(x) = \ln x$ in (2.38) and (2.39) we get Hadamard fractional integrals, and the choice $g(x) = \exp(-x)$ with its applications was considered in [107].

As A. M. Dzrbashian pointed out, operators of fractional derivatives of a function with respect to another function (2.38) even in some more general setting were introduced and studied by his father M. M. Dzrbashian (cf. [99–101,103,104] and the monograph [494]). In these papers integral representations of this operator class, their inversion, and corresponding integro-differential equations of fractional order were studied.

2.3.3 Averaged or distributed order fractional operators

Further generalizations of fractional integro-differential operators are connected with combinations and compositions of more standard fractional operators defined above.

The *averaged or distributed order fractional operator, associated with any given fractional operator R^t*, is introduced by the following formula:

$$I^{(a,b)}_{MR} f = \int\limits_{a}^{b} R^t f(t) dt, \tag{2.40}$$

where R^t is a given fractional operator of order t of any kind. In the case R^t is in particular the fractional Riemann–Liouville operator, the names *continued* or *distributed* fractional integrals or derivatives are often used. Such operators were studied by A. Pskhu and his disciples [459,460].

Note that one of the authors proposed the following modification of an averaged expression in (2.40) to be more convenient and also similar to an integral mean value:

$$\overline{I}^{(a,b)}_{MR} f = \frac{1}{b-a} \int\limits_{a}^{b} R^t f(t) dt.$$

This variant of modified definition interested and was approved by A. Nakhushev, but it is still not used unfortunately.

2.3.4 Saigo, Love, and other fractional operators with special function kernels

Saigo fractional integrals (see [487] and [468]) are

$$J_x^{\gamma,\beta,\eta} f(x) = \frac{1}{\Gamma(\gamma)} \int\limits_x^{\infty} (t-x)^{\gamma-1} t^{-\gamma-\beta} {}_2F_1\left(\gamma+\beta, -\eta; \gamma; 1-\frac{x}{t}\right) f(t)dt \tag{2.41}$$

and

$$I_x^{\gamma,\beta,\eta} f(x) = \frac{x^{-\gamma-\beta}}{\Gamma(\gamma)} \int\limits_0^{x} (x-t)^{\gamma-1} {}_2F_1\left(\gamma+\beta, -\eta; \gamma; 1-\frac{t}{x}\right) f(t)dt, \tag{2.42}$$

where $\gamma > 0$, β, θ are real numbers.

Another similar class of generalizations introduced by Love and more generalizations with special function kernels are mentioned in [494]. Also important generalizations of classical fractional operators were studied, namely, Buschman–Erdélyi operators and fractional Bessel operators, which we consider in detail in this book.

2.4 Integral transforms and basic differential equations of fractional order

One of the most popular methods to obtain explicit solutions to fractional differential equations is the integral transforms method. Usually Laplace, Mellin, and Fourier transforms are used.

2.4.1 Integral transforms of fractional integrals and derivatives

2.4.1.1 Laplace transform of Riemann–Liouville fractional integrals and derivatives on semiaxes

Theorem 22. *(see [241], p. 84) Let $\alpha > 0$ and $f \in L_1(0,b)$ for any $b > 0$. Also let the estimate*

$$|f(x)| \le Ae^{p_0 x}, \qquad x > b > 0,$$

hold for some constants $A > 0$ and $p_0 > 0$.

(1) If $f \in L_1(0,b)$ for any $b > 0$, then the relation

$$(LI_{0+}^{\alpha} f)(s) = s^{-\alpha}(Lf)(p)$$

is valid for $\operatorname{Re} s > p_0$.

(2) If $n = [\alpha] + 1$, $g \in AC^n[0, b]$ *for any* $b > 0$, *the estimate*

$$|g(x)| \le Be^{q_0 x}, \qquad x > b > 0,$$

holds for constants $B > 0$ *and* $q_0 > 0$, *and* $g^{(k)}(0) = 0$, $k = 0, 1, \ldots, n-1$, *then the relation*

$$(LD^{\alpha}_{0+} g)(s) = s^{\alpha}(Lg)(s)$$

is valid for $\operatorname{Re} s > q_0$.

Remark 5. *(see [241], p. 84) If* $\alpha > 0$, $n = [\alpha] + 1$, $g \in AC^n[0, b]$ *for any* $b > 0$, *the condition* $|g(x)| \le Be^{q_0 x}$, $x > b > 0$, *holds for constants* $B > 0$ *and* $q_0 > 0$, *and there exist the finite limits*

$$\lim_{x \to +0} [D^k I^{n-\alpha}_{0+}]$$

and

$$\lim_{x \to \infty} [D^k I^{n-\alpha}_{0+}] = 0, \qquad D = \frac{d}{dx}, \qquad k = 0, 1, \ldots, n-1,$$

then

$$(LD^{\alpha}_{0+} g)(s) = s^{\alpha}(Lg)(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^k (I^{n-\alpha}_{0+} g)(+0), \qquad \operatorname{Re} s > q_0. \tag{2.43}$$

In particular, when $0 < \alpha < 1$ *and* $g \in AC[0, b]$ *for any* $b > 0$, *then*

$$(LD^{\alpha}_{0+} g)(s) = s^{\alpha}(Lg)(s) - (I^{1-\alpha}_{0+} g)(+0).$$

2.4.1.2 Mellin transform of Riemann–Liouville fractional integrals and derivatives on semiaxes

The Mellin transform of the Riemann–Liouville fractional integrals I^{α}_{0+} and I^{α}_{-} and fractional derivatives D^{α}_{0+} and D^{α}_{-} are given by the following statements.

We need here the space $X^p_c(a, b)$, $c \in \mathbb{R}$, $1 \le p \le \infty$, consisting of those complex valued Lebesgue measurable functions f on (a, b) for which $||f||_{X^p_c} < \infty$, with

$$||f||_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

and

$$||f||_{X^{\infty}_c} = \operatorname*{ess\,sup}_{a \le x \le b} [x^c |f(x)|].$$

Theorem 23. *Let $\alpha > 0$, $s \in \mathbb{C}$, and $f \in X^1_{s+\alpha}(0,\infty)$.*
(1) If $\operatorname{Re} s > 0$, *then*

$$(MI^\alpha_- f)(s) = \frac{\Gamma(s)}{\Gamma(s+\alpha)}(Mf)(s+\alpha).$$

(2) If $\operatorname{Re} s < 1-\alpha$, *then*

$$(MI^\alpha_{0+} f)(s) = \frac{\Gamma(1-\alpha-s)}{\Gamma(1-s)}(Mf)(s+\alpha).$$

Theorem 24. *Let $\alpha > 0$, $n = [\alpha]+1$, $s \in \mathbb{C}$, and $g \in X^1_{s-\alpha}(0,\infty)$.*
(1) If $\operatorname{Re} s > 0$ *and the conditions*

$$\lim_{x\to+0}[x^{s-k-1}(I^{n-\alpha}_- g)(x)] = 0, \qquad k = 0,1,...,n-1,$$

and

$$\lim_{x\to\infty}[x^{s-k-1}(I^{n-\alpha}_- g)(x)] = 0, \qquad k = 0,1,...,n-1,$$

hold, then

$$(MD^\alpha_- g)(s) = \frac{\Gamma(s)}{\Gamma(s-\alpha)}(Mf)(s-\alpha).$$

(2) If $\operatorname{Re} s < 1+\alpha$ *and the conditions*

$$\lim_{x\to+0}[x^{s-k-1}(I^{n-\alpha}_{0+} g)(x)] = 0, \qquad k = 0,1,...,n-1,$$

and

$$\lim_{x\to\infty}[x^{s-k-1}(I^{n-\alpha}_{0+} g)(x)] = 0, \qquad k = 0,1,...,n-1,$$

hold, then

$$(MD^\alpha_{0+} g)(s) = \frac{\Gamma(1+\alpha-s)}{\Gamma(1-s)}(Mf)(s-\alpha).$$

Remark 6. *Let $\alpha > 0$, $n = [\alpha]+1$, $s \in \mathbb{C}$, and $g \in X^1_{s-\alpha}(0,\infty)$.*
(1) If $\operatorname{Re} s > 0$, *then*

$$(MD^\alpha_- g)(s) = \frac{\Gamma(s)}{\Gamma(s-\alpha)}(Mg)(s-\alpha) + \sum_{k=0}^{n-1}\frac{(-1)^{n-k}\Gamma(s)}{\Gamma(s-k)}[x^{s-k-1}(I^{n-\alpha}_- g)(x)]_0^\infty.$$

(2) If $\operatorname{Re} s < 1 + \alpha$*, then*

$$(MD^{\alpha}_{0+}g)(s) = \frac{\Gamma(1+\alpha-s)}{\Gamma(1-s)}(Mg)(s-\alpha) + \sum_{k=0}^{n-1} \frac{\Gamma(1+k-s)}{\Gamma(1-s)}[x^{s-k-1}(I^{n-\alpha}_{0+}g)(x)]^{\infty}_{0}.$$

In particular, when $0 < \alpha < 1$*, then*

$$(MD^{\alpha}_{-}g)(s) = \frac{\Gamma(s)}{\Gamma(s-\alpha)}(Mg)(s-\alpha) + [x^{s-1}(I^{1-\alpha}_{-}g)(x)]^{\infty}_{0}$$

and

$$(MD^{\alpha}_{0+}g)(s) = \frac{\Gamma(1+\alpha-s)}{\Gamma(1-s)}(Mg)(s-\alpha) + [x^{s-1}(I^{1-\alpha}_{0+}g)(x)]^{\infty}_{0}.$$

2.4.1.3 Laplace transform of Gerasimov–Caputo fractional derivatives on semiaxes

Theorem 25. *Let* $\alpha > 0$*,* $n - 1 < \alpha \leq n$*,* $n \in \mathbb{N}$*, such that* $y(x) \in C^n(0, \infty)$*,* $g^{(n)}(x) \in L_1(0, b)$ *for any* $b > 0$ *and the estimate*

$$|g(x)| \leq Be^{q_0 x}, \qquad b > 0, \qquad q_0 > 0,$$

is valid. Let the Laplace transforms Ly *and* $LD^n y$ *exist and*

$$\lim_{x \to +\infty} (D^k g)(x) = 0, \qquad k = 0, 1, ..., n-1.$$

Then

$$(\mathcal{L}\, {}^{GC}D^{\alpha}_{0+}g)(s) = s^{\alpha}(\mathcal{L}y)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}(D^k g)(0). \tag{2.44}$$

In particular, for $0 < \alpha \leq 1$*,*

$$(\mathcal{L}\, {}^{GC}D^{\alpha}_{0+}g)(s) = s^{\alpha}(\mathcal{L}y)(s) - s^{\alpha-1}g(0). \tag{2.45}$$

2.4.2 Laplace transform method for the homogeneous equations with constant coefficients with the left-sided Riemann–Liouville fractional derivatives of the order α on a semiaxis $(0, \infty)$

We first present a result from [241] which demonstrates how to use Laplace transform for solving the one-dimensional fractional nonhomogeneous differential equation with

constant coefficients of the form

$$\sum_{k=1}^{m} A_k (D_{0+}^{\alpha_k} y)(x) + A_0 y(x) = f(x), \qquad x>0, \qquad m \in \mathbb{N}, \tag{2.46}$$
$$0<\alpha_1<...<\alpha_m, \qquad A_0, A_1, ..., A_m \in \mathbb{R},$$

with the left-sided Riemann–Liouville fractional derivatives of the order α on a semi-axis $(0,\infty)$ (2.28).

Let $y_1(x), ..., y_l(x)$, $l-1<\alpha_m \le l$, $l \in \mathbb{N}$, are linearly independent solutions of (2.46) such that

$$(D_{0+}^{\alpha_m - k} y_k)(0+) = 1, \qquad k=1,...,l,$$
$$(D_{0+}^{\alpha_m - k} y_j)(0+) = 0, \qquad k,j=1,...,l; \qquad k \neq j,$$

where

$$(D_{0+}^{\beta} g)(0+) = \lim_{x\to+0} (D_{0+}^{\beta} g)(x).$$

Such system $y_1(x), ..., y_l(x)$ is the **fundamental system of solutions** of (2.46).

Following [241] first we consider Eq. (2.46) when $m=1$:

$$(D_{0+}^{\alpha} y)(x) - \lambda y(x) = 0, \qquad x>0, \qquad \lambda \in \mathbb{R},$$

with conditions

$$(D_{0+}^{\alpha - j} y)(0+) = d_j, \qquad j=1,...,l, \qquad j=1,...,l, \qquad l \in \mathbb{N}, \qquad d_j \in \mathbb{R}.$$

Applying the Laplace transform (1.54) to $(D_{0+}^{\alpha} y)(x) - \lambda y(x) = 0$, taking into account (2.43), we obtain

$$s^{\alpha} \mathcal{L}[y](s) - \sum_{j=1}^{l} d_j s^{j-1} = \lambda \mathcal{L}[y](s)$$

or

$$\mathcal{L}[y](s) = \sum_{j=1}^{l} d_j \frac{s^{j-1}}{s^{\alpha} - \lambda},$$

and (1.55) gives

$$y(x) = \sum_{j=1}^{l} d_j x^{\alpha - j} E_{\alpha,\alpha+1-j}(\lambda x^{\alpha}).$$

It is easily verified that the functions $y_j(x) = x^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda x^\alpha)$ are solutions to the equation $(D^\alpha_{0+} y)(x) - \lambda y(x) = 0$:

$$(D^\alpha_0[t^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda t^\alpha)])(x) = \lambda x^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda x^\alpha), \qquad j = 1, ..., l,$$

and by (1.39)

$$(D^{\alpha-k}_{0+} y_j)(x) = \sum_{n=0}^{\infty} \frac{\lambda^n x^{\alpha n+k-j}}{\Gamma(\alpha n + k + 1 - j)}.$$

Then

$$(D^{\alpha-k}_{0+} y_k)(0+) = 1, \qquad k = 1, ..., l,$$

and

$$(D^{\alpha-k}_{0+} y_j)(0+) = 0, \qquad k, j = 1, ..., l, \qquad k \neq j.$$

Thus the following result is valid.

Theorem 26. *[241] Let $l - 1 < \alpha \leq l$, $l \in \mathbb{N}$, $\lambda \in \mathbb{R}$. Then the functions*

$$y_j(x) = x^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda x^\alpha), \qquad j = 1, ..., l,$$

yield the fundamental system of solutions to the equation

$$(D^\alpha_{0+} y)(x) - \lambda y(x) = 0, \qquad x > 0,$$

and the solution to this equation satisfying conditions

$$(D^{\alpha-j}_{0+} y)(0+) = d_j, \qquad j = 1, ..., l, \qquad l \in \mathbb{N}, \qquad d_j \in \mathbb{R}$$

is

$$y(x) = \sum_{j=1}^{l} d_j x^{\alpha-j} E_{\alpha,\alpha+1-j}(\lambda x^\alpha).$$

Example 1. *Let us consider the case when $2 < \alpha \leq 3$. Then $l = 3$ and the solution to the problem*

$$\begin{aligned}
&(D^\alpha_{0+} y)(x) - \lambda y(x) = 0, \qquad x > 0,\\
&(D^{\alpha-1}_{0+} y)(0+) = d_1, \quad (D^{\alpha-2}_{0+} y)(0+) = d_2, \quad (D^{\alpha-3}_{0+} y)(0+) = d_2,\\
&d_1, d_2, d_3 \in \mathbb{R},
\end{aligned}$$

is

$$y(x) = d_1 x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha) + d_2 x^{\alpha-2} E_{\alpha,\alpha-1}(\lambda x^\alpha) + d_3 x^{\alpha-3} E_{\alpha,\alpha-2}(\lambda x^\alpha).$$

In [17] it was shown that if $f(x) \in L(a,b)$, $b_k \in \mathbb{C}$ the Cauchy type problem for the linear differential equation

$$(D^\alpha_{a+}y)(x) - \lambda y(x) = f(x),\ (D^{\alpha-k}_{a+}y)(a+) = b_k,\ k = 1, ..., n,\ n = [\mathrm{Re}\,\alpha] + 1,$$

has the unique solution $y(x)$ in some subspace of $L(a,b)$ given by

$$y(x) = \sum_{k=1}^{n} b_k x^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda(x-a)^\alpha) + \int_a^x (x-t)^{\alpha-1} E_{\alpha,\alpha}(\lambda(x-t)^\alpha) f(t)dt, \tag{2.47}$$

where $E_{\alpha,\alpha}$ is the Mittag-Leffler function defined by (1.39). In particular for $0 < \mathrm{Re}\,\alpha < 1$ and $f(x) = 0$ the function

$$y(x) = b_1 x^{\alpha-1} E_{\alpha,\alpha}(\lambda(x-a)^\alpha)$$

is the solution to

$$(D^\alpha_{a+}y)(x) = \lambda y(x), \qquad (I^{1-\alpha}_{a+}y)(a+) = b_1, \qquad b_1 \in \mathbb{C}.$$

2.4.3 Laplace transform method for homogeneous equations with constant coefficients with the left-sided Gerasimov–Caputo fractional derivatives of the order α on a semiaxis $[0, \infty)$

In [241], p. 312, the Laplace transform method was applied to derive explicit solutions to homogeneous equations of the form

$$({}^{GC}D^\alpha_{0+}f)(x) = \lambda f(x), \qquad x > 0, \qquad l-1 < \alpha \le l, \qquad l \in \mathbb{N}, \qquad \lambda \in \mathbb{R}, \tag{2.48}$$

where ${}^{GC}D^\alpha_{0+}f$ is the left-sided Gerasimov–Caputo fractional derivative on semi-axes (2.30). Gerasimov in [179] derived and solved fractional order partial differential equations with the derivative (2.30) for mechanical applied problems in 1948.

The conditions

$$f^k(0+) = d_k, \qquad k = 0, 1, ..., l-1, \qquad d_k \in \mathbb{R} \tag{2.49}$$

were added to Eq. (2.48). The solution to the problem (2.48)–(2.49) is (see [241], p. 312)

$$f(x) = \sum_{k=0}^{l-1} d_k\, x^k\, E_{\alpha,k+1}(\lambda x^\alpha), \tag{2.50}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function (1.39).

2.4.4 Mellin integral transform and nonhomogeneous linear differential equations of fractional order

Nonhomogeneous linear differential equations of fractional order with given functions $A_k(x)$, $k = 0, 1, ..., m$, and $f(x)$ have the form

$$A_0 y(x) + \sum_{k=1}^{m} A_k(x)(D^{\alpha_k} y)(x) = f(x). \tag{2.51}$$

Differential fractional order operators in (2.51) can have various forms.

For example we can take in (2.51) the right- and left-sided Riemann–Liouville fractional derivatives on semiaxes given by (2.27) and (2.28) and constant coefficients $A_k, B_k \in \mathbb{R}$, $k = 0, ..., m$:

$$\sum_{k=0}^{m} B_k (D_{-}^{\alpha+k} y)(x) = f(x), \qquad x > 0, \qquad \alpha > 0, \tag{2.52}$$

and

$$\sum_{k=0}^{m} A_k (D_{0+}^{\alpha+k} y)(x) = f(x), \qquad x > 0, \qquad \alpha > 0. \tag{2.53}$$

The Mellin transform method for solving Eqs. (2.52) and (2.53) is based on the relations following from Theorem 24:

$$(\mathcal{M} x^{\alpha+k} D_{-}^{\alpha+k} y)(s) = \frac{\Gamma(s+\alpha+k)}{\Gamma(s)} (\mathcal{M} y)(s)$$

and

$$(\mathcal{M} x^{\alpha+k} D_{0+}^{\alpha+k} y)(s) = \frac{\Gamma(1-s)}{\Gamma(1-s-\alpha-k)} (\mathcal{M} y)(s).$$

Applying the Mellin transform to (2.52) and (2.53) we obtain

$$\left[\sum_{k=0}^{m} B_k \frac{\Gamma(s+\alpha+k)}{\Gamma(s)} \right] (\mathcal{M} y)(s) = (\mathcal{M} f)(s)$$

and

$$\left[\sum_{k=0}^{m} A_k \frac{\Gamma(1-s)}{\Gamma(1-s-\alpha-k)} \right] (\mathcal{M} y)(s) = (\mathcal{M} f)(s),$$

respectively. Using formula (1.64), the solution to (2.52) is

$$y(x) = \int_0^{\infty} G_{\alpha}^{1} \left(\frac{x}{t} \right) f(t) \frac{dt}{t},$$

where

$$G^1_\alpha(x) = \left(\mathcal{M}^{-1}\left[\frac{1}{P^1_\alpha(s)}\right]\right)(x), \qquad P^1_\alpha(s) = \sum_{k=0}^{m} B_k \frac{\Gamma(s+\alpha+k)}{\Gamma(s)}.$$

Using the same formula (1.64), the solution to (2.53) is

$$y(x) = \int_0^\infty G^2_\alpha(t)\, f(xt)dt,$$

where

$$G^2_\alpha(x) = \left(\mathcal{M}^{-1}\left[\frac{1}{P^2_\alpha(1-s)}\right]\right)(x), \qquad P^2_\alpha(s) = \sum_{k=0}^{m} A_k \frac{\Gamma(s)}{\Gamma(s-\alpha-k)}.$$

Essentials of transmutations

3

3.1 Definition of the transmutation operator, some examples of classical transmutations

3.1.1 Introduction to transmutation theory

Following [229], we give a definition of the transmutation operator.

Definition 21. *Let us have two operators* (A, B). *The nonzero operator* T *is called the transmutation operator if the following relation is satisfied:*

$$T A = B T. \tag{3.1}$$

The relation (3.1) is also called *intertwining property*, as they say that the transmutation operator T *intertwines* the operators A and B or is an *intertwining operator*. To transform (3.1) into a strict definition, it is necessary to specify spaces or sets of functions on which the operators A, B, and, therefore, T act.

The method of solving problems based on the use of the operator T with the property (3.1) is called the **method of transmutation operators**.

It is obvious that the notion of transmutation is a direct and far reaching generalization of the matrix similarity from linear algebra. But the transmutations do not reduce to similar operators because intertwining operators often are not bounded in classical spaces and the inverse operator may not exist or not be bounded in the same space. As a consequence, spectra of intertwining operators are not the same as a rule. Moreover, transmutations may be unbounded. This is the case for the Darboux transformations, which are defined for a pair of differential operators and are differential operators themselves; in this case all three operators are unbounded in classical spaces. But the theory of Darboux transformations is included in transmutation theory too. A pair of intertwining operators may not be differential ones. In transmutation theory there are problems for the following various types of operators: integral, integro-differential, difference-differential (e.g., the Dunkl operator), differential or integro-differential of infinite order (e.g., in connection with Schur's lemma), general linear operators in functional spaces, and pseudodifferential and abstract differential operators.

All classical integral transforms due to Definition 1 are also special cases of transmutations, including the Fourier, Petzval (Laplace), Mellin, Hankel, Weierstrass, Kontorovich–Lebedev, Meyer, Stankovic, Obrechkoff, finite Grinberg, and other transforms.

In quantum physics, the study of the Shrödinger equation, and inverse scattering theory, the underlying transmutations are called wave operators.

The commuting operators are also a special class of transmutations. The most important class consists of operators commuting with derivatives. In this case transmutations as commutants are usually in the form of formal series, pseudodifferential,

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or infinite order differential operators. Finding commutants is directly connected with finding all transmutations in the given functional space. For these problems works a theory of operator convolutions, including the Berg–Dimovski convolutions [89]. Also, more and more applications are developed in connection with transmutation theory for commuting differential operators; such problems are based on classical results of J. L. Burchnall and T. W. Chaundy. The transmutations are also connected with factorization problems for integral and differential operators. Special class of transmutations are the so-called Dirichlet-to-Neumann and Neumann-to-Dirichlet operators, which link together solutions of the same equation but with different kinds of boundary conditions.

And how do the transmutations usually work? Suppose we study properties for a rather complicated operator A. But suppose also that we know the corresponding properties for a more simple model operator B and transmutation (3.1) readily exists. Then we usually may copy results for the model operator B to corresponding ones for the more complicated operator A. This is shortly the main idea of transmutations.

Let us consider for example an equation $Au = f$. Then applying to it a transmutation with property (3.1) we consider a new equation $Bv = g$, with $v = Tu$, $g = Tf$. So if we can solve the simpler equation $Bv = g$, then the initial one is also solved and has solution $u = T^{-1}v$. Of course, it is supposed that the inverse operator exists and its explicit form is known. This is a simple application of the transmutation technique for finding and proving formulas for solutions of ordinary and partial differential equations.

The monographs [22,51–53,139,571] are completely devoted to transmutation theory and its applications (note also surveys [234,532] and [377]). Moreover, essential parts of monographs [56,89,242,252,259,277,316,321,330,373,376,497,580], etc., include material on transmutations; the complete list of books which investigate some transmutational problems is now near 100 items.

We use the term “transmutation” due to [53]: “Such operators are often called transformation operators by the Russian school (Levitan, Naimark, Marchenko, etc.), but transformation seems a too broad term, and, since some of the machinery seems ‘magical’ at times, we have followed Lions and Delsarte in using the word *transmutation*.”

Now transmutation theory is a completely formed part of the mathematical world in which methods and ideas from different areas are used: differential and integral equations, functional analysis, function theory, complex analysis, special functions, and fractional integro-differentiation.

The transmutation theory is deeply connected with many applications in different fields of mathematics. The transmutation operators are applied in inverse problems via the generalized Fourier transform, spectral functions, and the famous Levitan equation; in scattering theory the Marchenko equation is formulated in terms of transmutations; in spectral theory transmutations help to prove trace formulas and asymptotics for spectral functions; estimates for transmutational kernels control stability in inverse and scattering problems; and for nonlinear equations via the Lax method transmutations for Sturm–Liouville problems lead to proving existence and explicit formulas for soliton solutions. Special kinds of transmutations are the generalized analytic functions, generalized translations and convolutions, and Darboux transformations. In the

theory of partial differential equations the transmutations work for proving explicit correspondence formulas among solutions of perturbed and nonperturbed equations, for singular and degenerate equations, pseudodifferential operators, problems with essential singularities at inner or corner points, and estimates of solution decay for elliptic and ultraelliptic equations. In function theory transmutations are applied to embedding theorems and generalizations of Hardy operators, Paley–Wiener theory, and generalizations of harmonic analysis based on generalized translations. Methods of transmutations are used in many applied problems: investigation of Jost solutions in scattering theory, inverse problems, Dirac and other matrix systems of differential equations, integral equations with special function kernels, probability theory and random processes, stochastic random equations, linear stochastic estimation, inverse problems of geophysics, and transsound gas dynamics. The number of applications of transmutations to nonlinear equations is constantly increasing.

In fact, the modern transmutation theory originated from two basic examples (see [532]). The first is the transmutation T for Sturm–Liouville problems with some potential $q(x)$ and natural boundary conditions

$$T(D^2 y(x) + q(x)y(x)) = D^2 (Ty(x)), D^2 y(x) = y''(x). \tag{3.2}$$

The second example is a problem of studying transmutations intertwining the Bessel operator B_ν and the second derivative:

$$TB_\nu f = D^2Tf, \quad B_\nu = D^2 + \frac{2\nu+1}{x}D, \quad D^2 = \frac{d^2}{dx^2}, \quad \nu \in \mathbb{C}. \tag{3.3}$$

This class of transmutations includes the Sonine–Poisson–Delsarte, Buschman–Erdélyi operators and their generalizations. Such transmutations found many applications for a special class of partial differential equations with singular coefficients. A typical equation of this class is the B-elliptic equation with the Bessel operator in some variables of the form

$$\sum_{k=1}^{n} B_{\nu,x_k} u(x_1,\dots,x_n) = f. \tag{3.4}$$

Analogously, B-hyperbolic and B-parabolic equations are considered; this terminology was proposed by I. Kipriyanov. This class of equations was first studied by Euler, Poisson, and Darboux and continued in Weinstein's theory of generalized axially symmetric potential (GASPT). These problems were further investigated by Zhitomirslii, Kudryavtsev, Lizorkin, Matiychuk, Mikhailov, Olevskii, Smirnov, Tersenov, He Kan Cher, Yanushauskas, Egorov, and others.

In the most detailed and complete way, equations with Bessel operators were studied by the Voronezh mathematician Kipriyanov and his disciples Ivanov, Ryzhkov, Katrakhov, Arhipov, Baidakov, Bogachov, Brodskii, Vinogradova, Zaitsev, Zasorin, Kagan, Katrakhova, Kipriyanova, Kononenko, Kluchantsev, Kulikov, Larin, Leizin, Lyakhov, Muravnik, Polovinkin, Sazonov, Sitnik, Shatskii, and Yaroslavtseva. The

essence of Kipriyanov's school results was published in [242]. For classes of equations with Bessel operators, Kipriyanov introduced special functional spaces which were named after him [243]. In this field interesting results were investigated by Katrakhov and his disciples; now these problems are considered by Gadjiev, Guliev, Glushak, and Lyakhov with their coauthors and students. Abstract equations of the form (3.4) originating from the monograph [56] were considered by Egorov, Repnikov, Kononenko, Glushak, Shmulevich, and others. Transmutations are one of the basic tools for equations with Bessel operators, and they are applied to the construction of solutions, fundamental solutions, the study of singularities, and new boundary value and other problems.

We must note that the term "operator" is used in this chapter for brevity in the broad and sometimes not exact meaning, so appropriate domains and function classes are not always specified. It is easy to complete and make strict every special result.

3.1.2 Some examples of classical transmutations

Let give some examples of classical transmutations.

Example 1. *Transmutation operator intertwining the second order and first order derivatives.*

If $u(t,x)$ is a function, satisfying the abstract Cauchy problem

$$u_{tt} = Au, \qquad u(0,x) = f(x), \qquad u_t(0,x) = 0,$$

then

$$v(t,x) = \widetilde{L}u(t,x) = \frac{1}{\sqrt{\pi t}} \int\limits_0^\infty u(s,x) e^{-\frac{s^2}{4t}} \, ds \tag{3.5}$$

satisfies the abstract Cauchy problem

$$v_t = Av, \qquad v(0,x) = f(x).$$

This example was given in lecture notes of R. Hersh [165] among the five examples of transmutations. R. Hersh mentioned that the transmutation operator $\widetilde{L}$ has been rediscovered repeatedly in [14,37,205,476,576].

Let consider how we can use (3.5) for obtaining the solution to the Cauchy problem for the diffusion equation if we know the solution to the Cauchy problem for the wave equation. We have

$$u_{tt} = u_{xx},$$
$$u(0,x) - f(x), \qquad u_t(0,x) = 0.$$

Then

$$u = \frac{f(x+t) + f(x-t)}{2}$$

and the solution to

$$v_t = v_{xx}, \qquad v(0,x) = f(x)$$

is

$$v(t,x) = \frac{1}{\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} f(s+x) e^{-\frac{s^2}{4t}} ds.$$

Example 2. *Let us consider the Poisson operator (see [317])*

$$\mathcal{P}_x^\gamma f(x) = \frac{2C(\gamma)}{x^{\gamma-1}} \int\limits_0^x \left(x^2 - t^2\right)^{\frac{\gamma}{2}-1} f(t)\, dt, \qquad C(\gamma) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)}.$$

For $f \in C^2_{ev}$ the Poisson operator acts as a transmutation operator by the formula

$$\mathcal{P}_x^\gamma D^2 f = B_\gamma \mathcal{P}_x^\gamma f, \qquad D^2 = \frac{d^2}{dx^2}, \qquad B_\gamma = \frac{d^2}{dx^2} + \frac{\gamma}{x}\frac{d}{dx}.$$

The Poisson operator will be considered in detail in Section 3.4.1.

Using the previous example we obtain that for the problem

$$w_t = w_{rr} + \frac{2}{r} w_r$$
$$w(0,r) = f(r), \qquad 0 \le r < \infty,$$

the bounded solution has the form

$$w(t,r) = 2C(2) \frac{1}{\sqrt{\pi t}} \frac{1}{r} \int\limits_0^r \int\limits_{-\infty}^{\infty} f(s+p) e^{-\frac{s^2}{4t}} ds dp.$$

Example 3. *The Radon transform (see Definition 14) intertwines a partial derivative with a univariate derivative:*

$$\mathcal{R}\left(\frac{\partial}{\partial x_i} f(x)\right) = \omega_i \frac{\partial}{\partial x_i} \mathcal{R} f(\omega, s).$$

Let Δ denote the Laplacian on $\mathbb{R}^n$:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

and let L denote the "radial" second derivative:

$$L = \frac{\partial^2}{\partial s^2}.$$

The Radon transform and its dual are intertwining operators for Δ *and* L *in the sense that*

$$\mathcal{R}(\Delta f) = L(\mathcal{R} f), \quad \mathcal{R}^*(Lg) = \Delta(\mathcal{R}^* g).$$

Example 4. *In [355] the transmutation operator*

$$(T_l g)(r) = \int_{-1}^{1} g(rt) C_l^{\frac{n-2}{2}}(t)(1-t^2)^{\frac{n-3}{2}} dt$$

was presented. For operators D_1 *and* D_2 *acting by formulas*

$$(D_1 g)(t) = (1-t^2) g''(t) + \alpha t g'(t) + \beta g(t)$$

and

$$(D_2 g)(t) = (1-t^2) g''(t) + \left(\alpha t + \frac{n-1}{t}\right) g'(t) + \left(\beta - \frac{l(l+n-2)}{t^2}\right) g(t),$$

the intertwining relation

$$D_2 T_l = T_l D_1$$

holds.

3.2 Transmutations for Sturm–Liouville operator

3.2.1 Description of the problem and terminology

The main problem of this subsection is the construction of different transmutation operators intertwining the simplest Sturm–Liouville operator

$$y''(x) + \lambda^2 y(x) = (L_0 y)(x) \tag{3.6}$$

with the Sturm–Liouville operator of the general kind

$$y''(x) + q(x) y(x) + \lambda^2 y(x) = (Ly)(x). \tag{3.7}$$

Function $q(x)$ in (3.7) is called a potential function. Appropriate functions can be complex, $\lambda \in C$, $x \in \mathbb{R}$. We are looking for a transformation operator satisfying the identity

$$SLf = L_0 S f \tag{3.8}$$

on suitable functions $f(x)$. A natural requirement for the transmutation operator S is linearity. It is very convenient if there is the invert to the S operator in some space.

The linearity requirement of S after substitution of (3.6) and (3.7) in (3.8) leads to a relation that is independent of λ:

$$S\left(D^2+q(x)\right) f = D^2 S f, \tag{3.9}$$

where $D = d/dx$. Invertibility of S naturally leads to the search of it as an integral operator

$$(Sf)(x) = \int\limits_{a(x)}^{b(x)} K(x,t) f(t)\, dt. \tag{3.10}$$

Here the kernel $K(x,t)$ in the general case can be distribution (for example, $K(x,t) = \delta(x-t) + G(x,t)$, G is a smooth function). In (3.10) $a(x), b(x)$ are some functions $\mathbb{R} \to \mathbb{R}$.

There are two different approaches to constructing transmutations. In the first, transmutations are built on arbitrary functions, possibly with some growth restrictions at various points. We will detail this particular approach but it is not generally accepted. Usually, the second approach is taken, which is based only on solutions of the equations $L_0 y = 0$, $Ly = 0$ with the operators (3.6) and (3.7). Such a method was adopted in pioneering classical works on transmutation operators. We will present it briefly.

Essentially both of these approaches are equivalent. Operators constructed on arbitrary functions are also defined on solutions with suitable boundary conditions. On the other hand, an operator built on eigenfunctions for any λ can be extended to fairly wide classes of functions.

Let us make some comments. We will usually call operators differential or integral expressions. When constructing transmutations it is assumed that functions $f(x)$ belong to some class Φ. In calculations, it is assumed that the kernels $K(x,t)$ have a certain smoothness in both variables. Exact definitions are given in each case.

Now we consider transmutation operators S of the forms

$$(Sf)(x) = \int\limits_a^b K(x,t) f(t)\, dt, \tag{3.11}$$

$$(Sf)(x) = f(x) + \int\limits_{-x}^{x} K(x,t) f(t)\, dt, \tag{3.12}$$

$$(Sf)(x) = f(x) + \int\limits_{c}^{x} K(x,t) f(t)\, dt, \tag{3.13}$$

$$(Sf)(x) = f(x) + \int\limits_{x}^{d} K(x,t) f(t)\, dt. \tag{3.14}$$

Here $K(x,t)$ is smooth in both variables, numbers a, b, and c belong to the extended numerical axis $\overline{\mathbb{R}}$. Fredholm operators (3.11) are the simplest. Volterra operators (3.12)–(3.14) are easily invertible in standard spaces. The freedom to choose the limits of integration allows in each case to choose those transmutations that are best suited for a concrete problem. In particular (3.12) preserves the asymptotic behavior of the function being converted and its derivatives for $x \to 0$ and the transmutation of the form (3.14) for $x \to d$. For example, transmutation (3.14) with $d = +\infty$ preserves the asymptotic behavior at infinity; this is a transmutation of the B. Ya. Levin type. Such operator is used to solve inverse problems of the quantum theory of scattering and problems of estimating the rate of decrease at infinity of solutions of differential equations, including partial differential equations. Operator (3.13) for $c = 0$ is called transmutation of the A. Povzner type (see [454]).

It is also possible to consider transmutations of other types, not like (3.11)–(3.14), for example, Fredholm and Volterra operators of the third kind. We show that in the case of a smooth potential $q(x)$ transmutations (3.11)–(3.14) are constructed most simply and "naturally." The construction of the transmutations of other types is difficult.

3.2.2 Transmutations in the form of the second kind Fredholm operators

Let consider here the construction of transmutation operators as the second kind Fredholm operators

$$(Sf)(x) = \int_a^b K(x,t) f(t)\, dt. \tag{3.15}$$

If such operator exists, then equality (3.9) is true. This equality in certain cases has the form

$$f''(x) + q(x)f(x) + \int_a^b K(x,t)[f''(t) + q(t)f(t)]\, dt$$
$$= f''(x) + \int_a^b \frac{\partial^2 K}{\partial x^2} f(t)\, dt.$$

Integrating by parts we obtain

$$q(x)f(x) + K(x,t)f'(t)\big|_a^b - \frac{\partial K}{\partial t} f(t)\Big|_a^b$$
$$= \int_a^b \left[\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - q(t)K\right] f(t)\, dt. \tag{3.16}$$

Therefore at least for finite functions $f \in C_0^\infty(a,b)$, the equality

$$\int_a^b G(x,t) f(t)\, dt = q(x) f(x) \tag{3.17}$$

must be fulfilled. In (3.17)

$$G(x,t) = \frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - q(t)K.$$

If we require that $G \in C\left([a,b]^2\right)$, then it is easy to see that equality (3.17) is impossible in the general case, which follows from a comparison of the spectra of the operators on the left and right sides of (3.17) (for example, in $L_2(a,b)$). Therefore, in the general case, it is impossible to construct transmutation of the form (3.15) satisfying the identity (3.9) with a smooth kernel.

Of course, the identities (3.16)–(3.17) can hold in the case of $q(x) \equiv 0$. In this case, the item weaves $D^2 \to D^2$, that is, commutes with the second derivative. Such operators exist. For this, it is enough, for example, that the core satisfies the wave equation

$$\frac{\partial^2 K}{\partial x^2} = \frac{\partial^2 K}{\partial t^2}$$

and boundary conditions

$$H(x) = K(x,t) f'(t)\Big|_a^b - \frac{\partial K}{\partial t} f(t)\Bigg|_a^b = 0. \tag{3.18}$$

Here after choosing the class Φ values $f(x)$, $f'(x)$ at $x=a$, $x=b$ are fixed. Usually, condition (3.18) gives

$$\Phi = \left\{ f(x) \in C^2(a,b) \mid f(a) = f(b) = f'(a) = f'(b) = 0 \right\}.$$

Remark 7. *(1) The kernel $K(x,t)$ independent of $f(x)$ does not exist.*

(2) When $q(x) = \mu = const$, the spectrum $Af = q \cdot f = muf(x)$ consists of one point.

The above reasoning justifies the impossibility of constructing a transmutation operator of the form (3.15) for sufficiently smooth potentials $q(x)$. If $q(x)$ has singularities for $x \in [a,b]$, then the comparison of the spectra on which we were based does not work. In this case, the operator $Af = q \cdot f$ is not defined in standard spaces. Therefore, the question of constructing a transmutation operator for potentials $q(x)$ with singularities remains open.

Therefore, it is interesting to note that for the potentials $q(x) \sim 1/x^2$ for $x \to 0$ there is an interesting class of transmutation operators of the form (3.15). These are

Kram–Krein transmutations. They are defined only on solutions of the equations $Ly = 0$ with regular and singular potentials and relate such solutions to each other. Note that the condition (3.18) of the form

$$W_t(K, f) = \begin{vmatrix} K(x,t) & f(t) \\ \frac{\partial K}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}$$

at points $x = a$ and $x = b$ arises naturally in the theory of Kram–Krein operators. It is these conditions that allow us to write the Kram–Krein transmutations in the integral form (3.15) (see [4]). In the general case, these operators are differential [376].

Now let us consider the problem of constructing a transmutation operator in the form (3.11):

$$(Sf)(x) = \int_a^b K(x,t) f(t)\, dt.$$

Substitution of this expression in (3.9) gives

$$\int_a^b K(x,t)[f''(t) + q(t)f(t)]\, dt = \frac{d^2}{dx^2} \int_a^b K(x,t) f(t)\, dt.$$

We assume that integration by parts and under the sign of the integral is possible. This leads to the equality

$$\int_a^b \left[\frac{\partial^2 K}{\partial x^2} - \frac{\partial^2 K}{\partial t^2} - q(t)K \right] f(t)]\, dt = K(x,t) f'(t)\Big|_a^b - \frac{\partial K}{\partial t} f(t)\Big|_a^b .$$

Let $f(t) \in \Phi$ and let Φ be dense in $L_2(a, b)$. Then it is enough to demand

$$\frac{\partial^2 K}{\partial x^2} = \frac{\partial^2 K}{\partial t^2} + q(t)K, \tag{3.19}$$

$$K(x,t) f'(t)\Big|_a^b - \frac{\partial K}{\partial t} f(t)\Big|_a^b = W_t(K, f)(x) = W(x) = 0. \tag{3.20}$$

Here we denote by $W_t(K, f)$ the Wronsky determinant

$$W_t(K, f) = \begin{vmatrix} K(x,t) & f(t) \\ \frac{\partial K(x,t)}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}. \tag{3.21}$$

Next, we consider solutions to Eq. (3.21). It is known that for the existence of a classical C^2-solution it is necessary to require $q(t) \in C^1(a, b)$. In this case, the Riemann function of Eq. (3.19) $R \in C^2$. If we restrict ourselves to the condition $q(t) \in C(a, b)$, then there is only a generalized solution to (3.19) from the class C^1 (in this

case, the Riemann function $R \in C^1$). Therefore, until the end of this subsection we will consider the potential $q \in C^1$ in the domain of our definition.

That gives us the next theorem.

Theorem 27. *Let $a, b, c, d \in \overline{R}$ and $K(x,t) \in C^2([c,d],[a,b])$. Class Φ is the set of functions $f(t)$ such that*

$$(1) \quad f(t) \in C^2(a,b),$$

$$(2) \quad W_t(K,f) = K(x,t) f'(t)\Big|_a^b - \frac{\partial K}{\partial t} f(t)\Big|_a^b = 0. \tag{3.22}$$

Then for existence of the transmutation operator of the form (3.11) for $f \in \Phi$ it is enough that the kernel $K(x,t)$ satisfies the hyperbolic Eq. (3.19).

Note that practically, except in very special cases, the condition (3.22) forces us to accept

$$\Phi = \left\{ f \in C^2(a,b) \mid f(a) = f(b) = f'(a) = f'(b) = 0 \right\}.$$

It follows from (3.22) that at least always Φ contains functions that are compactly supported on (a,b). Therefore, the assumption made about the density Φ in $L_2(a,b)$ is always fulfilled.

The authors did not find in the literature constructions of the type (3.11). Nevertheless, we will show further that in practice this kind exists. For some simple potentials, they will be constructed explicitly.

3.2.3 Transmutations in the form of the second kind Volterra operators

We proceed to the construction of one of the most important classes of transmutation operators. These are Volterra operators of the second kind, having the form (3.13)

$$(Sf)(x) = f(x) + \int_c^x K(x,t) f(t)\, dt.$$

Such operators in the case $c = 0$ were first obtained by Povzner (see [325,454]). The case $(Sf)(x) = f(x) + \int_x^\infty K(x,t) f(t)\, dt$ was introduced by Levin (see [313,314]). We will consider them in the form

$$(Sf)(x) = f(x) + \int_c^x K(x,t) f(t)\, dt,$$

where $c \in \overline{\mathbb{R}}$, which allows us to combine these cases and unify the text.

Substitution in formula (3.9) leads to the relation

$$f''(x)+q(x)f(x)+\int\limits_c^x K(x,t)[f''(t)+q(t)f(t)]\,dt$$
$$=f''(x)+\frac{d^2}{dx^2}\int\limits_c^x K(x,t)f(t)\,dt.$$

Transforming the first part of this formula, we obtain

$$\int\limits_c^x \left(\frac{\partial^2 K}{\partial t^2}+q(t)K\right)f(t)\,dt+f''(x)+q(x)f(x)+\left.K(x,t)f'(t)\right|_c^x$$
$$-\left.\frac{\partial K}{\partial t}f(t)\right|_c^x.$$

Transforming the remaining part,

$$\frac{d^2}{dx^2}\int\limits_c^x K(x,t)f(t)\,dt$$
$$=\int\limits_c^x \frac{\partial^2 K}{\partial x^2}f(t)\,dt+\frac{\partial K(x,x)}{\partial x}f(x)+K'(x,x)f(x)+K(x,x)f'(x).$$

Equating the corresponding terms, we obtain

$$\frac{\partial^2 K}{\partial t^2}+q(t)K=\frac{\partial^2}{\partial x},\tag{3.23}$$
$$\frac{d}{dx}K(x,x)+\lim_{t\to x}\left(\frac{\partial K}{\partial x}+\frac{\partial K}{\partial t}\right)=q(x),\quad (x,t)\in\Omega\subset\mathbb{R}^2,\tag{3.24}$$
$$\lim_{t\to c}W_t\,(f,K(x,t))=\lim_{t\to c}W(x)=0.\tag{3.25}$$

By Ω we denote the domain of the function $K(x,t)$, whose closure contains part of the diagonal $t=x$.

Eq. (3.23) is standard in the problem we are considering. The condition (3.25) highlights the point c. This condition reduces to weighted boundary conditions on the function $f(x)$ and its first derivative $f'(x)$ for $x\to c$. The relation (3.25) dictates the choice of the class Φ. Now we show that the condition (3.24) can be simplified if we assume that $K\in C^1$. For this, we prove two technical lemmas.

Lemma 10. *Let $K(x,t) \in C^1(\Omega)$, $\overline{\Omega} \cap \{(x,t) \mid x = t\} \neq \emptyset$. Then for $(t,x) \in \Omega$ the equality*

$$\frac{d}{dx} K(x,x) = \lim_{t \to x} \left(\frac{\partial K}{\partial x}(x,t) + \frac{\partial K}{\partial t}(x,t) \right) \tag{3.26}$$

is valid.

Proof. Consider the function of two variables $K(x,y)$, the variables themselves depend on the parameter t: $x = \phi(t)$, $y = \psi(t)$. Then using the formula

$$\frac{d}{dt} K(x,y) = \frac{\partial K}{\partial x}\frac{dx}{dt} + \frac{\partial K}{\partial y}\frac{dy}{dt} \tag{3.27}$$

from [143] and putting $x = y = t$ in (3.27) we obtain

$$\frac{d}{dt} K(t,t) = \left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial y} \right)\Bigg|_{x=y=t}.$$

This formula is equivalent to (3.26). □

The second lemma is proved similarly.

Lemma 11. *Under the conditions of Lemma 10 we have*

$$\frac{d}{dx} K(x,-x) = \lim_{t \to -x} \left(\frac{\partial K}{\partial x} - \frac{\partial K}{\partial t} \right). \tag{3.28}$$

So, by Lemma 10 the relation (3.24) takes the form

$$\frac{d}{dx} K(x,x) = \frac{1}{2} q(x). \tag{3.29}$$

Equality (3.29) shows that the value of the core $K(x,t)$ on the diagonal $t = x$ allows one to reconstruct the potential $q(x)$. This fact is fundamental in the theory of inverse problems. Therefore, the most common methods for solving inverse problems come down to finding the kernel of the transformation operator by the spectral function (as in the Gelfand–Levitan equation) or by scattering data (as in the Marchenko equation).

Assume that the core $K \in C^2(\Omega)$, where

$$\Omega = \{(x,t) \mid x \in [a,b], t \text{ is between } c \text{ and } x\}.$$

Define $\Phi \in C^2(a,b)$ as a set of functions f, satisfying the condition (3.25).

Let $c \in \overline{\mathbb{R}}$ and let kernel K and class Φ satisfy the conditions stated above. Then for existing transmutation operators of the form (3.14) for functions $f \in \Phi$ it is sufficient that the kernel K satisfies Eq. (3.23) and condition (3.29).

3.2.4 Transmutations in the form of the first kind Volterra operators

Let us consider the possibility of construction of transmutations in the form of the first kind Volterra operators, i.e., in the form

$$(Sf)(x) = \int_c^x K(x,t) f(t)\, dt. \tag{3.30}$$

Practically repeating the calculations of the previous subsections, Theorem 27 is valid and relations (3.23) and (3.25) are the same but instead of (3.24) we get

$$\frac{d}{dx} K(x,x) + \lim_{t \to x} \left(\frac{\partial K}{\partial t} + \frac{\partial K}{\partial x} \right) = 0. \tag{3.31}$$

Of course, for kernels smooth up to the diagonal $t = x$ this condition is replaced by

$$\frac{d}{dx} K(x,x) = 0, \qquad K(x,x) = \text{const}. \tag{3.32}$$

However, we prefer a more accurate notation (3.31), since the kernel may have a gap at $t \to x$. This is permissible.

An example of a transmutation operator of the form (3.30) is the fractional integral

$$(Sf)(x) = \left(I^{\alpha}_{c+}\right)(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t)\, dt, \qquad \alpha > 0.$$

This operator intertwines D^2 and D^2, i.e., commutes with D^2. Condition (3.31) obviously is valid for the kernel

$$K(x,t) = \frac{1}{\Gamma(\alpha)} (x-t)^{\alpha-1}.$$

Note that for $0 < \alpha < 1$, it is precisely (3.31) that is valid, not (3.32). For $\alpha > 1$, both of these relations hold. The condition (3.25) reduces to $f(c) = f'(c) = 0$. These conditions are rougher than the minimum sufficient. The latter are given, for example, in [494].

Instead of the introduced Riemann–Liouville fractional integration operators, we can consider the Weil operators or fractional integrals over an arbitrary function (2.38) and (2.39).

Let us consider an important case of transmutations of the form (3.14). Using integration by parts, we calculate the left side in (3.9):

$$f''(x)+q(x)f(x)+\int_{-x}^{x} K(f''(t)+q(t)f(t))\,dt$$
$$=\int_{-x}^{x}\left(\frac{\partial^2 K}{\partial t^2}+q\right) f\,dt+f''(x)+q(x)f(x)+\left. K(x,t)f'(t)\right|_{-x}^{x}-\left.\frac{\partial K}{\partial t}f(t)\right|_{-x}^{x}.$$

Calculating the right side in (3.9) we obtain

$$f''(x)+\frac{d^2}{dx^2}\int_{-x}^{x} K(x,t)f(t)\,dt$$
$$=f''(x)+\int_{-x}^{x}\frac{\partial^2 K}{\partial x^2}f\,dt+\left.\frac{\partial K}{\partial x}f(t)\right|_{-x}^{x}+\left(K(x,t)f(t)|_{-x}^{x}\right)'$$
$$=f''(x)+\int_{-x}^{x}\frac{\partial^2 K}{\partial x^2}f\,dt+\lim_{t\to x}\frac{\partial K}{\partial x}f(t)+\lim_{t\to -x}\frac{\partial K}{\partial x}f(t)$$
$$+K'(x,x)f(x)+K(x,x)f'(x)+K'(x,-x)f(-x)+K(x,-x)[f(-x)]'.$$

Therefore

$$\int_{-x}^{x}\left(\frac{\partial^2 K}{\partial t^2}+q(t)K\right) f\,dt+q(x)f(x)+K(x,x)f'(x)-\lim_{t\to -x}K(x,t)f'(t)$$
$$-\frac{\partial K}{\partial t}(x,x)f(x)+\lim_{t\to -x}\frac{\partial K}{\partial t}f(-x)$$
$$=\int_{-x}^{x}\frac{d^2 K}{dx^2}f\,dt+\lim_{t\to x}\frac{\partial K}{\partial x}f(x)+K'(x,x)f(x)$$
$$+K(x,x)f'(x)+\lim_{t\to -x}\frac{\partial K}{\partial x}f(t)+\frac{d}{dx}(K(x,-x))\,f(-x)$$
$$+K(x,-x)\frac{d}{dx}\lim_{t\to -x}f(t).$$

In order for the calculations to make sense, we assume that $f(x)\in\Phi\subset C^2(a,-a)$ for some $a>0$. Then it is enough that again the kernel satisfies the equation

$$\frac{\partial^2 K}{\partial x^2}=\frac{\partial^2 K}{\partial t^2}+q(t)K \tag{3.33}$$

and the additional condition

$$\left(K(x,-x)\frac{d}{dx}\lim_{t\to -x} f(t) + K(x,-x)\lim_{t\to -x} f'(t)\right)$$
$$+ f(x)\left[\frac{d}{dx}K(x,-x) + \lim_{t\to x}\left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial t}\right) - q(x)\right]$$
$$- f(-x)\left[\lim_{t\to x}\left(\frac{\partial K}{\partial t} - \frac{\partial K}{\partial x}\right) - \frac{d}{dx}K(x,-x)\right] = 0.$$

In the last expression, the term in parentheses is zero. Therefore, the equality

$$f(x)\left[\frac{d}{dx}K(x,-x) + \lim_{t\to x}\left(\frac{\partial K}{\partial x} + \frac{\partial K}{\partial t}\right) - q(x)\right]$$
$$= f(-x)\left[\lim_{t\to x}\left(\frac{\partial K}{\partial t} - \frac{\partial K}{\partial x}\right) - \frac{d}{dx}K(x,-x)\right] \tag{3.34}$$

is true in the case of the smooth down to the line $t = x$ kernel $K(x,t)$. Expression (3.34) according to (3.31) can be simplified to

$$f(x)\left[2K'(x,x) - q(x)\right] + f(-x)\left[2K'(x,-x)\right] = 0. \tag{3.35}$$

A simple analysis of the relation (3.35) shows that if $f(x)$ and $f(-x)$ are independent, then both equalities should be satisfied in the case of smooth down to the line

$$\begin{cases} K'(x,x) = \frac{1}{2}q(x), \\ K'(x,-x) = 0. \end{cases} \tag{3.36}$$

For even functions we get the equality

$$K'(x,x) + K'(x,-x) = \frac{1}{2}q(x), \tag{3.37}$$

and for odd functions we get

$$K'(x,x) - K'(x,-x) = \frac{1}{2}q(x). \tag{3.38}$$

Let the kernel $K(x,t) \in C^2$ in the considered domain for some $a > 0$. We define the class of functions $\Phi = C^2(-a,a)$ (without any boundary conditions!). Then, in order for the operator (3.14) to be transmutation operator on functions $f \in \Phi$, it is enough that the kernel satisfies Eq. (3.33) and the system (3.36). If the class Φ consists of even functions on $(-a,a)$, then (3.36) is replaced by (3.37). If the class Φ consists of odd functions on $(-a,a)$, then (3.36) is replaced by (3.38).

It is important to note that transmutations of the form (3.14) considered in this subsection essentially do not require any conditions on the function (of course, except

for smoothness). Apparently, it is precisely these operators that are the most general and, in a certain sense, "natural."

Now let us show that every transmutation of the form (3.14) generates transmutation (3.15) with $c = 0$. Moreover, the relations for kernels (3.33)–(3.36) give relations (3.23)–(3.25) for new kernels. In this case, the potential $q(x)$ will be assumed to be even.

So let the transmutation of the form (3.15) be defined on the set of even functions Φ. Then for $x > 0$

$$
\begin{aligned}
(Sf)(x) &= f(x) + \int\limits_{-x}^{x} K(x,t) f(t)\,dt = f(x) + \int\limits_{0}^{x} [K(x,t) + K(x,-t)] f(t)\,dt \\
&= f(x) + \int\limits_{0}^{x} G(x,t) f(t)\,dt.
\end{aligned} \tag{3.39}
$$

We obtained transmutation of the form (3.15) for $c = 0$. Let us show that the new kernel $G(x,t)$ satisfies the conditions of Theorem 27. It is clear that the smoothness conditions of G are fulfilled (we assume that they were satisfied for the original kernel K). Let us verify that for $x \in [0,a]$ and the functions $f(x) \in \Phi$, the class Φ is defined by equality (3.25). Actually,

$$
W_t(f, G) = f(t)\left[\frac{\partial}{\partial t} K(x,t) - \frac{\partial}{\partial t} K(x,-t)\right] - (K(x,t) + K(x,-t))\,[f'(t)].
$$

For $t \to 0$ both expressions in square brackets tend to zero (since $f(t)$ is even we have $f'(0) = 0$). Therefore condition (3.25) is satisfied. Condition (3.23) follows from (3.33) and (3.24) or in our case (3.29) follows from (3.37).

Thus, by Theorem 27, the operator (3.39) is a transmutation operator on functions from the class Φ defined on $[0,a]$ and admitting smooth even extension.

The question whether it is possible to construct a transmutation operator of the form (3.14) with $c = 0$ using an operator of the form (3.15) is more complicated. It is connected to the question whether it is possible to smoothly extend the solutions of the hyperbolic Eq. (3.23) from the segment $[0,a]$ to the segment $[-a,a]$. Obviously, an operator of the form (3.39) can be determined by the operator (3.14) and on functions that admit an odd continuation.

Finally, we consider the question of necessary and sufficient conditions of existing transmutation operators of the form (3.14) with the kernel $K \in C^2(\Omega)$.

Theorem 28. *We assume that the kernel $K(x,t)$ in (3.14) belongs to the class $C^2(\Omega)$ and $f \in \Phi$. Then in order that the transmutation operator of the form (3.14) exists it is necessary and sufficient that the relations (3.23) and (3.29) are satisfied.*

Proof. Sufficiency was proved in Theorem 27.

We turn our attention to necessity. Let the transmutation operator (3.14) with a kernel $K \in C^2$ exist on Φ. Let us show that (3.23) and (3.29) hold.

Calculations in (3.9) show that since $f \in \Phi$, we have

$$\int_c^x \left(\frac{\partial^2 K}{\partial t^2} + q(t)K - \frac{\partial^2 K}{\partial x^2} \right) f(t)\,dt + \left[q(x) - 2K'(x,x) \right] f(x) = 0$$

or

$$(Af)(x) = \int_c^x \left(\frac{\partial^2 K}{\partial t^2} + q(t)K - \frac{\partial^2 K}{\partial x^2} \right) f(t)\,dt = (Bf)(x) = g(x) f(x),$$

$$g(x) = 2K'(x,x) - q(x) \in C^1(a,b).$$

Both operators A and B act from $L_2(a,b)$ to $L_2(a,b)$, where (a,b) is an arbitrary segment on which $q(x)$, $f(x)$, and $K(x,t)$ are defined. We compare their spectra. Operator A is the Volterra operator. Operator A has only one point in the spectrum $\lambda = 0$, $\lambda \in \mathbb{C}$. The operator B has a continuous spectrum consisting of the set of values of the function $g(x)$ defined on the closed segment $[a,b]$. If $g(x)$ is defined on the open interval (a,b), then the spectrum coincides with the closure of its set of values. Therefore, the set of values of the function $g(x)$ consists of a single point, that is, $g(x) \equiv 0$. Therefore, we have (3.29) and the equality

$$\int_c^x \left(\frac{\partial^2 K}{\partial t^2} + q(t)K - \frac{\partial^2 K}{\partial x^2} \right) f(t)\,dt = 0.$$

The kernel of this operator is from $C(a,b) \Rightarrow$. Therefore, $K \in L_2(a,b)$, since $\operatorname{mes}(a,b) < \infty$. Function $f \in \Phi$, and Φ is dense in $L_2(a,b)$; therefore the kernel is equal to zero and (3.23) is true. □

For cases of nonsmooth potentials $q(x)$ or kernels $K(x,t)$ with singularities, for example, for $x = t$, this proof is not correct.

In essence, they come down to a superposition of the operator of multiplication by the function $p(x)$ and ordinary transmutations. Such a reduction is difficult if $p(x) = 0$ for some admissible x.

In conclusion of this section, we show that linear operators in the general case admit a natural construction for linear differential operators.

Theorem 29. *There is no nonzero linear operator S satisfying the identity*

$$S(D^2 y + y^2) = D^2 S y \tag{3.40}$$

for arbitrary $f \in C^2[a,b]$.

Proof. Let (3.40) be valid. Then we choose an arbitrary function $y(x)$ and in addition to the relation (3.40) for $y(x)$ we can write this relation for $f(x)=y(x)+1$:

$$S(D^2 y + y^2 + 2y + 1) = D^2 S(y+1).$$

Since S is linear we get

$$S(D^2y+y^2)(x)+2S(y)(x)+S(1)(x)=D^2S(y)(x)+D^2S(1)(x).$$

Now using (3.40) we can write

$$S(y)(x)=\frac{1}{2}\left(D^2S(1)(x)-S(1)(x)\right).$$

So operator S does not depend on y. Again due to linearity

$$S(y)(x)=S(0)(x)=0. \qquad \square$$

This theorem holds for any class of the function $\Phi \in C^2$, which, together with each $f(x)$, also contains $f(x)+1$. This condition is not satisfied for the class Φ if, for example, we fix the boundary condition at zero $f(0)=0$.

The following theorem shows, even on a very narrow class of functions with maximally restrictive boundary conditions, that there is no linear transmutation operator $D^2+(\cdot)\to D^2$.

Theorem 30. *On a class of functions Φ*

$$\Phi=\left\{f(x)\in C^2(a,b)\mid (D^nf)(c)=0, \quad n\in\mathbb{N}, \quad c\in(a,b)\right\}$$

there is no linear transmutation operator satisfying the relation (3.40).

Proof. Note that if $f\in\Phi$, then $\lambda f\in\Phi$, $\forall\lambda$ and $f^2\in\Phi$. Let us write (3.40) for $y(x)=\lambda f(x)$, where $\lambda\in\mathbb{C}$ but $\lambda\neq 0$. We have

$$S(D^2\lambda f+\lambda^2f^2)=D^2S(\lambda f).$$

By linearity, we obtain

$$S(D^2f+\lambda f^2)=D^2S(f).$$

Subtract from this equality (3.40) for $y=f$. We have

$$(\lambda-1)S(f^2)(x)=0, \quad \forall f\in\Phi.$$

Therefore $S(y)=0$ for $y\geq 0$. Due to linearity we get $S\equiv 0 \quad \forall y$. $\square$

However, it should be noted that linear operators find the most important applications in the theory of nonlinear equations. For example, they are an important part of the methods of the inverse problem of integrating nonlinear evolution equations.

Of course, the above considerations are applicable to $S(D^2y+y^\alpha)=D^2Sy$.

The following theorem holds for the general case. The meaning of this interesting result is that the existence of a good operator

$$S: D^2+H(y)\to D^2 \tag{3.41}$$

necessarily leads to linearity of the function itself $H(y)$.

Theorem 31. *Let Φ be some linear, $\Phi \subset C^2(a,b)$, and $H(y) : \Phi \to \Phi$. Then if the linear transmutation operator (3.41) $S : \Phi \to \Phi$ exists, then operator $SH(y)$ is also a linear operator from Φ to Φ.*

Proof. It is obvious that $SH(y) : \Phi \to \Phi$. Let us calculate $SH(\lambda y)$. Since $\lambda y \in \Phi$, from (3.41) we get

$$S\left(D^2\lambda y + H(\lambda y)\right) = D^2 S(\lambda y),$$
$$SH(\lambda y) = \lambda(D^2 S(y) - SD^2 y) = \lambda SH(y).$$

Additivity is verified in the same way:

$$SH(y_1 + y_2) = D^2 S(y_1) - SD^2 y_1 + D^2 S(y_2) - SD^2 y_2 = SHy_1 + SHy_2. \quad \square$$

The theorem is proved similarly.

Theorem 32. *Suppose that the conditions of Theorem 31 are satisfied. Then if S is invertible in some space L containing Φ, then the operator $H(y)$ is linear in y.*

3.3 Transmutations for different potentials

In this section we construct transmutation operators intertwining operators of the Sturm–Liouville type from the previous section for different concrete potentials q.

3.3.1 Kernel of transmutation intertwining operators of the Sturm–Liouville type

Note that in all the cases considered by us, the equations for the kernels of the transmutations coincide (see (3.19), (3.23), and (3.33)). Therefore, we should use some of the methods for solving this hyperbolic equation.

The following action plan is usually implemented. At the first step, we pass from the partial differential equation to the integral one. These equations are not equivalent, but each solution of the integral equation satisfies the original hyperbolic. At this step, the existence of a certain kernel of the transmutation operator and its certain smoothness are proved. At the second step, some additional conditions for the kernel are checked and the appropriate class of functions Φ is selected. This completes the construction of the transmutation operator.

So, let us move on to solving the equation

$$\frac{\partial^2 K}{\partial x^2} = \frac{\partial^2 K}{\partial t^2} + q(t)K \tag{3.42}$$

with an additional condition on the diagonal $x = t$ (see (3.24), (3.29), and (3.36)),

$$K(x,x) = \frac{1}{2}q(x). \tag{3.43}$$

We perform the standard change of variables by the formulas

$$u = \frac{1}{2}(x+t), \quad v = \frac{1}{2}(x-t). \tag{3.44}$$

The diagonal equality $x = t$ in the new variables takes the form $v = 0$.

We introduce the notation for the kernel in new variables

$$H(u,v) = K(u+v, u-v) = K(x,t). \tag{3.45}$$

For function H we pass from the relations (3.42) and (3.43) to the new

$$\frac{\partial^2 H}{\partial u \partial v} = q(u-v)H, \tag{3.46}$$

$$H(u,0) = \frac{1}{2}\int_c^u q(s)\,ds. \tag{3.47}$$

Here function $q(u-v)$ should be defined, c is an arbitrary number, possible $c = \pm\infty$.

An important point to make follows here. It follows from (3.44) that both variants, $u > 0$, $v > 0$ and $u < 0$, $v < 0$, are possible.

The system (3.46)–(3.47) is a Cauchy problem with only one initial condition. So under our assumptions on q $(q(x) \in C^1)$ this system has infinitely many solutions. Therefore, for every potential $q(x)$ there are infinitely many transmutations, for example, of the form (3.14). This is extremely convenient in applications where it is possible to choose, with the same potential, different operations most suitable for each specific problem.

One of the ways to construct kernels satisfying (3.33) and (3.34) is using the formula

$$H(u,v) = \frac{1}{2}\int_c^u q(s)\,ds \int_d^u d\alpha \int_0^v q(\alpha-\beta)H(\alpha,\beta)\,d\beta. \tag{3.48}$$

We should check that each C^2-function of the form (3.48) satisfies (3.46) and (3.47). Arbitrary numbers c, d are from $\overline{R}$.

We note again that u, v, α, and β in (3.48) can have any sign.

Another way of solving (3.48) is using a Riemann function.

The importance of studying Eq. (3.48) with different c, d is that we can simultaneously study the case $c = d = 0$, which arises when constructing operators of the Povzner–Levitan type and the case $c = d = +\infty$, which arises when constructing Levin type operators, and the case of arbitrary different c, d. Usually, these types of operators were studied separately, and the case of arbitrary c $(c \neq 0, c \neq \pm\infty)$ was not considered.

The further content of this chapter essentially consists in studying Eq. (3.48) under various assumptions. The most important case for us will be $c = d$. In specific

examples, it is usually convenient to select one of the following values: $-\infty$, 0, or $+\infty$. When constructing transmutations of the form (3.15) it is usually also convenient (although not necessary) if the values of c in formulas (3.15), (3.47), and (3.48) coincide.

Next, we consider those simplest potentials $q(x)$ that admit the construction of transmutations explicitly. Here, various solutions of Eq. (3.48) will be denoted identically by $H(u, v)$. We will only be interested in deriving formulas for the nuclei themselves. From these formulas, their continuity and the existence of the desired number of continuous derivatives will automatically follow.

3.3.2 Cases when potential $q(x)$ is an exponential function

Consider the following problem. Find a solution to Eq. (3.48) under the conditions

$$q(x) = e^x, \quad c = d = -\infty.$$

Then Eq. (3.48) takes the form

$$H(u, v) = \frac{1}{2}e^u + \int_{-\infty}^{u} e^\alpha \, d\alpha \int_0^v e^{-\beta} H(\alpha, \beta) \, d\beta. \tag{3.49}$$

Using the method of successive approximations, we set

$$H_0(u, v) = \frac{1}{2}e^u,$$
$$H_{n+1}(u, v) = \int_{-\infty}^{u} e^\alpha \, d\alpha \int_0^v e^{-\beta} H_n(\alpha, \beta) \, d\beta.$$

We obtain the first interactions

$$H_1(u, v) = \int_{-\infty}^{u} e^\alpha \, d\alpha \int_0^v e^{-\beta} \frac{1}{2} e^\alpha \, d\beta = \frac{1}{2} \int_{-\infty}^{u} e^{2\alpha} \, d\alpha \int_0^v e^{-\beta} \, d\beta$$
$$= \frac{1}{2} \frac{e^{2u}}{2!} \int_0^v e^{-\beta} \, d\beta,$$
$$H_2(u, v) = \frac{1}{2} \frac{e^{3u}}{3!} \int_0^v e^{-\beta_2} \int_0^{\beta_2} e^{-\beta_1} \, d\beta_1 \, d\beta_2.$$

Using mathematical induction, it is easy to show that

$$H_n(u, v) = \frac{1}{2} \frac{e^{(n+1)u}}{(n+1)!} y_n(v), \tag{3.50}$$

where $y_n(v)$ is defined by

$$y_n(v) = \int_0^v e^{-\beta_n} \int_0^{\beta_n} e^{-\beta_{n-1}} \dots \int_0^{\beta_2} e^{-\beta_1}\, d\beta_1. \tag{3.51}$$

The formula

$$y_n(v) = \frac{\left(1 - e^{-v}\right)^n}{n!} \tag{3.52}$$

is valid.

Using mathematical induction we get

$$\begin{aligned}
y_0(v) &= 1,\\
y_{n+1}(v) &= \int_0^v e^{-\beta} y_n(\beta)\, d\beta = \int_0^v e^{-\beta} \frac{\left(1 - e^{-v}\right)^n}{n!}\, d\beta\\
&= \frac{1}{n!} \int_0^v \left(1 - e^{-\beta}\right)^n (-1)\, de^{-\beta} = \frac{(-1)}{n!} \int_1^{e^{-v}} (1-t)^n\, dt\\
&= \left\{ \begin{array}{l} e^{-\beta} = t, \\ -e^{-\beta}\, d\beta = dt \end{array} \right\} = \frac{1}{n!} \frac{(1-t)^{n+1}}{(n+1)} \bigg|_1^{e^{-v}} = \frac{\left(1 - e^{-v}\right)^{n+1}}{(n+1)!}.
\end{aligned}$$

Substitution of (3.52) in (3.50) gives

$$H_n(u, v) = \frac{1}{2} \frac{e^{(n+1)u}}{(n+1)!} \frac{\left(1 - e^{-v}\right)^n}{n!} = \frac{1}{2n!(n+1)!} e^u [q(u, v)]^n,$$

where

$$q(u, v) = q = e^u \left(1 - e^{-v}\right).$$

From here we obtain solutions of Eq. (3.49) in the form of the Neumann series:

$$H(u, v) = \sum_{n=0}^{\infty} H_n(u, v) = \frac{e^u}{2} \sum_{n=0}^{\infty} \frac{q^n}{n!(n+1)!}.$$

The identity

$$\sum_{n=0}^{\infty} \frac{q^n}{n!(n+1)!} = \frac{1}{\sqrt{q}} I_1\left(2\sqrt{q}\right) \tag{3.53}$$

is valid. Here $q \in \mathbb{C}$, $I(\cdot)$ is a modified Bessel function of the first kind (1.16).

By formula (1.16) we have

$$I_1(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+1}}{n!(n+1)!} = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(x^2/4)^n}{n!(n+1)!}.$$

Denoting $q = x^2/4$, $x = 2\sqrt{q}$ we can write

$$I_1\left(2\sqrt{q}\right) = \sqrt{q} \sum_{n=0}^{\infty} \frac{q^n}{n!(n+1)!}.$$

That gives (3.53).

Now from (3.53) we obtain a solution to (3.49) in the form

$$H(u, v) = \frac{e^u}{2} \frac{1}{\sqrt{q}} I_1\left(2\sqrt{q}\right) \tag{3.54}$$

$$= \frac{1}{2} \frac{e^u}{\sqrt{e^u\left(1 - e^{-v}\right)}} I_1\left(2\sqrt{e^u\left(1 - e^{-v}\right)}\right)$$

$$= \frac{1}{2} \sqrt{\frac{e^u}{1 - e^{-v}}}\, I_1\left(2\sqrt{e^u\left(1 - e^{-v}\right)}\right). \tag{3.55}$$

Now consider the same problem, but under the conditions

$$q(x) = e^x, \quad c = d = 0.$$

Since all arguments are similar, we will outline them briefly. We have

$$H_0(u, v) = \frac{1}{2} \int_0^u e^s\, ds,$$

$$H_{n+1}(u, v) = \int_0^u e^{\alpha}\, d\alpha \int_0^v e^{-\beta} H_n(\alpha, \beta)\, d\beta,$$

$$H_1(u, v) = \frac{1}{2} \int_0^u e^{\alpha_2}\, d\alpha_2 \int_0^{\alpha_2} e^{\alpha_1}\, d\alpha_1 \int_0^v e^{-\beta_1}\, d\beta_1.$$

So

$$H_n(u, v) = \frac{1}{2} z_n(u) y_n(v),$$

where $y_n(\cdot)$ is defined by (3.52). Let us find a formula for $z_n(u)$:

$$z_0(u) = e^u - 1,$$

$$z_1(u) = \int_0^u e^{\alpha}\left(e^{\alpha} - 1\right) d\alpha = \frac{(e^u - 1)^2}{2!},$$

$$z_{n+1}(u) = \int_0^u e^{\alpha} z_n(u) = \int_0^u e^{\alpha_{n+1}}\, d\alpha_{n+1} \int_0^{\alpha_{n+1}} \ldots \int_0^{\alpha_2} e^{\alpha_1}\, d\alpha_1. \tag{3.56}$$

We obtain the statement proved by induction:

$$z_n(u) = \frac{(e^u - 1)^{n+1}}{(n+1)!}. \tag{3.57}$$

Therefore,

$$\begin{aligned} H(u, v) &= \frac{1}{2}\left(e^u - 1\right)\frac{1}{\sqrt{q}} I_1\left(2\sqrt{q}\right) \\ &= \frac{1}{2}\sqrt{\frac{e^u - 1}{1 - e^{-v}}}\, I_1\left(2\sqrt{(e^u - 1)\left(1 - e^{-v}\right)}\right), \end{aligned} \tag{3.58}$$

where

$$q = \left(e^u - 1\right)\left(1 - e^{-v}\right).$$

Finally, we consider the general case, including both previous ones as particulars. We have

$$H(u, v) = \frac{1}{2}\int_c^u e^s\, ds + \int_c^u e^{\alpha}\, d\alpha \int_0^v e^{-\beta} H(\alpha, \beta)\, d\beta.$$

The first iterations have the forms

$$H_0(u, v) = \frac{1}{2}\int_c^x e^s\, ds,$$

$$H_1(u, v) = \frac{1}{2}\int_c^u e^{\alpha_2}\, d\alpha_2 \int_c^{\alpha_1} e^{\alpha_1}\, d\alpha_1 \int_0^v e^{-\beta}\, d\beta.$$

We obtain the statement proved by induction:

$$H_n(u, v) = \frac{1}{2} z_n(u) y_n(v),$$

$$y_n(v) = \frac{\left(1 - e^{-v}\right)^n}{n!} = \frac{\left[\int_0^v e^{-s}\, ds\right]^n}{n!},$$

$$z_n(u) = \int\limits_c^u e^{\alpha_{n+1}}\, d\alpha_{n+1} \int\limits_0^{\alpha_{n+1}} e^{\alpha_n}\, d\alpha_n \dots \int\limits_0^{\alpha_2} e^{\alpha_1}\, d\alpha_1 = \frac{\left[\int\limits_c^u e^s\, ds\right]^{n+1}}{(n+1)!}. \tag{3.59}$$

Therefore

$$\begin{aligned} H(u, v) &= \frac{1}{2} \left(\int\limits_c^u e^s\, ds \right) \frac{1}{\sqrt{q}} I_1\left(2\sqrt{q}\right) \\ &= \frac{1}{2}\sqrt{\frac{e^u - e^c}{1 - e^{-v}}}\, I_1\left(2\sqrt{(e^u - e^c)\left(1 - e^{-v}\right)}\right), \end{aligned} \tag{3.60}$$

where

$$q = \left(e^u - e^c\right)\left(1 - e^{-v}\right).$$

Special cases of formula (3.60) are (3.58) for $c = 0$ and (3.55) for $c = -\infty$.

Now we consider the problem of solving Eq. (3.48) (and, therefore, the construction of the transformation operator) for the case of the potential of the opposite sign

$$q(x) = -e^x, \quad c = d, \quad c < \infty.$$

The equation takes the form

$$H(u, v) = -\frac{1}{2} \int\limits_c^u e^s\, ds - \int\limits_c^u e^\alpha\, d\alpha \int\limits_0^v e^{-\beta} H(\alpha, \beta)\, d\beta.$$

The first iteration and general formulas are as follows:

$$H_0(u, v) = -\frac{1}{2} \int\limits_c^u e^s\, ds,$$

$$H_{n+1}(u, v) = -\int\limits_c^u e^\alpha\, d\alpha \int\limits_0^v e^{-\beta} H(\alpha, \beta)\, d\beta.$$

The assumption proved by induction is

$$H_n(u, v) = (-1)^{n+1} \frac{1}{2} z_n(u) y_n(v).$$

The kernel is

$$H(u, v) = \left(-\frac{1}{2}\right)\left(e^u - e^c\right) \sum_{n=0}^{\infty} \frac{(-q)^n}{n!(n+1)!}, \tag{3.61}$$

where

$$q = \int_c^u e^s\,ds \cdot \int_0^v e^{-s}\,ds = \left(e^u - e^c\right)\left(1 - e^{-v}\right).$$

For $q \in \mathbb{C}$ we have (see (1.13))

$$\sum_{n=0}^{\infty} \frac{(-q)^n}{n!(n+1)!} = \frac{1}{\sqrt{q}} J_1\left(2\sqrt{q}\right), \tag{3.62}$$

where $J_1(\cdot)$ is the Bessel function of the first kind.

From (1.13) we have

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^n}{n!\Gamma(\nu+n+1)},$$

and putting $\nu = 1$, $q = \frac{z^2}{4}$, $z = 2\sqrt{q}$ we get (3.62).

Formula (3.62) is also immediately obtained from (3.53) after replacing q by $-q$. Therefore, for the kernel in this case, we have the formula

$$H(u,v) = \left(-\frac{1}{2}\right)\sqrt{\frac{e^u - e^c}{1 - e^{-v}}}\, J_1\left(2\sqrt{\left(e^u - e^c\right)\left(1 - e^{-v}\right)}\right). \tag{3.63}$$

Let us solve Eq. (3.48) in the case

$$q(x) = e^{-x}, \quad c = d, \quad c \neq -\infty.$$

We have

$$H(u,v) = \frac{1}{2}\int_c^u e^{-s}\,ds + \int_c^u e^{-\alpha}\,d\alpha \int_0^v e^{\beta} H(\alpha,\beta)\,d\beta,$$

$$H_n(u,v) = \frac{1}{2} y_n(u) z_n(v),$$

$$y_n(u) = \frac{\left(e^{-c} - e^{-u}\right)^{n+1}}{(n+1)!},$$

$$z_n(v) = \frac{(e^v - 1)^n}{n!},$$

$$H(u,v) = \frac{1}{2}\left(e^{-c} - e^{-u}\right) \cdot \sum_{n=0}^{\infty} \frac{q^n}{n!(n+1)!} = \frac{1}{2}\frac{\left(e^{-c} - e^{-u}\right)}{\sqrt{q}} I_1\left(2\sqrt{q}\right),$$

where $q = \left(e^{-c} - e^{-u}\right)(e^{v} - 1)$ or

$$H(u,v) = \frac{1}{2}\sqrt{\frac{\left(e^{-c} - e^{-u}\right)}{(e^{v} - 1)}}\, I_1\left(2\sqrt{\left(e^{-c} - e^{-u}\right)(e^{v} - 1)}\right). \tag{3.64}$$

Similarly, in the case $q(x) = -e^{-x}$, $c \neq d = -\infty$, solution to (3.48) has the form

$$H(u,v) = \frac{1}{2}\sqrt{\frac{\left(e^{-c} - e^{-u}\right)}{(e^{v} - 1)}}\, J_1\left(2\sqrt{\left(e^{-c} - e^{-u}\right)(e^{v} - 1)}\right). \tag{3.65}$$

Applying the well-known formulas

$$\int_a^t dt_1 \int_a^{t_1} dt_2 \dots \int_a^{t_{n-1}} f(t_1) f(t_2) \dots f(t_n)\, dt_n = \frac{\left(\int_a^t f(s)\, ds\right)^n}{n!}, \tag{3.66}$$

$$\int_a^x dx_1 \int_a^{x_1} dx_2 \dots \int_a^{x_{n-1}} f(x_n)\, dx_n = \int_a^x f(t) \frac{(x-t)^{n-1}}{(n-1)!}\, dt, \tag{3.67}$$

we obtain (3.59), (3.57), and (3.52) without using mathematical induction. Usually formulas (3.66) and (3.67) are associated with the name of Dirichlet.

3.3.3 Cases when potential $q(x)$ is constant

Now we turn to the consideration of the important case when the potential is constant,

$$q(x) = \lambda^2 = \text{const}, \quad \lambda \in \mathbb{C}.$$

Solution to Eq. (3.48) in this case are kernels of transmutations intertwining $D^2 \pm \lambda^2$ and D^2. These operators are particular cases of more general operators intertwining the Bessel operator with a spectral parameter and the second derivative

$$S : B_\nu \pm \lambda^2 \to D^2, \quad B_\nu = \frac{d^2}{dx^2} + \frac{2\nu + 1}{x}\frac{d}{dx}.$$

Such operators appear in papers of A. Erdélyi [122–126], I. N. Vekua [581], and J. S. Lowndes [337–339]. Therefore, it is natural to call them Erdélyi–Vekua–Lowndes (EVL). Here we got the simplest EVL operators.

So, consider Eq. (3.48) for $q(x) = \lambda^2$, $c = d \in R$, i.e., c is a finite number:

$$H(u,v) = \frac{1}{2}\lambda^2(u-c) + \lambda^2 \int_c^u d\alpha \int_0^v H(\alpha,\beta)\, d\beta.$$

Iterations are determined by the formulas

$$H_0(u,v) = \frac{1}{2}\lambda^2(u-c),$$
$$H_{n+1}(u,v) = \lambda^2 \int\limits_c^u d\alpha \int\limits_0^v H_n(\alpha,\beta)\,d\beta.$$

From the first iterations we get

$$H_1(u,v) = \frac{1}{2}\left(\lambda^2\right)^2 \frac{(u-c)^2}{2!} v,$$
$$H_2(u,v) = \frac{1}{2}\left(\lambda^2\right)^3 \frac{(u-c)^3}{3!}\frac{v^2}{2!},$$

and

$$H(u,v) = \frac{1}{2}\frac{\left(\lambda^2\right)^{n+1}}{n!(n+1)!}(u-c)^{n+1}v^n. \tag{3.68}$$

Summing up the Neumann series using formula (3.53) we obtain

$$H(u,v) = \frac{\lambda^2}{2}(u-c)\frac{1}{\sqrt{\lambda^2(u-c)v}} I_1\left(2\sqrt{\lambda^2(u-c)v}\right). \tag{3.69}$$

In particular, for real $\lambda > 0$ formula (3.69) for the kernel can be transformed to

$$H(u,v) = \frac{\lambda}{2}\sqrt{\frac{u-c}{v}}\, I_1\left(2\lambda\sqrt{(u-c)v}\right). \tag{3.70}$$

It has the simplest form when $c = 0$:

$$H(u,v) = \frac{\lambda}{2}\sqrt{\frac{u}{v}}\, I_1\left(2\lambda\sqrt{uv}\right). \tag{3.71}$$

Similar results hold for a negative constant potential. We have

$$q(x) = -\lambda^2, \quad \lambda \in \mathbb{C}.$$

If $\lambda \in \mathbb{C}$, $c \in R$, then

$$H(u,v) = \left(-\frac{1}{2}\right)\frac{\lambda^2}{\sqrt{\lambda^2}}\sqrt{\frac{u-c}{v}}\, J_1\left(2\sqrt{\lambda^2(u-c)v}\right). \tag{3.72}$$

If $\lambda \in R_+$ $(\lambda > 0)$, $c \in R$, then

$$H(u,v) = \left(-\frac{1}{2}\right)\lambda\sqrt{\frac{u-c}{v}}\, J_1\left(2\lambda\sqrt{(u-c)v}\right). \tag{3.73}$$

Finally, if $\lambda > 0$, $c = 0$, then

$$H(u, v) = -\frac{\lambda}{2}\sqrt{\frac{u}{v}}\, J_1\left(2\lambda\sqrt{uv}\right). \tag{3.74}$$

The formulas for the kernels for $q(x) = -\lambda^2$ can be obtained from the formulas for $q(x) = \lambda^2$ by the formal replacement $\lambda \to i\lambda$. Note that for $\lambda = 0$, all the kernels considered vanish, and transmutations of the form (3.6)–(3.15) are identical.

Now we consider the general case when Eq. (3.48) is taken with $q(x) = \lambda^2$ for different c, d,

$$H(u, v) = \frac{1}{2}\lambda^2(u - c) + \lambda^2 \int\limits_d^u d\alpha \int\limits_0^v H(\alpha, \beta)\, d\beta.$$

From the first iterations, we easily obtain

$$H_n(u, v) = \frac{1}{2}\left(\lambda^2\right)^{n+1} \frac{v^n}{n!} y_n(u), \tag{3.75}$$

where $y_n(u)$ are defined by

$$y_0(u) = u - c,$$

$$y_1(u) = \int\limits_d^u (x_1 - c)\, dx_1,$$

$$y_{n+1}(u) = \int\limits_d^u y_n(s)\, ds.$$

Therefore, the formula

$$y_n(u) = \int\limits_d^u dx_1 \int\limits_d^{x_1} dx_2 \ldots \int\limits_d^{x_{n-1}} (x_n - c)\, dx_n$$

is valid. To rewrite this expression we use (3.67). So

$$y_n(u) = \int\limits_d^u (t - c)\frac{(u - t)^{n-1}}{(n-1)!}\, dt, \quad n \geq 1,$$

$$y_n(u) = \int\limits_d^u (t - c)\left[(-1)\frac{(u - t)^n}{n!}\right]' dt$$

$$= (-1)(t - c)\, \frac{(u - t)^n}{n!}\bigg|_d^u - \int\limits_d^u (-1)\frac{(u - t)^n}{n!}\, dt$$

$$= (d-c)\frac{(u-d)^n}{n!} - \left.\frac{(u-t)^{n+1}}{(n+1)!}\right|_d^u = \frac{(u-d)^{n+1}}{(n+1)!} + (d-c)\frac{(u-d)^n}{n!}.$$

Obviously, the last formula retains meaning even for $n=0$. Substituting it into (3.75), we get

$$\begin{aligned} H_n(u,v) &= \frac{1}{2}\left(\lambda^2\right)^{n+1}\frac{v^n}{n!}\left[\frac{(u-d)^{n+1}}{(n+1)!} + (d-c)\frac{(u-d)^n}{n!}\right] \\ &= \frac{1}{2}\left(\lambda^2\right)(u-d)\frac{\left[\lambda^2(u-d)v\right]^n}{n!(n+1)!} + \frac{1}{2}\left(\lambda^2\right)(d-c)\frac{\left[\lambda^2(u-d)v\right]^n}{(n!)^2}. \end{aligned}$$

Using formula (3.53) and the relation for $I_0(\cdot)$ of the form (see [2])

$$I_0(z) = \sum_{n=0}^{\infty}\frac{\left(\frac{z^2}{4}\right)^n}{(n!)^2}, \quad \sum_{n=0}^{\infty}\frac{q^n}{(n!)^2} = I_0\left(2\sqrt{q}\right),$$

we obtain the Neumann series for the sum

$$H(u,v) = \frac{1}{2}\lambda^2(u-d)\frac{1}{\sqrt{q}}I_1\left(2\sqrt{q}\right) + (d-c)\frac{1}{2}\lambda^2 I_0\left(2\sqrt{q}\right). \tag{3.76}$$

Here $q = \lambda^2(u-d)v$.

For $\lambda > 0$ expression (3.76) allows

$$H(u,v) = \frac{\lambda}{2}\sqrt{\frac{u-d}{v}}\,I_1\left(2\lambda\sqrt{(u-d)v}\right) + (d-c)\frac{\lambda^2}{2}I_0\left(2\lambda\sqrt{(u-d)v}\right), \tag{3.77}$$

which for $d=c$ gives (3.70).

Another interesting feature is the choice $d=0$:

$$H(u,v) = \frac{\lambda}{2}\sqrt{\frac{u}{v}}\,I_1\left(2\lambda\sqrt{uv}\right) - \frac{c\lambda^2}{2}I_0\left(2\lambda\sqrt{uv}\right), \tag{3.78}$$

where the parameter c can take any values.

Finally, for the potential $q(x) = -\lambda^2$, taking into account the relations for the Bessel function of the imaginary argument, we obtain the following formulas:

$$H(u,v) = -\frac{1}{2}\lambda^2(u-d)\frac{1}{\sqrt{q}}J_1(2\sqrt{q}) - (d-c)\frac{\lambda^2}{2}J_0(2\sqrt{q}), \tag{3.79}$$

under conditions $\lambda \in \mathbb{C}$, $q = \lambda^2(u-d)v$;

$$H(u,v) = -\frac{\lambda}{2}\sqrt{\frac{u-d}{v}}\,J_1\left(2\lambda\sqrt{(u-d)v}\right) - (d-c)\frac{\lambda^2}{2}J_0\left(2\lambda\sqrt{(u-d)v}\right) \tag{3.80}$$

under the condition $\lambda > 0$, and

$$H(u,v) = -\frac{\lambda}{2}\sqrt{\frac{u}{v}}\, J_1\left(2\lambda\sqrt{uv}\right) + \frac{c\lambda^2}{2} J_0\left(2\lambda\sqrt{uv}\right) \tag{3.81}$$

for $\lambda > 0$, $d = 0$.

Formulas (3.79)–(3.81) give kernels of transmutations intertwining $D^2 - \lambda^2 \to D^2$.

Now we consider some other relations valid for any linear EVL operators satisfying the identity

$$S_\alpha (D^2 + \alpha) f = D^2 S_\alpha f, \quad \alpha \in \mathbb{C}, \tag{3.82}$$

for arbitrary functions f. These are formal equalities. Now we consider some other relations valid for any linear EVL operators satisfying

$$S_\alpha D^2 = (D^2 - \alpha) S_\alpha f, \tag{3.83}$$

$$(S_\alpha)^2 (D^2 + \alpha) = S_\alpha S_\alpha (D^2 + \alpha) = (D^2 - \alpha) S_\alpha S_\alpha, \tag{3.84}$$

$$(S_\alpha)^2 D^2 = (D^2 - 2\alpha)(S_\alpha)^2, \tag{3.85}$$

$$(S_\alpha)^n D^2 = (D^2 - n\alpha)(S_\alpha)^n, \quad n \in \mathbb{N}, \tag{3.86}$$

$$(S_{-\alpha})^2 D^2 = (D^2 + 2\alpha)(S_{-\alpha})^2, \tag{3.87}$$

$$(S_{-\alpha})^n D^2 = (D^2 + n\alpha)(S_{-\alpha})^n, \quad n \in \mathbb{N}, \tag{3.88}$$

$$(S_\alpha S_\beta) D^4 = (D^2 - \alpha - \beta)^2 (S_\alpha S_\beta), \tag{3.89}$$

$$S_\alpha D^4 = (D^2 - \alpha) S_\alpha, \tag{3.90}$$

$$S_\alpha (D^2)^n = (D^2 - \alpha)^n S_\alpha, \quad n \in \mathbb{N}, \tag{3.91}$$

$$\left(S_{\alpha_1} S_{\alpha_2} \cdots S_{\alpha_k}\right) D^2 = \left(D^2 - \alpha_1 - \alpha_2 - \cdots - \alpha_k\right)\left(S_{\alpha_1} S_{\alpha_2} S_{\alpha_k}\right), \; k \in \mathbb{N}, \tag{3.92}$$

$$\left(\prod_{n=1}^{k} S_{\alpha_n}\right)\left(D^2\right)^m = \left(D^2 - \sum_{n=1}^{k} \alpha_n\right)^m \left(\prod_{n=1}^{k} S_{\alpha_n}\right), \quad k, n, m \in \mathbb{N}. \tag{3.93}$$

They turn into identities for functions from the class Φ and each such a class Φ should be defined for each case. The relations (3.83)–(3.93) show that each EVL operator satisfying the property (3.82) generates many different families of other operators intertwining differential expressions of the second and higher orders.

In conclusion, we note that each of the operators intertwining differential expressions (3.83)–(3.93) leads to correspondence formulas between the solutions of some

differential equations (including those with partial derivatives). Fixing the class Φ leads to the fact that some correspondence between the boundary conditions is added to this correspondence.

3.3.4 Estimates of kernels and point formulas for estimating the error for calculating transmutation operators

In this subsection, we consider potentials satisfying various uniform and integral inequalities.

Let the continuous potential q satisfy in the domain of definition the inequality

$$|q(\alpha-\beta)| \le R(\alpha)T(\beta). \tag{3.94}$$

Here continuous functions $R(\alpha)$ and $T(\beta)$ are nonnegative and locally integrable.

Let us consider (3.48) for $c=d$. We have

$$H(u,v)=\frac{1}{2}\int_c^u q(s)\,ds+\int_c^u d\alpha\int_0^v q(\alpha-\beta)H(\alpha,\beta)\,d\beta. \tag{3.95}$$

We estimate successively terms in the Neumann series

$$|H_0(u,v)| \le \frac{1}{2}\left|\int_c^u |q(s)|\,ds\right| \le \frac{1}{2}T(0)\left|\int_c^u R(s)\,ds\right|.$$

The module in front of the integral is needed, because we do not know whether the limits of integration are in the natural order. For example, this will certainly not be the case when $c=+\infty$. We have

$$\begin{aligned}|H_1(u,v)| &\le \left|\int_c^u d\alpha\int_0^v |q(\alpha-\beta)|\frac{1}{2}T(0)\left|\int_c^\alpha R(s)\,ds\right|\,d\beta\right|\\ &\le \frac{1}{2}T(0)\left|\int_c^u R(\alpha)\left|\int_c^\alpha R(s)\,ds\,|d\alpha|\right|\int_0^v T(\beta)\,d\beta\right|.\end{aligned}$$

We fixed u,v,c. Number α is between u and c. So

$$|H_1| \le \pm\frac{1}{2}T(0)\int_0^v T(\beta)\,d\beta\,\frac{\left[\int_c^u R(\alpha)\,d\alpha\right]^2}{2!}. \tag{3.96}$$

Signs $\pm$ are chosen so that the common sign of the expression (3.96) will be a plus. Similarly

$$|H_n| \le \pm\frac{1}{2}T(0)\frac{\left[\int\limits_0^v T(\beta)\,d\beta\right]^n}{n!}\cdot\frac{\left[\int\limits_u^c R(\alpha)\,d\alpha\right]^{n+1}}{(n+1)!}.$$

Using (3.53) we can estimate a kernel as

$$|H(u,v)| \le \frac{1}{2}T(0)\left(\int\limits_{(u,c)}^{(u,c)_+} R(\alpha)\,d\alpha\right)\frac{1}{\sqrt{\left(\int\limits_{(v,0)_-}^{(v,o)_+} T(\beta)d\beta\right)\left(\int\limits_{(u,c)_-}^{(u,c)_+} R(\alpha)\,d\alpha\right)}}$$
$$\times I_1\left(2\sqrt{\left(\int\limits_{(v,0)_-}^{(v,o)_+} T(\beta)\,d\beta\right)\left(\int\limits_{(u,c)_-}^{(u,c)_+} R(\alpha)\,d\alpha\right)}\right), \tag{3.97}$$

where $I_1(\cdot)$ is a modified Bessel function of the first kind (1.16). Let

$$(a,b)_- = \min(a,b), \quad (a,b)_+ = \max(a,b).$$

Consequently, under the condition (3.95), there exists a transmutation operator with a kernel for which inequality (3.97) holds. Next, a standard piece of statements is needed:

(a) From the integral Eq. (3.95), existence of the continuous function $\frac{\partial^2 H}{\partial u \partial v}$ follows. Estimates for this function can be obtained from (3.95), (3.94), and (3.97).

(b) Differentiating (3.95), we obtain the existence of $\frac{\partial H}{\partial v}$ and $\frac{\partial H}{\partial u}$, and estimates for them of the form (3.97).

(c) Existence of $\frac{\partial^2 H}{\partial u^2}$ and $\frac{\partial^2 H}{\partial v^2}$ and other higher derivatives are obtained if additional smoothness is required from $q(x)$.

(d) Solving (3.95) for locally integrable q, we approximate it by continuous potentials:

$$q_n(x) \in C,$$
$$\int\limits_0^u d\alpha \int\limits_0^v \{q_n(\alpha-\beta) - q(\alpha-\beta)\}\, d\beta \xrightarrow[n\to\infty]{} 0.$$

We consider particular cases of formula (3.97).

(1) We have $S = S_2$, $q(x) = e^x$, $\left|e^{\alpha-\beta}\right| \le e^{\alpha}e^{-\beta}$,
$R(\alpha) = e^{\alpha}$, $T(\beta) = e^{-\beta}$, $T(0) = 1$,
$c = -\infty$, $(u,c)_- = -\infty$, $(u,c)_+ = u$,

$v > 0$, $(v,0)_- = 0$, $(v,0)_+ = v$.
So we have an estimate (3.97) for S_2.

(2) We have $S = S_4$, $q(x) = e^x$, R, $R(\alpha) = e^\alpha$, $T(\beta) = e^{-\beta}$, $c = -\infty$, $u > 0$, $v > 0$.
So we have an estimate (3.97) for S_4.
We also can easily verify that the estimate (3.97) holds for S_{15}, S_{17}.

(3) Let us consider the case when $c = +\infty$, $v < 0$ $(a = \infty) \Rightarrow \beta < 0$, and

$$|q(x)| \le \frac{c}{x^v}, \quad v > 1.$$

Then

$$|q(\alpha - \beta)| \le \frac{c}{(|\alpha| + |\beta|)^v}.$$

Therefore, we can accept

$$R(\alpha) = \frac{1}{\alpha^v}, \quad T(\beta) = C.$$

Therefore, for such a singular potential there exists a transmutation operator of the form

$$(Sf) = f(x) - \int_x^\infty K(x,t) f(t)\, dt$$

with kernel $K(x,t) = H(u,v)$ satisfying the estimate

$$\begin{aligned} |H(u,v)| &\le \frac{1}{2} C \frac{1}{\sqrt{c|v|\frac{u^{1-v}}{(1-v)}}} I_1\left(2\sqrt{\frac{c}{1-v}|v| u^{1-v}}\right) \\ &= \frac{1}{2}\sqrt{\frac{c}{c-v}} \frac{u^{\frac{1-v}{2}}}{|v|^{\frac{1}{2}}} I_1\left(2\sqrt{\frac{c}{c-v}} u^{\frac{1-v}{2}} |v|^{\frac{1}{2}}\right). \end{aligned} \tag{3.98}$$

If $v = 0$ we can obtain an estimate with $c = \lambda^2$ for constant potential.

Remark 8. *If q is a monotonous function, $|q(\alpha - \beta)| \le |q(\alpha)|$, then (3.94) is valid. Also condition (3.94) holds if $|q(\alpha - \beta)| \le C|q(\alpha)|$.*

Now consider a slightly weaker integral condition

$$\left| \int_0^v |q(\alpha - \beta)|\, d\beta \right| \le R(\alpha) \left| \int_0^v T(\beta)\, d\beta \right| \tag{3.99}$$

where conditions on the functions $R(\alpha)$ and $T(\beta)$ are the same.

Let

$$W(u) = \sup_{\forall \xi} \left| \int_c^u q(s)\, ds \right|, \quad (u,s)_- \leq (u,c)_+.$$

We estimate the terms of the Neumann series

$$\begin{aligned}
|H_0(u,v)| &\leq \frac{1}{2} W(u), \\
|H_1(u,v)| &\leq \pm \int_c^u d\alpha \int_0^v |q(\alpha-\beta)| \frac{1}{2} W(\alpha)\, d\beta \\
&\qquad \leq \pm W(u) \left(\int_c^u R(\alpha)\, d\alpha \right) \left(\int_0^v T(\beta)\, d\beta \right).
\end{aligned}$$

We used the inequality $W(u) \geq W(\alpha)$ since α is between u and c. Therefore

$$|H(u,v)| \leq \pm W(u) \sum_{n=0}^{\infty} \frac{q^n}{(n!)^2},$$

where

$$q = \left| \int_c^u R(\alpha)\, d\alpha \cdot \int_0^v T(\beta)\, d\beta \right|.$$

Using formula (1.16)*, which takes the form*

$$I_0(z) = \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{(k!)^2}, \quad q = \frac{z^2}{4}, \quad z = 2\sqrt{q},$$

$$I_0\left(2\sqrt{q}\right) = \sum \frac{q^k}{(k!)^2} \tag{3.100}$$

as a result, we obtain that there exists a kernel satisfying the inequality

$$|H(u,v)| \leq W(u) I_1 \left(2 \sqrt{\int_{(0,v)_-}^{(0,v)_+} T(\beta)\, d\beta \int_{(c,u)_-}^{(c,u)_+} R(\alpha)\, d\alpha} \right). \tag{3.101}$$

We consider estimates for a power potential. One of these estimates in the case of an operator of the Levin type will be given later. Unfortunately, in this case it is not possible to precisely construct the potential. Therefore, various estimates are interesting.

So, let the inequality

$$|q(x)| \le A|x|^{\nu}, \quad \nu > -1, \tag{3.102}$$

hold. Consider the simplest equation when $c = 0$. We estimate the members of the Neumann series, assuming that $u, v > 0$. We have

$$\begin{aligned}
|H_0(u,v)| &\le \frac{1}{2} A \frac{u^{\nu+1}}{\nu+1} = \frac{1}{2} A \frac{\Gamma(\nu+1)}{\Gamma(\nu+2)} u^{\nu+1}, \\
|H_1(u,v)| &\le \frac{1}{2} A (u+v) \frac{u^{\nu+2}}{(\nu+1)(\nu+2)} v \\
&= \frac{1}{2} A \frac{\Gamma(\nu+1) u^{\nu+1}}{\Gamma(\nu+3)} \left[A(u+v)^{\nu} \cdot uv\right]^1, \\
|H_n(u,v)| &\le \frac{1}{2} A \frac{\Gamma(\nu+1)}{\Gamma(\nu+n+2)\Gamma(n+1)} \cdot u^{\nu+1} \left[Auv(u+v)^{\nu}\right]^n.
\end{aligned}$$

Taking into account formula (7) from [455], p. 708, of the form

$$\sum \frac{q^n}{n!\Gamma(\nu+n+2)} = (\sqrt{q})^{-(\nu+1)} I_{\nu+1}\left(2\sqrt{q}\right), \tag{3.103}$$

we obtain

$$\begin{aligned}
|H(u,v)| &\le \frac{A}{2} \Gamma(\nu+1) u^{\nu+1} \left[Auv(u+v)^{\nu}\right]^{-\frac{(\nu+1)}{2}} \cdot I_{\nu+1}\left(2(u+v)^{\frac{\nu}{2}}\sqrt{Auv}\right) \\
&= \frac{A^{\frac{1-\nu}{2}} \Gamma(\nu+1)}{2} \left(\frac{u}{v}\right)^{\frac{(\nu+1)}{2}} (u+v)^{-\frac{\nu(\nu-1)}{2}} I_{\nu+1}\left(2(u+v)^{\frac{1}{2}}\sqrt{Auv}\right).
\end{aligned} \tag{3.104}$$

Of course, this is a rough estimate. However, we note that at a constant potential ($\nu = 0$, $A = \lambda^2$), it turns into exact equality.

Consider special estimates valid for the negative potential

$$q(x) \le 0. \tag{3.105}$$

In the equation

$$H(u,v) = \frac{1}{2} \int_c^u q(s)\, ds + \int_c^u d\alpha \int_0^v q(\alpha-\beta) H(\alpha,\beta)\, d\beta$$

we put $u > c$, $v > 0$. We define the signs of the members of the Neumann series

$$H_0(u,v) = \frac{1}{2} \int_c^u q(s)\, ds \le 0,$$

$$H_1(u,v) = \int\limits_c^u d\alpha \int\limits_0^v q(\alpha-\beta) H_0(\alpha,\beta)\, d\beta \geq 0,$$

and so on. Obviously the signs of the kernels will alternate.

Suppose that in the region of variation of the variables u, v, the inequalities

$$\left| \int\limits_c^u d\alpha \int\limits_0^v q(\alpha-\beta)\, d\beta \right| < A < 1 \tag{3.106}$$

hold. Then:

(a) the integral equation under consideration has a unique solution, and this solution is bounded;

(b) the estimates

$$|H(u,v)| \leq \frac{1}{2(1-A)} \cdot \int\limits_c^u |q(s)|\, ds,$$

$$|H(u,v)| \leq \frac{1}{2} \int\limits_c^u |q(s)|\, ds \leq \text{ const}$$

are valid;

(c) kernel $H(u,v)$ is negative and

$$-\frac{1}{2} \int\limits_c^u |q(s)|\, ds \leq H(u,v) \leq -\frac{(1-A)}{2} \int\limits_c^u |q(s)|\, ds.$$

So, the Neumann series is alternating, all members of the series are sign-definite. From the iterative formula

$$|H_{n+1}(u,v)| = \left| \int\limits_0^u d\alpha \int\limits_0^v q(\alpha-\beta) H_n(\alpha,\beta)\, d\beta \right|$$

it follows that all functions $H_n(u,v)$ increase in u and v. Then from the same formula we get

$$|H_{n+1}(u,v)| \leq |H_n(u,v)| \left| \int\limits_0^u d\alpha \int\limits_0^v q(\alpha-\beta)\, d\beta \right| \leq A\, |H_n(u,v)|,$$
$$0 < A < 1.$$

Therefore, the inequality

$$|H_n(u,v)| \leq A^n\, |H_0(u,v)|, \quad \forall n,$$

holds. Consequently, the series converges at least with the speed of geometric progression, and the kernel satisfies the estimate

$$|H_n(u,v)| \le A^n \, |H_0(u,v)|\,, \qquad \forall n. \tag{3.107}$$

It follows from Fubini's theorem that the integral for H_0 converges,

$$\left|\int_c^u q(s)\,ds\right| < \infty,$$

since the integral

$$\int_c^u q(\alpha-\beta)\,d\alpha < \infty$$

is finite. We can put $\beta = 0$ in the last integral.

Estimate (3.107) can be clarified. Indeed, in the condition of the theorem all the requirements of the Leibnitz theorem are satisfied. Therefore

$$H_0(u,v) \le H(u,v) \le H_0(u,v) + H_1(u,v).$$

It is equivalent to

$$\begin{aligned} -\frac{1}{2}\left|\int_c^u q(s)\,ds\right| &\le H(u,v) \le -\frac{1}{2}\left|\int_c^u q(s)\,ds\right| \\ &+\frac{1}{2}\int_c^u d\alpha \int_0^v |q(\alpha-\beta)| \int_c^u |q(t)|\,dt\,d\beta \le \frac{1}{2}\int_c^u |q(t)|\,dt\,(A-1). \end{aligned}$$

Therefore, the kernel H in this case is negative and the estimate

$$-\frac{1}{2}\int_c^u |q(s)\,ds| \le H(u,v) \le -\frac{(1-A)}{2}\int_c^u |q(s)|\,ds \tag{3.108}$$

is correct. The lower A, the higher the accuracy.

It follows that

$$|H(u,v)| \le \frac{1}{2}\int_c^u |q(s)|\,ds.$$

This estimate is always more accurate than (3.107).

In addition, we obtain estimates of derivatives

$$\frac{\partial H}{\partial u} = \frac{1}{2}q(u) + \int\limits_0^v q(u-\beta)H(u,\beta)\,d\beta,$$

$$\frac{\partial H}{\partial u} \geq \frac{1}{2}q(u) + \int\limits_0^v q(u-\beta)\left[\left(-\frac{1}{2}\right)^{(1-A)}\int\limits_c^u |q(s)|\,ds\right]d\beta$$

$$= \frac{(1-A)}{2}\int\limits_c^u |q(s)|\,ds \cdot \int\limits_0^v |q(u-\beta)|\,d\beta - \frac{1}{2}|q(u)|.$$

Another derivative is estimated similarly.

Remark 9. *Most of the previous estimates are also true in the case of the potential $q = q(x,\lambda)$, in particular, with the condition of boundedness.*

Let us consider now the following problem having great practical value.

Suppose that the kernel of the transmutation operator obtained as a solution of the integral Eq. (3.48) in the form of a Neumann series is approximated by partial sums of this Neumann series in the calculations. As a result, the operator S is replaced by the operator S_n. It is required to estimate the error introduced by such a replacement of S by S_n in the transmutation identity

$$\Delta S_n f = S_n(D^2 + q(x))f - D^2 S_n f, \tag{3.109}$$

$$\Delta S f = S(D^2 + q(x))f - D^2 S f = 0. \tag{3.110}$$

Note that a priori we cannot expect that the residual (3.109), which we designated as ΔS_n, will be small in standard norms. The difference $(S_n - S)f$ is really small, but (3.109) contains differentiation operators that are unbounded in standard spaces like C^k.

It is all the more surprising that we not only prove the smallness of the residual (3.109), but also derive a simple exact formula for it in terms of the same operators S_n. This formula allows one to evaluate the residual (3.109) in any norms.

To highlight the main idea, we first consider the case of a model transmutation operator $q(x) = \lambda^2$. Then the main result will be proved, its consequences will be obtained, and possible generalizations will be outlined.

Let us consider a transmutation operator intertwining $D^2 + \lambda^2$ and D^2 with kernel (3.71). Integrated kernels in variables x, t have the forms (see (3.68) with $c = 0$)

$$K_0(x,t) = \frac{1}{2}\left(\frac{x+t}{2}\right),$$

$$K_1(x,t) = \frac{1}{32}(x+t)^2(x-t),$$

and generally

$$K_n(x,t) = \frac{1}{2^{2n+2}} \frac{1}{n!(n+1)!}(x+t)^{n+1}(x-t)^n.$$

Without loss of generality we put $\lambda = 1$.

We introduce the definition of a "defect" of the operator A. By definition, we accept

$$(\Delta Af)(x) = \left(A\left(D^2+1\right)f\right)(x) - \left(D^2Af\right)(x).$$

We calculate the defect for operators

$$(T_n f)(x) = \int_0^x K_n(x,t) f(t)\, dt.$$

We start with the obvious formula for the defect of the unit operator

$$(\Delta If)(x) = (If)(x). \tag{3.111}$$

Simple calculations using integration by parts lead to formulas

$$\begin{aligned}(\Delta T_0 f)(x) &= (T_0 f)(x) - f(x) + \frac{1}{c_0} f(0) - \frac{1}{c_0} x f'(0),\\ (\Delta T_1 f)(x) &= (T_1 f)(x) - (T_0 f)(x) + \frac{1}{c_1} x^2 f(0) - \frac{1}{c_1} x^3 f'(0),\end{aligned} \tag{3.112}$$

where

$$c_n = \frac{1}{2^{2n+2} n!(n+1)!}, \quad c_0 = \frac{1}{4}, \quad c_1 = \frac{1}{32}.$$

A similar formula holds for a defect of an operator T_n:

$$(\Delta T_n f)(x) = (T_n f)(x) - (T_{n-1} f)(x) + \frac{x^{2n}}{c_n} f(0) - \frac{x^{2n+1}}{c_n} f'(0). \tag{3.113}$$

Now we sum up formulas (3.111)–(3.113). As a result, for the operator S_n defined by the formula

$$S_n = I + \sum_{k=0}^{n} k,$$

we obtain the following formula for a defect:

$$(\Delta S_n f)(x) = (T_n f)(x) + f(0) \sum_{k=0}^{n} \frac{x^{2k}}{c_k} - f'(0) \sum_{k=0}^{n} \frac{x^{2k+1}}{c_k}. \tag{3.114}$$

This is the desired expression. It allows one to evaluate the error from replacing operator S by S_n in any norm. Actually,

$$\|\Delta S_n f\| \leq \|T_n f\| + |f(0)| \cdot \left\| \sum_{k=0}^{n} \frac{x^{2k}}{c_k} \right\| + |f'(0)| \cdot \left\| \sum_{k=0}^{n} \frac{x^{2k+1}}{c_k} \right\|. \tag{3.115}$$

In particular, on functions satisfying the boundary condition $f(0) = f'(0) = 0$ we get

$$\|\Delta S_n f\| = \|T_n f\| \tag{3.116}$$

with an exact equal sign.

Since, for example, in the uniform norm, $T_n f$ is a common term of the Neumann series, which is majorized by rapidly converging series, the value of the defect (3.116) decreases just as rapidly with increasing n. From the same formula (3.116) it can be seen that on the functions $f(0) = f'(0) = 0$,

$$\lim_{n\to\infty} \Delta S_n = \Delta S = 0.$$

On arbitrary functions, from (3.114) we obtain

$$\lim_{n\to\infty} \Delta S_n = \frac{1}{2x} I_1(x) f(0) - \frac{1}{2} I_1(x) f'(0), \tag{3.117}$$

where $I_1(\cdot)$ is a modified Bessel function of the first kind (1.16).

If the space under consideration is not only normalized, but also forms a Banach algebra in multiplication (such as, for example, the space C of continuous functions), then formula (3.115) can be simplified as follows:

$$\|\Delta S_n f\| \leq \|T_n f\| + \frac{1}{2\|x\|} I_1(\|x\|)|f(0)| + \frac{1}{2} I_1(\|x\|)|f'(0)|. \tag{3.118}$$

We also note the importance of the general formula for the error (3.115). In the practical calculation of operator S on a computer by replacing it with S_n, the quantities $f(0)$ and $f'(0)$ are required. For example, they are necessary for the approximate calculation of the integral $T_n f$. Even if we assume in the calculations that $f(0) = f'(0) = 0$, then in a computer these quantities will be replaced by machine zero. As a result of such a replacement, an error occurs, which will then accumulate in the calculations. This error is controlled by the corresponding terms in formula (3.115).

3.4 Transmutations for singular Bessel operator

One of the most important transmutation operators for the singular Bessel operator (9.1) are the Poisson operator, generalized translation, and weighted spherical mean. All them are generated by the singular differential Bessel operator. These classes of transmutations may be used for deriving explicit formulas for solutions of partial

differential equations with Bessel operators via unperturbed equation solutions. An example is the B-elliptic equation of the form

$$\sum_{k=1}^{n}(B_\gamma)_{x_k}u(x_1,\dots,x_n)=f, \tag{3.119}$$

and similar B-hyperbolic and B-parabolic equations. This idea works by the Sonine–Poisson–Delsarte transmutations (cf. [51–53,56,242]). New results follow automatically for new classes of transmutations.

3.4.1 One-dimensional Poisson operator

In this section, we consider the one-dimensional Poisson operator. This operator is the one of the most important transmutation operators connected with the Bessel operator.

Definition 22. *Let $\gamma>0$. The one-dimensional Poisson operator is defined for integrable function f by the equality*

$$\mathcal{P}_x^\gamma f(x)=\frac{2C(\gamma)}{x^{\gamma-1}}\int\limits_0^x\left(x^2-t^2\right)^{\frac{\gamma}{2}-1}f(t)\,dt,\qquad C(\gamma)=\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)}, \tag{3.120}$$

or

$$\mathcal{P}_x^\gamma f(x)=C(\gamma)\int\limits_0^\pi f(x\cos\varphi)\sin^{\gamma-1}\varphi\,d\varphi. \tag{3.121}$$

The constant $C(\gamma)$ is chosen so that $\mathcal{P}_x^\gamma[1]=1$.

Using the example of the Poisson operator, we show how to find out its intertwining property.

Let us consider the equation

$$(B_\gamma)_y u(x,y)=D_x^2u(x,y). \tag{3.122}$$

Looking for the solution to this equation in the form

$$u=\Omega(r),\qquad r=\sqrt{y^2-(x-\xi)^2},$$

where ξ is some variable, we obtain for $\Omega(r)$

$$\begin{aligned}D_x^2\Omega(r)&=D_x\Omega'(r)\left(-\frac{x-\xi}{\sqrt{y^2-(x-\xi)^2}}\right)\\&=\Omega''(r)\left(-\frac{x-\xi}{\sqrt{y^2-(x-\xi)^2}}\right)^2\end{aligned}$$

$$+\,\Omega'(r)\left(-\frac{\sqrt{y^2-(x-\xi)^2}+\frac{(x-\xi)^2}{\sqrt{y^2-(x-\xi)^2}}}{y^2-(x-\xi)^2}\right)$$

$$=\Omega''(r)\,\frac{(x-\xi)^2}{r^2}-\Omega'(r)\,\frac{y^2}{r^3},$$

$$(B_\gamma)_y\Omega(r)=\frac{1}{y^\gamma}D_y y^\gamma D_y\Omega(r)=\frac{1}{y^\gamma}D_y\Omega'(r)\,\frac{y^{\gamma+1}}{\sqrt{y^2-(x-\xi)^2}}$$

$$=\Omega''(r)\,\frac{y^2}{y^2-(x-\xi)^2}$$

$$+\,\Omega'(r)\,\frac{1}{y^\gamma}\,\frac{(\gamma+1)y^\gamma\sqrt{y^2-(x-\xi)^2}-\frac{y^{\gamma+2}}{\sqrt{y^2-(x-\xi)^2}}}{y^2-(x-\xi)^2}$$

$$=\Omega''(r)\,\frac{y^2}{r^2}+\Omega'(r)\,\frac{(\gamma+1)r^2-y^2}{r^3},$$

$$(B_\gamma)_y\Omega(r)-D_x^2\Omega(r)=\Omega''(r)+\frac{\gamma+1}{r}\Omega'(r)=0.$$

One solution is

$$u_1=\Omega(r)=r^{-\gamma}.$$

Taking into account the recurrent formula (1.109) the function

$$u_2=y^{1-\gamma}r^{\gamma-2}$$

is also a solution to (3.122). It is easy to see that

$$u_1=\int\limits_{x-y}^{x+y}\frac{\Psi(\xi)}{r^\gamma}d\xi=\int\limits_{x-y}^{x+y}\frac{\Psi(\xi)}{(y^2-(x-\xi)^2)^{\frac{\gamma}{2}}}d\xi,$$

$$u_2=y^{1-\gamma}\int\limits_{x-y}^{x+y}\Phi(\xi)r^{\gamma-2}d\xi=y^{1-\gamma}\int\limits_{x-y}^{x+y}\Phi(\xi)(y^2-(x-\xi)^2)^{\frac{\gamma}{2}-1}d\xi,$$

where Φ and Ψ are arbitrary functions with suitable properties. Putting $\xi=x+y(2t-1)$, summing u_1 and u_2, we get a general solution to (3.122),

$$u(x,y)=\int\limits_0^1\frac{\Phi(x+y(2t-1))}{(t(1-t))^{1-\frac{\gamma}{2}}}dt+y^{1-\gamma}\int\limits_0^1\frac{\Psi(x+y(2t-1))}{(t(1-t))^{\frac{\gamma}{2}}}dt.$$

This solution is valid for $0 < \gamma < 1$. Here we change $2^{\gamma-1}\Phi$ to Φ and $2^{1-\gamma}\Psi$ to Ψ.

Adding to (3.122) the initial conditions

$$u(x,0) = f(x), \qquad u_y(x,0) = 0,$$

we get

$$u(x,y) = \frac{\Gamma(\gamma)}{\Gamma^2\left(\frac{\gamma}{2}\right)} \int\limits_0^1 \frac{f(x+y(2t-1))}{(t(1-t))^{1-\frac{\gamma}{2}}} dt.$$

Now we can introduce the new variable $z = y(2t-1)$ and write

$$\begin{aligned} u(x,y) &= \frac{\Gamma(\gamma)}{(2y)^{\gamma-1}\Gamma^2\left(\frac{\gamma}{2}\right)} \int\limits_{-y}^{y} \frac{f(x+z)}{(t(1-t))^{1-\frac{\gamma}{2}}} dt \\ &= \frac{\Gamma(\gamma)}{(2y)^{\gamma-1}\Gamma^2\left(\frac{\gamma}{2}\right)} \left(\int\limits_0^y \frac{f(x+z)}{(t(1-t))^{1-\frac{\gamma}{2}}} dt + \int\limits_{-y}^{0} \frac{f(x+z)}{(t(1-t))^{1-\frac{\gamma}{2}}} dt \right) \\ &= \frac{\Gamma(\gamma)}{(2y)^{\gamma-1}\Gamma^2\left(\frac{\gamma}{2}\right)} \int\limits_0^y \frac{f(x+z)+f(x-z)}{(t(1-t))^{1-\frac{\gamma}{2}}} dz. \end{aligned}$$

Using the Legendre duplication formula (1.7) we obtain

$$\frac{\Gamma(\gamma)}{\Gamma^2\left(\frac{\gamma}{2}\right)2^{\gamma-1}} = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} = C(\gamma)$$

and

$$u(x,y) = \frac{C(\gamma)}{y^{\gamma-1}} \int\limits_0^y \frac{f(x+z)+f(x-z)}{(t(1-t))^{1-\frac{\gamma}{2}}} dz.$$

We note here that we came to the Poisson operator acting on $\frac{f(x+y)+f(x-y)}{2}$:

$$u(x,y) = \mathcal{P}_y^\gamma \left(\frac{f(x+y)+f(x-y)}{2} \right).$$

Therefore we obtain that the Poisson operator intertwines the solution of the Cauchy problem for the wave equation

$$D_y^2 v(x,y) = D_x^2 v(x,y), \qquad v(x,0) = f(x), \qquad v_y(x,0) = 0$$

and the solution of the Cauchy problem for the hyperbolic equation with the Bessel operator

$$(B_\gamma)_y u(x,y) = D_x^2 u(x,y), \qquad u(x,0) = f(x), \qquad u_y(x,0) = 0.$$

So we have $u(x,y) = \mathcal{P}_y^\gamma v(x,y)$ for $\gamma > 0$. It is easy to see that if $f(x) \in C^2$, then $u(x,y) \in C^2$ too. We can formulate this result in the form of a statement.

Statement 3. *For $f \in C^2_{ev}$ operator (3.120) acts as a transmutation operator by the formula*

$$\mathcal{P}_x^\gamma D^2 f = B_\gamma \mathcal{P}_x^\gamma f, \qquad D^2 = \frac{d^2}{dx^2}, \qquad B_\gamma = \frac{d^2}{dx^2} + \frac{\gamma}{x}\frac{d}{dx}. \tag{3.123}$$

Theorem 33. *The left inverse operator for (3.120) for $\gamma > 0$ for any summable function $H(x)$ is defined by*

$$(\mathcal{P}_x^\gamma)^{-1} H(x) = \frac{2\sqrt{\pi}x}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(n-\frac{\gamma}{2}\right)}\left(\frac{d}{2x dx}\right)^n \int\limits_0^x H(z)(x^2-z^2)^{n-\frac{\gamma}{2}-1} z^\gamma \, dz, \tag{3.124}$$

where $n = \left[\frac{\gamma}{2}\right] + 1$.

Proof. Let us find an operator $(\mathcal{P}_x^\gamma)^{-1}$, such that $(\mathcal{P}_x^\gamma)^{-1} H(x) = H(x)$, where $H(x) = \mathcal{P}_x^\gamma G(x)$.

Change of variables by the formulas $t^2 = w$, $x^2 = \xi$ drive the operator $\mathcal{P}_x^\gamma$ to the fractional Riemann–Liouville integral:

$$(I_{a+}^\alpha \varphi)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \qquad x > a.$$

So we have

$$\begin{aligned}
\mathcal{P}_x^\gamma G(x) &= \frac{2C(\gamma)}{x^{\gamma-1}} \int\limits_0^x \left(x^2 - t^2\right)^{\frac{\gamma}{2}-1} G(t)\, dt \\
&= \frac{C(\gamma)}{\xi^{\frac{\gamma-1}{2}}} \int\limits_0^\xi \frac{G(\sqrt{w})}{\sqrt{w}} (\xi - w)^{\frac{\gamma}{2}-1} dz \\
&= \frac{\Gamma\left(\frac{\gamma}{2}\right) C(\gamma)}{\xi^{\frac{\gamma-1}{2}}} (I_{0+}^{\frac{\gamma}{2}})_\xi \left(\frac{G(\sqrt{\xi})}{\sqrt{\xi}}\right) = H(\sqrt{\xi}).
\end{aligned}$$

Then

$$(I_{0+}^{\frac{\gamma}{2}})_\xi \left(\frac{G(\sqrt{\xi})}{\sqrt{\xi}}\right) = \frac{\xi^{\frac{\gamma-1}{2}}}{\Gamma\left(\frac{\gamma}{2}\right) C(\gamma)} H(\sqrt{\xi}).$$

Using formula (2.14) we obtain

$$G(\sqrt{\xi}) = \frac{\sqrt{\xi}}{\Gamma\left(\frac{\gamma}{2}\right) C(\gamma)} (D_{0+}^{\frac{\gamma}{2}})_\xi \left(\xi^{\frac{\gamma-1}{2}} H(\sqrt{\xi})\right)$$

$$= \frac{\sqrt{\xi}}{\Gamma\left(\frac{\gamma}{2}\right) C(\gamma)} \frac{1}{\Gamma\left(n-\frac{\gamma}{2}\right)} \left(\frac{d}{d\xi}\right)^n \int\limits_0^\xi t^{\frac{\gamma-1}{2}} H(\sqrt{t})(\xi-t)^{n-\frac{\gamma}{2}-1} dt,$$

$$n = \left[\frac{\gamma}{2}\right] + 1.$$

Returning to x by the formula $\xi = x^2$ and putting $t = z^2$, we get

$$G(x) = \frac{2\sqrt{\pi} x}{\Gamma\left(\frac{\gamma+1}{2}\right) \Gamma\left(n-\frac{\gamma}{2}\right)} \left(\frac{d}{2x dx}\right)^n \int\limits_0^x H(z)(x^2-z^2)^{n-\frac{\gamma}{2}-1} z^\gamma dz,$$

$$n = \left[\frac{\gamma}{2}\right] + 1.$$

The proof is complete. $\square$

For the Bessel function of the first kind J_ν the integral representation using the Poisson integral with $\nu > -\frac{1}{2}$ (see formula (1) in [591], p. 58)

$$J_\nu(x) = \frac{x^\nu}{\sqrt{\pi} 2^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \int\limits_0^\pi e^{ix\cos\varphi} \sin^{2\nu}\varphi d\varphi$$

is valid. So we can write for $\nu = \frac{\gamma-1}{2}$

$$j_{\frac{\gamma-1}{2}}(x) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} \int\limits_0^\pi e^{ix\cos\varphi} \sin^{\gamma-1}\varphi d\varphi = \mathcal{P}_x^\gamma e^{ix}. \tag{3.125}$$

For the Bessel function of the second kind I_ν the integral representation using the Poisson integral with $\nu > -\frac{1}{2}$ (see formula (9) in [591], p. 94)

$$I_\nu(x) = \frac{x^\nu}{\sqrt{\pi} 2^\nu \Gamma\left(\nu+\frac{1}{2}\right)} \int\limits_0^\pi e^{\pm x\cos\varphi} \sin^{2\nu}\varphi d\varphi$$

is valid. So we can write for $\nu = \frac{\gamma-1}{2}$

$$i_{\frac{\gamma-1}{2}}(x) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} \int\limits_0^\pi e^{\pm x\cos\varphi} \sin^{\gamma-1}\varphi d\varphi = \mathcal{P}_\gamma e^{\pm x}. \tag{3.126}$$

Now we consider how the Poisson operator intertwines solutions to equations with the Bessel operator and with the second derivative.

The abstract Euler–Poisson–Darboux equation has the form

$$Au = (B_\gamma)_t\, u, \qquad u = u(x,t;\gamma),$$

where A is a linear operator acting only by variable $x = (x_1, ..., x_n)$.

Since the Poisson operator (3.120) intertwines the second derivative and the Bessel operator, i.e., $\mathcal{P}_t^\gamma D_t^2 = (B_\gamma)_t \mathcal{P}_t^\gamma$, it is possible to use it for finding a solution to the abstract Euler–Poisson–Darboux equation (1.108) using known a solution to the equation $Aw{=}w_{tt}$, $w{=}w(x,t)$, $x{=}(x_1, ..., x_n)$, $t \in \mathbb{R}$.

Theorem 34. *Let $\gamma > 0$. We have a twice continuously differentiable for $t > 0$ solution $u{=}u(x,t;\gamma)$ to the equation*

$$Au = (B_\gamma)_t u, \quad u = u(x,t;\gamma), \qquad x = (x_1, ..., x_n), \quad t > 0, \tag{3.127}$$

associated with the twice continuously differentiable solution to the equation

$$Aw = w_{tt}, \quad w = w(x,t), \qquad x = (x_1, ..., x_n), \quad t \in \mathbb{R}, \tag{3.128}$$

by the formula

$$u(x,t;\gamma) = \mathcal{P}_t^\gamma\, w(x,t), \tag{3.129}$$

where $\mathcal{P}_t^\gamma$ is the Poisson operator (3.120) acting by the variable t.

Proof. Let us show that function u defined by formula (3.129) satisfies Eq. (3.127). We have

$$u = \frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} \int\limits_0^1 w(x,\alpha t)[1-\alpha^2]^{\frac{\gamma}{2}-1}\, d\alpha.$$

Let $\xi = \alpha t$. Integrating by parts we obtain

$$u = w_\xi(x,\alpha t), dv = \alpha[1-\alpha^2]^{\frac{\gamma}{2}-1}\, d\alpha, du = t w_{\xi\xi}(x,\alpha t) d\alpha, v = -\frac{1}{\gamma}[1-\alpha^2]^{\frac{\gamma}{2}}$$

and

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} \int\limits_0^1 \alpha\, w_\xi(x,\alpha t)[1-\alpha^2]^{\frac{\gamma}{2}-1}\, d\alpha \\ &= \frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} \frac{t}{\gamma} \int\limits_0^1 w_{\xi\xi}(x,\alpha t)[1-\alpha^2]^{\frac{\gamma}{2}}\, d\alpha\end{aligned}$$

$$= \frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \frac{t}{\gamma} \int_0^1 w_{\xi\xi}(x,\alpha t)[1-\alpha^2]^{\frac{\gamma}{2}}\,d\alpha$$

$$= \frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \frac{t}{\gamma} \int_0^1 Aw(x,\alpha t)[1-\alpha^2]^{\frac{\gamma}{2}}\,d\alpha.$$

For $\frac{\partial^2 u}{\partial t^2}$ we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \int_0^1 \alpha^2\, w_{\xi\xi}(x,\alpha t)[1-\alpha^2]^{\frac{\gamma}{2}-1}\,d\alpha$$

$$= \frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \int_0^1 Aw(x,\alpha t)\alpha^2\,[1-\alpha^2]^{\frac{\gamma}{2}-1}\,d\alpha.$$

So

$$(B_\gamma)_t u = \frac{\partial^2 u}{\partial t^2} + \frac{\gamma}{t}\frac{\partial u}{\partial t} =$$

$$\frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \left[\int_0^1 Aw(x,\alpha t)\alpha^2\,[1-\alpha^2]^{\frac{\gamma}{2}-1}\,d\alpha + \int_0^1 Aw(x,\alpha t)[1-\alpha^2]^{\frac{\gamma}{2}}\,d\alpha\right] =$$

$$\frac{2\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)} \int_0^1 Aw(x,\alpha t)[1-\alpha^2]^{\frac{\gamma}{2}-1}\,d\alpha = A\mathcal{P}_t^\gamma w(x,t) = Au.$$

Hence the function u satisfies Eq. (3.129).

The proof is complete. □

Corollary 3. *For* $0<\gamma<1$ *the function*

$$u(t,x;\gamma) = \mathcal{P}_t^\gamma\left[\frac{f(x+t)+f(x-t)}{2}\right] + t^{1-\gamma}\mathcal{P}_t^{2-\gamma}\left[\frac{g(x+t)+g(x-t)}{2}\right]$$

satisfies the Cauchy problem

$$u_{xx} = (B_\gamma)_t u, \tag{3.130}$$

$$u(0,x;\gamma) = f(x), \qquad t^\gamma u_t(t,x;\gamma)|_{t=0} = g(x). \tag{3.131}$$

Proof. Let us consider the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{3.132}$$

and its general solution

$$F(x+t)+G(x-t), \tag{3.133}$$

where F and G are arbitrary functions. Applying Theorem 34 we obtain that one of the solutions to

$$\frac{\partial^2 u}{\partial x^2} = (B_\gamma)_t u, \qquad u = u(x,t;\gamma) \tag{3.134}$$

is a function

$$u_1 = 2C(\gamma)\frac{1}{t^{\gamma-1}}\int\limits_0^t [F(x+z)+G(x-z)](t^2-z^2)^{\frac{\gamma}{2}-1}\,dz.$$

We transform the resulting general solution as follows:

$$u_1 = \frac{C(\gamma)}{t^{\gamma-1}}\int\limits_{-t}^t \frac{F(x+z)+F(x-z)+G(x+z)+G(x-z)}{(t^2-z^2)^{1-\frac{\gamma}{2}}}\,dz.$$

We introduce a new variable p by the formula $z = t(2p-1)$, and we obtain

$$u_1 = 2^{\gamma-1}C(\gamma)\int\limits_0^1 \frac{\Phi(x+t(2p-1))}{(p(1-p))^{1-\frac{\gamma}{2}}}\,dp,$$

where

$$\Phi(x+z) = [F(x+z)+F(x-z)+G(x+z)+G(x-z)]\,.$$

From Lemma 7 it follows that if the function $u(x,t;\gamma)$ is the solution to (3.134), then the function $t^{1-\gamma}u(x,t;2-\gamma)$ will also be a solution to (3.134). Therefore, the second solution to (3.134) is a function

$$u_2 = 2^{1-\gamma}C(2-\gamma)t^{1-\gamma}\int\limits_0^1 \frac{\Psi(x+t(2p-1))}{(p(1-p))^{\frac{\gamma}{2}}}\,dp,$$

where Ψ is an arbitrary function that, generally speaking, does not coincide with Φ. Composing the values of u_1 and u_2 we obtain that the general solution to (3.134) has

the form

$$u = 2^{\gamma-1}C(\gamma)\int_0^1 \frac{\Phi(x+t(2p-1))}{(p(1-p))^{1-\frac{\gamma}{2}}}\,dp$$
$$+\,2^{1-\gamma}C(2-\gamma)t^{1-\gamma}\int_0^1 \frac{\Psi(x+t(2p-1))}{(p(1-p))^{\frac{\gamma}{2}}}\,dp. \tag{3.135}$$

From the conditions $u(0,x;\gamma) = f(x)$ and $t^\gamma u_t(t,x;\gamma)|_{t=0} = g(x)$ it is easy to find Φ and Ψ. We have for $0 < \gamma < 1$

$$u(x,0;\gamma) = 2^{\gamma-1}C(\gamma)\Phi(x)\int_0^1 (p(1-p))^{\frac{\gamma}{2}-1}\,dp = f(x),$$

$$\int_0^1 (p(1-p))^{\frac{\gamma}{2}-1}\,dp = \frac{\Gamma\left(\frac{\gamma}{2}\right)^2}{\Gamma(\gamma)}.$$

Using the Legendre duplication formula (1.7), we obtain

$$2^{\gamma-1}C(\gamma)\int_0^1 (p(1-p))^{\frac{\gamma}{2}-1}\,dp$$
$$= 2^{\gamma-1}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)}\frac{\Gamma\left(\frac{\gamma}{2}\right)^2}{\Gamma(\gamma)} = \frac{2^{\gamma-1}\,\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}{\sqrt{\pi}\,\Gamma(\gamma)} = 1,$$

and therefore

$$\Phi(x) = f(x).$$

Let us now find $\left(t^\gamma \frac{\partial u}{\partial t}\right)|_{t=0}$. We have

$$\left(t^\gamma \frac{\partial u}{\partial t}\right)\Big|_{t=0} = (1-\gamma)2^{1-\gamma}C(2-\gamma)\Psi(x)\int_0^1 (p(1-p))^{-\frac{\gamma}{2}}\,dp = g(x),$$

$$\int_0^1 (p(1-p))^{-\frac{\gamma}{2}}\,dp = \frac{\Gamma\left(1-\frac{\gamma}{2}\right)^2}{\Gamma(2-\gamma)}.$$

Using the Legendre duplication formula (1.7), we obtain

$$(1-\gamma)2^{1-\gamma}C(2-\gamma)\int\limits_0^1 (p(1-p))^{-\frac{\gamma}{2}}\,dp$$
$$=(1-\gamma)2^{1-\gamma}\,\frac{\Gamma\left(\frac{3-\gamma}{2}\right)}{\sqrt{\pi}\,\Gamma\left(1-\frac{\gamma}{2}\right)}\,\frac{\Gamma\left(1-\frac{\gamma}{2}\right)^2}{\Gamma(2-\gamma)}$$
$$=(1-\gamma)\frac{2^{1-\gamma}\Gamma\left(\frac{3-\gamma}{2}\right)}{\sqrt{\pi}}\,\frac{\Gamma\left(1-\frac{\gamma}{2}\right)}{\Gamma(2-\gamma)}=1-\gamma,$$

and then

$$\Psi(x)=\frac{g(x)}{1-\gamma}.$$

We obtain that the solution to (3.130)–(3.131) for $0<k<1$ is

$$u=2^{\gamma-1}C(\gamma)\int\limits_0^1 \frac{f(x+t(2p-1))}{(p(1-p))^{1-\frac{\gamma}{2}}}\,dp$$
$$+\frac{2^{1-\gamma}C(2-\gamma)}{1-\gamma}t^{1-\gamma}\int\limits_0^1 \frac{g(x+t(2p-1))}{(p(1-p))^{\frac{\gamma}{2}}}\,dp. \tag{3.136}$$

Putting in (3.136) $t(2p-1)=z$, we get

$$u=\frac{C(\gamma)}{t^{\gamma-1}}\int\limits_{-t}^{t} f(x+z)(t^2-z^2)^{\frac{\gamma}{2}-1}\,dz$$
$$+\frac{C(2-\gamma)}{1-\gamma}\int\limits_{-t}^{t} g(x+z)(t^2-z^2)^{-\frac{\gamma}{2}}\,dp$$
$$=\frac{2C(\gamma)}{t^{\gamma-1}}\int\limits_0^t \frac{f(x+z)+f(x-z)}{2}(t^2-z^2)^{\frac{\gamma}{2}-1}\,dz$$
$$+\frac{2C(2-\gamma)}{1-\gamma}\int\limits_0^t \frac{g(x+z)+g(x-z)}{2}(t^2-z^2)^{-\frac{\gamma}{2}}\,dp$$
$$=\mathcal{P}_t^{\gamma}\left[\frac{f(x+t)+f(x-t)}{2}\right]+t^{1-\gamma}\mathcal{P}_t^{2-\gamma}\left[\frac{g(x+t)+g(x-t)}{2}\right].\qquad\square$$

3.4.2 Multi-dimensional Poisson operator

In this section, we consider the multi-dimensional Poisson operator and calculate two integrals $\int\limits_{S_1^+(n)} \mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\, dS$ and $\int\limits_{S_1^+(n)} \mathbf{i}_\gamma(r\theta,\xi)\theta^\gamma\, dS$.

Definition 23. *The multi-dimensional Poisson operator $\mathbf{P}_x^\gamma$ acts on the integrable function f by the formula*

$$\mathbf{P}_x^\gamma f(x){=}C(\gamma)\int\limits_0^\pi ... \int\limits_0^\pi f(x_1\cos\alpha_1,...,x_n\cos\alpha_n)\prod_{i=1}^n \sin^{\gamma_i-1}\alpha_i\, d\alpha_i, \qquad (3.137)$$

where

$$C(\gamma)=\pi^{-\frac{n}{2}}\prod_{i=1}^n\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}$$

such that $\mathbf{P}_x^\gamma[1]=1$.

From (3.123) we get the statement.

Statement 4. *For $f\in C_{ev}^2$ operator (3.137) acts as a transmutation operator by the formula*

$$\mathbf{P}_x^\gamma\Delta f=\Delta_\gamma\mathbf{P}_x^\gamma f,$$

where Δ is the Laplace operator and $\Delta_\gamma=\sum\limits_{i=1}^n B_{\gamma_i}$.

From formulas (3.125) and (3.126) it is easy to obtain representations for (1.30) and (1.31) of the form

$$\mathbf{j}_\gamma(x,\xi)=\mathbf{P}_\xi^\gamma[e^{-i\langle x,\xi\rangle}] \qquad (3.138)$$

and

$$\mathbf{i}_\gamma(x,\xi)=\mathbf{P}_\xi^\gamma[e^{\pm\langle x,\xi\rangle}], \qquad (3.139)$$

where $\langle x,\xi\rangle=\sum\limits_{i=1}^n x_i\xi_i$.

The part of a sphere of radius r with center at the origin belonging to $\mathbb{R}_+^n$ we will denote by $S_r^+(n)$:

$$S_r^+(n){=}\{x\in\mathbb{R}_+^n : |x|{=}r\}.$$

We obtain the formulas expressing the weight integrals on the part of the sphere $S_1^+(n)$ of functions (3.138) and (3.139).

Statement 5. *The integral* $\int\limits_{S_1^+(n)} \mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS$ *is calculated by the formula*

$$\int\limits_{S_1^+(n)} \mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\,j_{\frac{n+|\gamma|}{2}-1}(r|\xi|). \tag{3.140}$$

Proof. Using formula (3.138) we can write

$$\int\limits_{S_1^+(n)} \mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS = \int\limits_{S_1^+(n)} \mathbf{P}_\xi^\gamma\left(e^{-ir\langle\theta,\xi\rangle}\right)\theta^\gamma\,dS.$$

Applying formula (3.143) to the last integral we obtain

$$\int\limits_{S_1^+(n)} \mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\int\limits_{-1}^{1} e^{-irp|\xi|}(1-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp.$$

Replacing p by $-p$ we get

$$\int\limits_{-1}^{1} e^{-irp|\xi|}(1-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp = \int\limits_{-1}^{1} e^{irp|\xi|}(1-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp.$$

The last integral is found by formula (2.3.5.3) from [455] of the form

$$\int\limits_{-a}^{a} e^{itp}(a^2-p^2)^{\beta-1}\,dp = \frac{\sqrt{\pi}\,(2a)^{\beta-\frac{1}{2}}\Gamma(\beta)}{t^{\beta-\frac{1}{2}}}\,J_{\beta-\frac{1}{2}}(at). \tag{3.141}$$

Therefore

$$\begin{aligned}
&\int\limits_{S_1^+(n)} \mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma\,dS \\
&= \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)}\,\frac{\sqrt{\pi}2^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)}{(r|\xi|)^{\frac{n+|\gamma|}{2}-1}}\,J_{\frac{n+|\gamma|}{2}-1}(r|\xi|) \\
&= \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\,j_{\frac{n+|\gamma|}{2}-1}(r|\xi|),
\end{aligned}$$

which gives (3.140). The proof is complete. □

Statement 6. *The integral* $\int\limits_{S_1^+(n)} \mathbf{i}_\gamma(r\theta,\xi)\theta^\gamma\,dS$ *is calculated by the formula*

$$\int\limits_{S_1^+(n)} \mathbf{i}_\gamma(r\theta,\xi)\theta^\gamma\,dS = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\, i_{\frac{n+|\gamma|}{2}-1}(r|\xi|). \tag{3.142}$$

Proof. Using (3.139) we obtain

$$\int\limits_{S_1^+(n)} \mathbf{i}_\gamma(r\theta,\xi)\theta^\gamma\,dS = \int\limits_{S_1^+(n)} \mathbf{P}_\xi^\gamma\left(-e^{r\langle\theta,\xi\rangle}\right)\theta^\gamma\,dS.$$

Applying formula (3.143) we get

$$\int\limits_{S_1^+(n)} \mathbf{i}_\gamma(r\theta,\xi)\theta^\gamma\,dS = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\,2^{n-1}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\int\limits_{-1}^{1} e^{-rp|\xi|}(1-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp.$$

The last integral is found by formula (2.3.5.1) from [455] of the form

$$\int\limits_{-a}^{a} e^{-tp}(a^2-p^2)^{\beta-1}\,dp = \frac{\sqrt{\pi}\,(2a)^{\beta-\frac{1}{2}}\Gamma(\beta)}{t^{\beta-\frac{1}{2}}}\, I_{\beta-\frac{1}{2}}(at).$$

So

$$\int\limits_{S_1^+(n)} \mathbf{i}_\gamma(r\theta,\xi)\theta^\gamma\,dS = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\, i_{\frac{n+|\gamma|}{2}-1}(r|\xi|),$$

which gives (3.142). The proof is complete. □

Now we present a known result received by I. A. Kipriyanov and L. A. Ivanov in [247] (see also [242,251] for the particular case). Since this article is difficult to get and it is not translated into English, we present here a formula with proof.

The function of the scalar product $f(\langle\xi,x\rangle)$ is usually called the function of the "plane wave" type, since it gives a constant value on the plane $\langle\xi,x\rangle = p$. We will call function $\mathbf{P}_\xi^\gamma f(\langle\xi,x\rangle)$ the function of the "weight plane wave" type because in the corresponding integral expressions it will become a function of an ordinary plane wave.

Theorem 35. *Let $f(s)$ be the integrable function on $(-1,1)$ of one variable. Then the following formula is valid:*

$$\int\limits_{S_1^+(n)} \mathbf{P}_\xi^\gamma f(\langle\sigma,\xi\rangle)\sigma^\gamma dS_\sigma = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)} \int\limits_{-1}^1 f(|\xi|p)(1-p^2)^{\frac{n+|\gamma|-3}{2}} dp. \tag{3.143}$$

Proof. Let us consider the integral

$$\mathcal{J} = \int\limits_{B_R^+(n)} \mathbf{P}_\xi^\gamma f(\langle\xi,x\rangle)x^\gamma\, dx,$$

where $B_R^+(n)=\{x : |x|<R; x_1\dots x_n>0\}$, $\xi\in\mathbb{R}_+^n$. On this integral $\mathcal{J}$ we introduce the new coordinates:

$$\widetilde{x}_1 = x_1\cos\alpha_1, \qquad \widetilde{x}_2 = x_1\sin\alpha_1,$$

$$\widetilde{x}_3 = x_2\cos\alpha_2, \qquad \widetilde{x}_4 = x_2\sin\alpha_2, \dots,$$

$$\widetilde{x}_{2n-1} = x_n\cos\alpha_n, \qquad \widetilde{x}_{2n} = x_n\sin\alpha_n.$$

In these coordinates

$$\mathcal{J} = C(\gamma) \int\limits_{\widetilde{B}_R^+(2n)} f(\langle\widetilde{\xi}',\widetilde{x}\rangle)\prod_{i=1}^n \widetilde{x}_{2i}^{\gamma_i-1}\, d\widetilde{x},$$

where $\widetilde{x}=(\widetilde{x}_1,\dots,\widetilde{x}_{2n})\in\mathbb{R}^{2n}$, $\widetilde{x}_{2i}>0$, $i=\overline{1,n}$, $\widetilde{\xi}'=(\xi_1,0,\xi_2,0,\dots,\xi_n,0)\in\mathbb{R}^{2n}$, $\widetilde{B}_R^+(2n)=\{\widetilde{x} : |\widetilde{x}|<R; \widetilde{x}_{2i}>0, i=\overline{1,n}\}$.

Now we consider under the integral $\mathcal{J}$ the function of the "plane wave" type. Selecting integration by $\widetilde{x}_1=p$ and noting that $\langle\widetilde{\xi}',\widetilde{x}\rangle = p|\xi|$, we obtain

$$\mathcal{J} = C(\gamma)\int\limits_{-R}^{R} f(p|\xi|)\, dp \int\limits_{\widetilde{B}^+_{\sqrt{R^2-p^2}}(2n-1)} \prod_{i=1}^n \widetilde{x}_{2i}^{\gamma_i-1}\, d\widetilde{x}_2\dots d\widetilde{x}_{2n},$$

where $\widetilde{B}^+_{\sqrt{R^2-p^2}}(2n-1)$ is a part of a ball centered at the origin with the radius $\sqrt{R^2-p^2}$ in $\mathbb{R}^{2n-1}$ with $\widetilde{x}_{2i}>0$, $i=\overline{1,n}$. Now we introduce a spherical coordinate transformation in the inner integral:

$$\widetilde{x}_1 = \rho\vartheta_1, \qquad \widetilde{x}_2 = \rho\vartheta_2, \qquad \dots, \qquad \widetilde{x}_{n-1} = \rho\vartheta_{n-1}$$

with Jacobian $I = \rho^{2n-2}$. In terms of the new coordinates we obtain

$$\mathcal{J} = C(\gamma) \int\limits_{-R}^{R} f(p|\xi|)\, dp \int\limits_{\widetilde{B}^+_{\sqrt{R^2-p^2}}(2n-1)} \rho^{2n-2} \prod_{i=1}^{n} \rho^{\gamma_i - 1} \vartheta_{2i}^{\gamma_i - 1}\, d\rho\, dS$$

$$= C(\gamma) \int\limits_{-R}^{R} f(p|\xi|)\, dp \int\limits_{0}^{\sqrt{R^2-p^2}} \rho^{n+|\gamma|-2}\, d\rho \int\limits_{\widetilde{S}^+_1(2n-1)} \prod_{i=1}^{n} \vartheta_{2i}^{\gamma_i - 1}\, dS,$$

where $\widetilde{S}^+_1(2n-1)$ is a part of a unit sphere centered at the origin in $\mathbb{R}^{2n-1}$ with $\widetilde{x}_{2i} > 0$, $i = \overline{1, n}$.

Applying formula (1.107) we get

$$\int\limits_{\widetilde{S}^+_1(2n-1)} \prod_{i=1}^{n} \vartheta_{2i}^{\gamma_i - 1}\, dS = \frac{\Gamma^{n-1}\left(\frac{1}{2}\right) \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)} = \frac{\pi^{\frac{n-1}{2}} \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)}$$

$$= |\widetilde{S}^+_1(2n-1)|_{|\gamma|-1}$$

and

$$\int\limits_{0}^{\sqrt{R^2-p^2}} \rho^{n+|\gamma|-2}\, d\rho = \frac{(R^2 - p^2)^{\frac{n+|\gamma|-1}{2}}}{n + |\gamma| - 1}.$$

So we obtain

$$\mathcal{J} = \int\limits_{B^+_R(n)} \mathbf{P}^{\gamma}_{\xi} f(\langle \xi, x\rangle) x^{\gamma}\, dx$$

$$= \frac{C(\gamma)}{n + |\gamma| - 1} |\widetilde{S}^+_1(2n-1)|_{|\gamma|-1} \int\limits_{-R}^{R} f(|\xi|p)(R^2 - p^2)^{\frac{n+|\gamma|-1}{2}}\, dp.$$

Now on the left side of this equality turning to the spherical coordinates

$$x_1 = \rho\vartheta_1 \ldots x_n = \rho\vartheta_n,$$

we get

$$\mathcal{J} = \int\limits_0^R \rho^{n+|\gamma|-1}\,d\rho \int\limits_{S_1^+(n)} \mathbf{P}_\xi^\gamma f(\langle\vartheta,\xi\rangle)\vartheta^\gamma\,dS$$

$$= \frac{C(\gamma)}{n+|\gamma|-1}|\widetilde{S}_1^+(2n-1)|_{|\gamma|-1} \int\limits_{-R}^{R} f(|\xi|p)(R^2-p^2)^{\frac{n+|\gamma|-1}{2}}\,dp.$$

Differentiating the last equality by R we write

$$R^{n+|\gamma|-1} \int\limits_{S_1^+(n)} \mathbf{P}_\xi^\gamma f(\langle\vartheta,\xi\rangle)\vartheta^\gamma\,dS$$

$$= \frac{C(\gamma)}{n+|\gamma|-1}|\widetilde{S}_1^+(2n-1)|_{|\gamma|-1}\, 2\,\frac{n+|\gamma|-1}{2}$$

$$\times \int\limits_{-R}^{R} f(|\xi|p)R(R^2-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp$$

$$= C(\gamma)|\widetilde{S}_1^+(2n-1)|_{|\gamma|-1} \int\limits_{-R}^{R} f(|\xi|p)R(R^2-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp.$$

Putting $R=1$, we get

$$\int\limits_{S_1^+(n)} \mathbf{P}_\xi^\gamma f(\langle\vartheta,\xi\rangle)\vartheta^\gamma\,dS = C(\gamma)|\widetilde{S}_1^+(2n-1)|_{|\gamma|-1} \int\limits_{-1}^{1} f(|\xi|p)(1-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp,$$

where

$$C(\gamma)|\widetilde{S}_1^+(2n-1)|_{|\gamma|-1} = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)}.$$

That gives (3.143).

□

Example 5. *Let us consider the case when $f(t)=|t|^k$, $k>-1$. We have*

$$\int\limits_{S_1^+(n)} \mathbf{P}_\xi^\gamma |\langle\vartheta,\xi\rangle|^k\vartheta^\gamma\,dS = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)} \int\limits_{-1}^{1} |\xi|^k|p|^k(1-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp$$

$$= \frac{\prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)} \frac{\Gamma\left(\frac{n+|\gamma|-1}{2}\right)\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|+k}{2}\right)}|\xi|^k$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)\prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{n+|\gamma|+k}{2}\right)} |\xi|^k.$$

3.4.3 Generalized translation

In this section we consider the transmutation operator called the *generalized translation*.

Definition 24. *Let $f = f(x)$, $x \in \mathbb{R}$, $\gamma > 0$. The* ***generalized translation*** *is defined by the equality*

$$({}^\gamma T_x^y f)(x) = {}^\gamma T_x^y f = C(\gamma) \int\limits_0^\pi f(\sqrt{x^2 + y^2 - 2xy\cos\varphi}) \sin^{\gamma-1}\varphi\, d\varphi, \quad (3.144)$$

where $C(\gamma) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)}$. For $\gamma = 0$ the generalized translation ${}^\gamma T_x^y$ is

$$ {}^0T_x^y = T_x^y f(x) = \frac{f(x+y) - f(x-y)}{2}.$$

The generalized translation ${}^\gamma T_x^y$ was introduced in the paper [83] and then studied in detail in [317], see also [320]. In particular, in [317] it was shown that $u(x,y) = ({}^\gamma T_x^y f)(x)$ is a unique solution to the Cauchy problem

$$(B_\gamma)_x u(x,y) = (B_\gamma)_y u(x,y), \quad (3.145)$$

$$u(x,0) = f(x), \qquad \left.\frac{\partial}{\partial y} u(x,y)\right|_{y=0} = 0.$$

The operator ${}^\gamma T_x^y$ of function $f \in C^2_{ev}$ is a transmutation operator with the following intertwining property:

$$ {}^\gamma T_x^y (B_\gamma)_x f(x) = (B_\gamma)_y {}^\gamma T_x^y f(x). \quad (3.146)$$

In addition, it satisfies the conditions

$$({}^\gamma T_x^y f)(x)|_{y=0} = f(x), \qquad \left.\frac{\partial}{\partial y}({}^\gamma T_x^y f)(x)\right|_{y=0} = 0. \quad (3.147)$$

Replacing a variable $\varphi \to \pi - \varphi$ it is easy to see that ${}^\gamma T_x^y$ can be written in the form

$$({}^\gamma T_x^y f)(x) = {}^\gamma T_x^y f(x) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} \int\limits_0^\pi f(\sqrt{x^2+y^2+2xy\cos\varphi}) \sin^{\gamma-1}\varphi d\varphi. \tag{3.148}$$

Let us present the elementary properties of the generalized translation from [317].

Elementary properties of the generalized translation

1. Linearity and uniformity:

$$ {}^\gamma T_x^y [af(x) + bg(x)] = a^\gamma T_x^y f(x) + b^\gamma T^y g(x), \qquad a, b \in \mathbb{R}.$$

2. Nonnegativity: ${}^\gamma T_x^y f(x) \geq 0$ if $f(x) \geq 0$.
3. ${}^\gamma T_x^y [1] = 1$.
4. ${}^\gamma T_x^0 f(x) = f(x)$.
5. ${}^\gamma T_x^y f(x) = {}^\gamma T_y^x f(y)$.
6. If $f(x) \equiv 0$ for $x \geq a$, then ${}^\gamma T_x^y f(x) \equiv 0$ for $|x - y| \geq a$.
7. If a sequence of continuous functions $f_n(x)$ converges uniformly in each finite interval to $f(x)$, then the sequence of functions of the two variables ${}^\gamma T_x^y f_n(x)$ converges uniformly in each finite area to ${}^\gamma T_x^y f(x)$.
8. Operator ${}^\gamma T_x^y$ is bounded:

$$|{}^\gamma T_x^y f(x)| \leq {}^\gamma T_x^y |f(x)| \leq \sup_{x\geq 0} |f(x)|. \tag{3.149}$$

9. Commutativity of operators ${}^\gamma T_x^y$:

$$ {}^\gamma T_x^y {}^\gamma T_x^z f(x) = {}^\gamma T_x^z {}^\gamma T_x^y f(x). \tag{3.150}$$

10. Associativity of operators ${}^\gamma T_x^y$:

$$ {}^\gamma T_y^z {}^\gamma T_x^y f(x) = {}^\gamma T_x^z {}^\gamma T_x^y f(x). \tag{3.151}$$

11. The function $j_{\frac{\gamma-1}{2}}$ is an eigenfunction of the generalized translation ${}^\gamma T_x^y$:

$$ {}^\gamma T_x^y j_{\frac{\gamma-1}{2}}(x\xi) = j_{\frac{\gamma-1}{2}}(x\xi)\, j_{\frac{\gamma-1}{2}}(y\xi). \tag{3.152}$$

We now give some more properties of generalized translation.

Statement 7. *For the generalized translation operator ${}^\gamma T_x^y$ the representation*

$$ {}^\gamma T_x^y f(x) = 2^{\gamma-1} C(\gamma) \int\limits_0^1 f\left((x+y)\sqrt{1 - \frac{4xy}{(x+y)^2} z}\right) \times z^{\frac{\gamma}{2}-1}(1-z)^{\frac{\gamma}{2}-1} dz \tag{3.153}$$

is valid.

Proof. We transform the generalized translation operator as follows. First putting $\varphi = 2\alpha$ in (3.148) we obtain

$$
\begin{aligned}
{}^{\gamma}T_x^y f(x) &= 2C(\gamma)\int\limits_0^{\pi/2} f(\sqrt{x^2+y^2+2xy\cos 2\alpha})\sin^{\gamma-1}(2\alpha)d\alpha \\
&= 2^{\gamma}C(\gamma)\int\limits_0^{\pi/2} f\left(\sqrt{x^2+y^2+2xy(\cos^2\alpha-\sin^2\alpha)}\right) \\
&\quad \times \sin^{\gamma-1}\alpha\cos^{\gamma-1}\alpha d\alpha \\
&= 2^{\gamma}C(\gamma)\int\limits_0^{\pi/2} f\left(\sqrt{x^2+y^2+2xy(1-2\sin^2\alpha)}\right) \\
&\quad \times \sin^{\gamma-1}\alpha(1-\sin^2\alpha)^{\frac{\gamma-1}{2}}d\alpha.
\end{aligned}
$$

Now let $\sin\alpha = t$. Then α=0 for t=0, α=$\pi/2$ for t=1, $d\alpha=\frac{dt}{(1-t^2)^{1/2}}$, and

$$
\begin{aligned}
{}^{\gamma}T_x^y f(x) &= 2^{\gamma}C(\gamma)\int\limits_0^1 f(\sqrt{x^2+y^2+2xy(1-2t^2)})t^{\gamma-1}(1-t^2)^{\frac{\gamma}{2}-1}dt \\
&= \{t^2 = z\} \\
&= 2^{\gamma-1}C(\gamma)\int\limits_0^1 f(\sqrt{x^2+y^2+2xy(1-2z)})z^{\frac{\gamma}{2}-1}(1-z)^{\frac{\gamma}{2}-1}dz \\
&= 2^{\gamma-1}C(\gamma)\int\limits_0^1 f(\sqrt{(x+y)^2-4xyz})z^{\frac{\gamma}{2}-1}(1-z)^{\frac{\gamma}{2}-1}dz \\
&= 2^{\gamma-1}C(\gamma)\int\limits_0^1 f\left((x+y)\sqrt{1-\frac{4xy}{(x+y)^2}z}\right)z^{\frac{\gamma}{2}-1}(1-z)^{\frac{\gamma}{2}-1}dz.
\end{aligned}
$$

The proof is complete. $\square$

Statement 8. *For the generalized translation operator ${}^{\gamma}T_x^y$ the representation*

$$
({}^{\gamma}T_x^y f)(x)=\frac{2^{\gamma}C(\gamma)}{(4xy)^{\gamma-1}}\int\limits_{|x-y|}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz \tag{3.154}
$$

is valid.

Proof. Changing the variable φ to 2α in (3.144), we obtain

$$
\begin{aligned}
{}^{\gamma}T_x^y f(x) &= 2C(\gamma)\int\limits_0^{\pi/2} f(\sqrt{x^2+y^2-2xy\cos 2\alpha})\sin^{\gamma-1}(2\alpha)d\alpha \\
&= 2^{\gamma}C(\gamma)\int\limits_0^{\pi/2} f\left(\sqrt{x^2+y^2-2xy(\cos^2\alpha-\sin^2\alpha)}\right) \\
&\quad \times \sin^{\gamma-1}\alpha\cos^{\gamma-1}\alpha d\alpha \\
&= 2^{\gamma}C(\gamma)\int\limits_0^{\pi/2} f\left(\sqrt{x^2+y^2-2xy(1-2\sin^2\alpha)}\right) \\
&\quad \times \sin^{\gamma-1}\alpha(1-\sin^2\alpha)^{\frac{\gamma-1}{2}}d\alpha.
\end{aligned}
$$

Now putting $\sin\alpha{=}t$ we get for $\alpha{=}0$, $t{=}0$, for $\alpha{=}\pi/2$, $t{=}1$, $d\alpha{=}\frac{dt}{(1-t^2)^{1/2}}$, and

$$
{}^{\gamma}T_x^y f(x) = 2^{\gamma}C(\gamma)\int\limits_0^1 f(\sqrt{x^2+y^2-2xy(1-2t^2)})t^{\gamma-1}(1-t^2)^{\frac{\gamma}{2}-1}dt.
$$

Introducing the variable z by the equality $\sqrt{x^2+y^2-2xy(1-2t^2)}=z$, we obtain

$$
t=\left(\frac{z^2-(x-y)^2}{4xy}\right)^{1/2}, \qquad dt=\frac{zdz}{(4xy)^{1/2}(z^2-(x-y)^2)^{1/2}},
$$

$z=|x-y|$ when $t=0$, $z=x+y$ when $t=1$, and

$$
{}^{\gamma}T_x^y f(x) = \frac{2^{\gamma}C(\gamma)}{(4xy)^{\gamma-1}}\int\limits_{|x-y|}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz.
$$

The proof is complete. □

Statement 9. *The generalized translation ${}^{\gamma}T_x^y$ can be written in the form*

$$
{}^{\gamma}T_x^y f(x) = \frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int\limits_{|x-y|}^{x+y} z^{2-\gamma}f(z)dz\int\limits_0^{\infty} j_{\frac{\gamma-1}{2}}(\lambda x)j_{\frac{\gamma-1}{2}}(\lambda y)j_{\frac{\gamma-1}{2}}(\lambda z)\lambda^{\gamma}d\lambda. \tag{3.155}
$$

Proof. The next formula is valid:

$$\frac{[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}}{(xy)^{\gamma-1}}$$
$$= z^{1-\gamma}\frac{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)}{2\Gamma^3\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(\lambda x)j_{\frac{\gamma-1}{2}}(\lambda y)j_{\frac{\gamma-1}{2}}(\lambda z)\lambda^\gamma d\lambda. \tag{3.156}$$

This formula follows from formula (2.12.42.14) from [456], p. 204, of the form

$$\int\limits_0^\infty \lambda^{1-\nu}J_\nu(x\lambda)J_\nu(y\lambda)J_\nu(z\lambda)d\lambda$$
$$= \frac{2^{1-3\nu}}{\sqrt{\pi}(xyz)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\nu-\frac{1}{2}}$$
$$= \frac{2^{\nu-1}\Delta^{2\nu-1}}{\sqrt{\pi}(xyz)^\nu\Gamma\left(\nu+\frac{1}{2}\right)},$$

where $|x-y|<z<x+y$, $x,y,z>0$, $\operatorname{Re}\nu>-\frac{1}{2}$, and Δ is the area of a triangle whose sides are equal to x, y, and z. Using (3.154) and (3.156) we get (3.155). □

Statement 9 is given in [220].

Statement 10. *If $f(x)$ is a continuous function such that*

$$\int\limits_0^\infty |f(x)|x^\gamma dx<\infty$$

and $g(x)$ is continuous and bounded for all $x\geq 0$, then

$$\int\limits_0^\infty {}^\gamma T_x^y f(x)g(y)y^\gamma dy = \int\limits_0^\infty f(y)^\gamma T_x^y g(x)y^\gamma dy. \tag{3.157}$$

Proof. Applying to $\int\limits_0^\infty {}^\gamma T_x^y f(x)g(y)y^\gamma dy$ representation (3.154), we obtain

$$\int\limits_0^\infty {}^\gamma T_x^y f(x)g(y)y^\gamma dy$$
$$= (4x)^{1-\gamma}2^\gamma C(\gamma)\int\limits_0^\infty yg(y)dy$$

$$
\times \int\limits_{|x-y|}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz
$$

$$
= (4x)^{1-\gamma}2^{\gamma}C(\gamma)\Bigg[\int\limits_0^x yg(y)dy\int\limits_{x-y}^{x+y} zf(z)
$$

$$
\times [(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz
$$

$$
+\int\limits_x^{\infty} yg(y)dy\int\limits_{y-x}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz\Bigg].
$$

Converting an expression $(z^2-(x-y)^2)((x+y)^2-z^2)$ and changing the order of integration we get

$$
\int\limits_0^{\infty} {}^{\gamma}T_x^y f(x)g(y)y^{\gamma}dy
$$

$$
= (4x)^{1-\gamma}2^{\gamma}C(\gamma)\Bigg[\int\limits_0^x zf(z)dz
$$

$$
\times\int\limits_{x-z}^{x+z} yg(y)[((z+x)^2-y^2)(y^2-(z-x)^2)]^{\frac{\gamma}{2}-1}dy
$$

$$
+\int\limits_x^{\infty} zf(z)dz\int\limits_{z-x}^{x+z} yg(y)[((z+x)^2-y^2)(y^2-(z-x)^2)]^{\frac{\gamma}{2}-1}dy\Bigg]
$$

$$
=\int\limits_0^{\infty} f(z)^{\gamma}T_x^z g(y)z^{\gamma}dz.
$$

The commutation is proved. □

Statement 10 was proved in [317] in another way.

Statement 11. *The Hankel transform from generalized translation of the function $f\in S_{ev}(\mathbb{R}_+)$ has a form*

$$
F_{\gamma}[{}^{\gamma}T_x^y f(x)](\xi) = j_{\frac{\gamma-1}{2}}(y\xi)\,F_{\gamma}[f](\xi). \tag{3.158}
$$

Proof. Using the property of self-adjointness of the generalized translation and (3.152), we obtain

$$
\begin{aligned}
F_\gamma[{}^\gamma T_x^y f(x)](\xi) &= \int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\, {}^\gamma T_x^y f(x) x^\gamma\, dx \\
&= \int\limits_0^\infty {}^\gamma T_x^y j_{\frac{\gamma-1}{2}}(x\xi)\, f(x) x^\gamma\, dx \\
&= j_{\frac{\gamma-1}{2}}(y) \int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\, f(x) x^\gamma\, dx \\
&= j_{\frac{\gamma-1}{2}}(y\xi)\, F_\gamma[f](\xi).
\end{aligned}
$$

□

Statement 11 was given in [610].

We obtain the formulas for the action of a generalized translation on some elementary and special functions.

(1) For $x>0$ the formula representing a generalized translation ${}^\gamma T_x^y$ of power function x^α is

$$
{}^\gamma T_x^y x^\alpha = \begin{cases} |x-y|^\alpha\, {}_2F_1\left(-\frac{\alpha}{2}, \frac{\gamma}{2}, \gamma; -\frac{4xy}{(x-y)^2}\right) & x \neq y, \\ x^\alpha \dfrac{2^{\alpha+\gamma-1}\Gamma\left(\frac{\gamma+\alpha}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\gamma+\frac{\alpha}{2}\right)} & x = y \end{cases} \tag{3.159}
$$

or

$$
{}^\gamma T_x^y x^\alpha = \begin{cases} x^\alpha\, {}_2F_1\left(-\frac{\alpha}{2}, \frac{1-\alpha-\gamma}{2}, \frac{\gamma+1}{2}; \frac{y^2}{x^2}\right) & x > y, \\ x^\alpha \dfrac{2^{\alpha+\gamma-1}\Gamma\left(\frac{\gamma+\alpha}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\gamma+\frac{\alpha}{2}\right)} & x = y, \\ y^\alpha\, {}_2F_1\left(-\frac{\alpha}{2}, \frac{1-\alpha-\gamma}{2}, \frac{\gamma+1}{2}; \frac{x^2}{y^2}\right) & x < y, \end{cases} \tag{3.160}
$$

where ${}_2F_1$ is a Gaussian hypergeometric function (1.33).

Proof. Let first $x \neq y$. Using formula (3.153) let us find ${}^\gamma T_x^y$ of x^α. We have

$$
{}^\gamma T_x^y x^\alpha = \frac{2^{\gamma-1}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} |x-y|^\alpha \int\limits_0^1 \left(1+\frac{4xy}{(x-y)^2} z\right)^{\frac{\alpha}{2}} (1-z)^{\frac{\gamma}{2}-1} z^{\frac{\gamma}{2}-1} dz.
$$

The last integral is a Gaussian hypergeometric function (1.33) for $z = -\frac{4xy}{(x-y)^2}$, $a = -\frac{\alpha}{2}$, $b = \frac{\gamma}{2}$, $c = 2b = \gamma$ $(c > b > 0)$, thus

$$
{}^\gamma T_x^y x^\alpha = \frac{2^{\gamma-1}\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}{\sqrt{\pi}\,\Gamma(\gamma)} |x-y|^\alpha\, {}_2F_1\left(-\frac{\alpha}{2}, \frac{\gamma}{2}, \gamma; -\frac{4xy}{(x-y)^2}\right).
$$

Using the doubling formula for the gamma function (1.7), we obtain (3.159).

For the proof of (3.160) we use the definition of the absolute value in (3.159) and get

$$
{}^{\gamma}T_x^y x^\alpha = \begin{cases} (x-y)^\alpha \, {}_2F_1\left(-\frac{\alpha}{2}, \frac{\gamma}{2}, \gamma; -\frac{4xy}{(x-y)^2}\right) & x > y, \\ {}_2F_1\left(-\frac{\alpha}{2}, \frac{\gamma}{2}, \gamma; -\frac{4xy}{(x-y)^2}\right) & x < y. \end{cases} \tag{3.161}
$$

In [295] the following formula is given:

$$
{}_2F_1\left(a, b, 2b; \frac{4z}{(1+z)^2}\right) = (1+z)^{2a} \, {}_2F_1\left(a, a-b+\frac{1}{2}, b+\frac{1}{2}; z^2\right),
$$

using which we get (3.160) for $x \neq y$.

For $x = y$ we have

$$
\begin{aligned}
{}^{\gamma}T_x^y x^\alpha &= (2x)^\alpha \frac{2^{\gamma-1}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)} \int\limits_0^1 (1-z)^{\frac{\gamma}{2}-1} z^{\frac{\gamma+\alpha}{2}-1} dz \\
&= (2x)^\alpha \frac{2^{\gamma-1}\Gamma\left(\frac{\gamma+\alpha}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\gamma+\frac{\alpha}{2}\right)}.
\end{aligned}
$$

That completes the proof. □

(2) The generalized translation ${}^{\gamma}T_x^y$ of e^{-x^2}, $x > 0$, is

$$
{}^{\gamma}T_x^y e^{-x^2} = \Gamma\left(\frac{\gamma+1}{2}\right)(xy)^{\frac{1-\gamma}{2}} e^{-x^2-y^2} I_{\frac{\gamma-1}{2}}(2xy). \tag{3.162}
$$

Proof. Using formula (3.154) we obtain

$$
{}^{\gamma}T_x^y e^{-x^2} = \frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}} \int\limits_{|x-y|}^{x+y} z e^{-z^2} [(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1} dz.
$$

We find the integral

$$
\begin{aligned}
I &= \int\limits_{|x-y|}^{x+y} z e^{-z^2} [(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1} dz = \{z^2 = t\} \\
&= \frac{1}{2} \int\limits_{(x-y)^2}^{(x+y)^2} e^{-t} [(t-(x-y)^2)((x+y)^2-t)]^{\frac{\gamma}{2}-1} dt = \{t-(x-y)^2 = w\} \\
&= \frac{1}{2} e^{-(x-y)^2} \int\limits_0^{4xy} e^{-w} [w(4xy-w)]^{\frac{\gamma}{2}-1} dw.
\end{aligned}
$$

Applying formula (2.3.6.2) from [455] of the form

$$\int_0^a x^{\alpha-1}(a-x)^{\alpha-1}e^{-px}dx = \sqrt{\pi}\,\Gamma(\alpha)\left(\frac{a}{p}\right)^{\alpha-1/2} e^{-ap/2}I_{\alpha-1/2}(ap/2), \tag{3.163}$$

$$\operatorname{Re}\alpha > 0,$$

we get

$$I = 2^{\gamma-2}\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right) e^{-x^2-y^2}(xy)^{\frac{\gamma-1}{2}} I_{\frac{\gamma-1}{2}}(2xy)\,.$$

Then

$$\begin{aligned} {}^{\gamma}T_x^y e^{-x^2} &= \frac{2^{\gamma}}{(4xy)^{\gamma-1}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\ \Gamma\left(\frac{\gamma}{2}\right)} \\ &\quad \times \int_{|x-y|}^{x+y} ze^{-z^2}[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz \\ &= \frac{2^{\gamma}}{(4xy)^{\gamma-1}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\ \Gamma\left(\frac{\gamma}{2}\right)} 2^{\gamma-2}\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right) e^{-x^2-y^2}(xy)^{\frac{\gamma-1}{2}} I_{\frac{\gamma-1}{2}}(2xy)\,. \end{aligned}$$

After simplification we get (3.162). □

(3) The generalized translation ${}^{\gamma}T_x^y$ of $x^2e^{-x^2}$, $x>0$, is

$$\begin{aligned} {}^{\gamma}T_x^y x^2e^{-x^2} &= \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{e^{x^2+y^2}} \\ &\times \left(\frac{2^{\gamma+1}e^{2xy}\,xy}{\Gamma(\gamma+1)}\,{}_1F_1\left(\frac{\gamma}{2}+1;\gamma+1;-4xy\right) + (xy)^{\frac{1-\gamma}{2}} I_{\frac{\gamma-1}{2}}(2xy)\right). \end{aligned} \tag{3.164}$$

Proof. Using formula (3.154), let us find ${}^{\gamma}T_x^y x^2e^{-x^2}$:

$${}^{\gamma}T_x^y e^{-x^2} = \frac{2^{\gamma}C(\gamma)}{(4xy)^{\gamma-1}} \int_{|x-y|}^{x+y} z^3e^{-z^2}[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz.$$

We find the integral

$$I = \int_{|x-y|}^{x+y} z^3e^{-z^2}[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz = \{z^2=t\}$$

$$
= \frac{1}{2} \int\limits_{(x-y)^2}^{(x+y)^2} t e^{-t} [(t-(x-y)^2)((x+y)^2-t)]^{\frac{\gamma}{2}-1} dt = \{t-(x-y)^2 = w\}
$$

$$
= \frac{1}{2} e^{-(x-y)^2} \int\limits_0^{4xy} (w+(x-y)^2) e^{-w} [w(4xy-w)]^{\frac{\gamma}{2}-1} dw = \frac{1}{2} e^{-(x-y)^2}
$$

$$
\times \left(\int\limits_0^{4xy} e^{-w} w^{\frac{\gamma}{2}} (4xy-w)^{\frac{\gamma}{2}-1} dw + (x-y)^2 \right.
$$

$$
\left. \times \int\limits_0^{4xy} e^{-w} [w(4xy-w)]^{\frac{\gamma}{2}-1} dw \right).
$$

Applying formulas (3.163) and (2.3.6.1) from [455] of the form

$$
\int\limits_0^a x^{\alpha-1} (a-x)^{\beta-1} e^{-px} dx = B(\alpha,\beta) a^{\alpha+\beta-1} {}_1F_1(\alpha;\alpha+\beta;-ap),
$$

$$
\operatorname{Re}\alpha, \qquad \operatorname{Re}\beta > 0,
$$

we get

$$
I = \frac{1}{2} e^{-(x-y)^2} B\left(\frac{\gamma}{2}, \frac{\gamma}{2}+1\right) (4xy)^{\gamma} {}_1F_1\left(\frac{\gamma}{2}+1; \gamma+1; -4xy\right)
$$
$$
+ 2^{\gamma-2} \sqrt{\pi} \Gamma\left(\frac{\gamma}{2}\right) (x-y)^2 e^{-x^2-y^2} (xy)^{\frac{\gamma-1}{2}} I_{\frac{\gamma-1}{2}}(2xy).
$$

Then

$$
{}^{\gamma}T_x^y x^2 e^{-x^2}
$$
$$
= \frac{2^{\gamma}}{(4xy)^{\gamma-1}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\ \Gamma\left(\frac{\gamma}{2}\right)} \int\limits_{|x-y|}^{x+y} z e^{-z^2} [(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1} dz
$$
$$
= \frac{2^{\gamma}}{(4xy)^{\gamma-1}} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\ \Gamma\left(\frac{\gamma}{2}\right)}
$$
$$
\times \left(\frac{1}{2} e^{-(x-y)^2} B\left(\frac{\gamma}{2}, \frac{\gamma}{2}+1\right) (4xy)^{\gamma} {}_1F_1\left(\frac{\gamma}{2}+1; \gamma+1; -4xy\right) \right.
$$
$$
\left. + 2^{\gamma-2} \sqrt{\pi} \Gamma\left(\frac{\gamma}{2}\right) (x-y)^2 e^{-x^2-y^2} (xy)^{\frac{\gamma-1}{2}} I_{\frac{\gamma-1}{2}}(2xy) \right)
$$

$$= \frac{2^{\gamma+1}\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma(\gamma+1)} e^{-(x-y)^2}\, xy\; {}_1F_1\left(\frac{\gamma}{2}+1; \gamma+1; -4xy\right)$$
$$+\Gamma\left(\frac{\gamma+1}{2}\right)(xy)^{\frac{1-\gamma}{2}}\, e^{-x^2-y^2} I_{\frac{\gamma-1}{2}}(2xy)\,.$$

Transforming we get (3.164). □

(4) The generalized translation ${}^{\gamma}T_x^y$ of $j_{\frac{\gamma-1}{2}}(x)$ is

$$ {}^{\gamma}T_x^y\, j_{\frac{\gamma-1}{2}}(x) = j_{\frac{\gamma-1}{2}}(x)\, j_{\frac{\gamma-1}{2}}(y). \tag{3.165}$$

Proof. Using the presentation (1.19) of $j_{\frac{\gamma-1}{2}}(x)$, formulas (3.154) and (2.12.6.1) from [456] of the form

$$\int\limits_a^b x^{1-\nu}(b^2-x^2)^{\nu-\frac{1}{2}}(x^2-a^2)^{\nu-\frac{1}{2}} J_\nu(cx)dx$$
$$= 2^{\nu-1}\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)(b^2-a^2)^\nu c^{-\nu}\, J_\nu\left(\frac{cb+ca}{2}\right) J_\nu\left(\frac{cb-ca}{2}\right),$$
$$0<a<b, \qquad \nu>-\frac{1}{2}, \qquad |\mathrm{Re}\, c|<\pi,$$

we get

$$ {}^{\gamma}T_x^y\, j_{\frac{\gamma-1}{2}}(x) = \frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}} \int\limits_{|x-y|}^{x+y} z j_{\frac{\gamma-1}{2}}(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz$$
$$= \frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}} 2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)$$
$$\times \int\limits_{|x-y|}^{x+y} z^{1-\frac{\gamma-1}{2}} J_{\frac{\gamma-1}{2}}(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz$$
$$= \frac{2^\gamma}{(4xy)^{\gamma-1}} 2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right) 2^{\frac{\gamma-1}{2}-1} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\frac{\gamma}{2}\right)} \sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)(4xy)^{\frac{\gamma-1}{2}}$$
$$\times J_{\frac{\gamma-1}{2}}(x) J_{\frac{\gamma-1}{2}}(y)$$
$$= j_{\frac{\gamma-1}{2}}(x) j_{\frac{\gamma-1}{2}}(y),$$

which completes the proof. □

(5) The generalized translation of $i_{\frac{\gamma-1}{2}}(x)$ is

$$ {}^{\gamma}T_x^y\, i_{\frac{\gamma-1}{2}}(x) = i_{\frac{\gamma-1}{2}}(x) i_{\frac{\gamma-1}{2}}(y). \tag{3.166}$$

Proof. Using the presentation (1.20) of $i_{\frac{\gamma-1}{2}}(x)$ and formulas (3.154) and (2.15.3.13) from [455] of the form

$$\int_a^b x^{1-\nu}(b^2-x^2)^{\nu-\frac{1}{2}}(x^2-a^2)^{\nu-\frac{1}{2}} I_\nu(cx)dx$$
$$= 2^{\nu-1}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)(b^2-a^2)^\nu c^{-\nu} I_\nu\left(\frac{cb+ca}{2}\right) I_\nu\left(\frac{cb-ca}{2}\right),$$
$$a,b>0, \qquad \nu > -\frac{1}{2},$$

we get

$$^\gamma T_x^y i_{\frac{\gamma-1}{2}}(x) = \frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}} \int_{|x-y|}^{x+y} z i_{\frac{\gamma-1}{2}}(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz$$
$$= \frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}} 2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right) \int_{|x-y|}^{x+y} z^{1-\frac{\gamma-1}{2}} I_{\frac{\gamma-1}{2}}(z)$$
$$\times [(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz$$
$$= \frac{2^\gamma}{(4xy)^{\gamma-1}} 2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right) 2^{\frac{\gamma-1}{2}-1} \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\frac{\gamma}{2}\right)} \sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)(4xy)^{\frac{\gamma-1}{2}}$$
$$\times I_{\frac{\gamma-1}{2}}(x) I_{\frac{\gamma-1}{2}}(y)$$
$$= i_{\frac{\gamma-1}{2}}(x) i_{\frac{\gamma-1}{2}}(y),$$

which completes the proof. □

The generalized translation $^\gamma \mathbf{T}_x^y$ responds to the definition of the generalized convolution as it allows one to correctly generalize distributions theory to the case when instead of the second derivative the Bessel operator is used.

Definition 25. *The generalized convolution (one-dimensional), generated by the generalized translation $^\gamma T_x^y$, is (see [242,610])*

$$(f*g)_\gamma(x) = \int_0^\infty f(y)^\gamma T_x^y g(x) y^\gamma \, dy. \tag{3.167}$$

Statement 12. *Let $f, g \in S_{ev}(\mathbb{R}_+)$. The Hankel transform applied to the generalized convolution (3.167) is*

$$F_\gamma[(f*g)_\gamma(x)](\xi) = F_\gamma[f(x)](\xi) F_\gamma[g(x)](\xi). \tag{3.168}$$

Proof. Using properties of generalized translation we have

$$F_\gamma[(f*g)_\gamma(x)](\xi)=\int\limits_0^\infty (f*g)_\gamma(x)\, j_{\frac{\gamma-1}{2}}(x\xi)x^\gamma\, dx$$

$$=\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)x^\gamma\, dx\int\limits_0^\infty f(y)\,{}^\gamma T_x^y g(x)\, y^\gamma\, dy$$

$$=\int\limits_0^\infty f(y)y^\gamma\, dy\int\limits_0^\infty {}^\gamma T_x^y g(x)\, j_{\frac{\gamma-1}{2}}(x\xi)x^\gamma\, dx$$

$$=\int\limits_0^\infty f(y)y^\gamma\, dy\int\limits_0^\infty g(x)\,{}^\gamma T_x^y\, j_{\frac{\gamma-1}{2}}(x\xi)x^\gamma\, dx$$

$$=\int\limits_0^\infty f(y)\, j_{\frac{\gamma-1}{2}}(x\xi)y^\gamma\, dy\int\limits_0^\infty g(x)\, j_{\frac{\gamma-1}{2}}(x\xi)x^\gamma\, dx=F_\gamma[f](\xi)F_\gamma[g](\xi). \qquad \square$$

Statement 12 is given in [610].

If in the definitions and statements of classical harmonic analysis we replace the ordinary shift with a generalized translation, then we get **weighted harmonic analysis** (see, for example, [242,348,442–444,610]) due to the power weight x^γ under the sign of integrals.

Definition 26. *The* ***multi-dimensional generalized translation*** *is defined by the equality*

$$({}^\gamma\mathbf{T}_x^y f)(x)={}^\gamma\mathbf{T}_x^y f(x)=({}^{\gamma_1}T_{x_1}^{y_1}\ldots{}^{\gamma_n}T_{x_n}^{y_n} f)(x), \tag{3.169}$$

where each one-dimensional generalized translation ${}^{\gamma_i}T_{x_i}^{y_i}$ *acts for* $i=1,\ldots,n$ *according to*

$$({}^{\gamma_i}T_{x_i}^{y_i} f)(x)=\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma_i}{2}\right)}$$
$$\times\int\limits_0^\pi f\left(x_1,\ldots,x_{i-1},\sqrt{x_i^2+\tau_i^2-2x_iy_i\cos\varphi_i},x_{i+1},\ldots,x_n\right)\,\sin^{\gamma_i-1}\varphi_i\, d\varphi_i.$$

Obviously, the properties of a one-dimensional generalized translation are transferred to the multi-dimensional one. So, for example, the following equalities are true

$${}^\gamma\mathbf{T}_x^y\,\mathbf{j}_\gamma(x;\xi)=\mathbf{j}_\gamma(x;\xi)\mathbf{j}_\gamma(y;\xi), \tag{3.170}$$

$$ {}^{\gamma}\mathbf{T}_x^y \, \mathbf{i}_\gamma(x;\xi) = \mathbf{i}_\gamma(x;\xi)\mathbf{i}_\gamma(y;\xi), \tag{3.171}$$

where $\mathbf{j}_\gamma(x;\xi) = \prod\limits_{i=1}^{n} j_{\frac{\gamma_i-1}{2}}(x_i\xi_i)$, $\mathbf{i}_\gamma(x;\xi) = \prod\limits_{i=1}^{n} i_{\frac{\gamma_i-1}{2}}(x_i\xi_i)$, $\gamma_1>0,...,\gamma_n>0$, function j_ν is given by (1.19), and i_ν is given by (1.20).

From Statement 10 if follows that if $f(x)$ is continuous on $\mathbb{R}^n_+$ such that

$$\int\limits_0^\infty |f(x)|x^\gamma dx < \infty$$

and $g(x)$ is continuous and bounded on $\mathbb{R}^n_+$, then

$$\int\limits_{\mathbb{R}^n_+} {}^{\gamma}\mathbf{T}_x^y f(x) g(y) y^\gamma dy = \int\limits_{\mathbb{R}^n_+} f(y) {}^{\gamma}\mathbf{T}_x^y g(x) y^\gamma dy. \tag{3.172}$$

Definition 27. ***Generalized convolution*** *generated by a multi-dimensional generalized translation* ${}^{\gamma}\mathbf{T}_x^y$ *is given by*

$$(f*g)_\gamma(x) = (f*g)_\gamma = \int\limits_{\mathbb{R}^n_+} f(y)({}^{\gamma}\mathbf{T}_x^y g)(x) y^\gamma \, dy. \tag{3.173}$$

We will use also the mixed generalized convolution product defined by the formula

$$\langle f*g\rangle_\gamma = \int\limits_{\mathbb{R}^{n+1}_+} f(\tau,y)({}^{\gamma}\mathbf{T}_x^y g)(t-\tau,x) y^\gamma d\tau dy, \tag{3.174}$$

where ${}^{\gamma}\mathbf{T}_x^y$ is the multi-dimensional generalized translation

$$({}^{\gamma}\mathbf{T}_x^y f)(t,x) = ({}^{\gamma_1}T_{x_1}^{y_1} ... {}^{\gamma_n}T_{x_n}^{y_n} f)(t,x). \tag{3.175}$$

Each one-dimensional generalized translation ${}^{\gamma_i}T_{x_i}^{y_i}$ is defined for $i=1,...,n$ by formula (3.144).

From (12) it follows that for $f,g \in S_{ev}(\mathbb{R}^n_+)$ the Hankel multi-dimensional transform applied to the generalized convolution (3.173) is

$$\mathbf{F}_\gamma[(f*g)_\gamma(x)](\xi) = \mathbf{F}_\gamma[f(x)](\xi)\mathbf{F}_\gamma[g(x)](\xi). \tag{3.176}$$

For the generalized convolution (3.173), Young's inequality is known, which we present for convenience with the proof.

Statement 13. *Let* $p,q,r \in [1,\infty]$ *and*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{3.177}$$

If $f \in L_p^\gamma$, $g \in L_q^\gamma$, $1 \le p, q, r \le \infty$, and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, then a generalized convolution $(f*g)_\gamma$ is bounded almost everywhere and the Hausdorff–Young inequality is valid,

$$||(f*g)_\gamma||_{r,\gamma} \le ||f||_{p,\gamma}||g||_{q,\gamma}. \tag{3.178}$$

If $\frac{1}{p} + \frac{1}{q} = 1$, then

$$||(f*g)_\gamma||_{\infty,\gamma} \le ||f||_{p,\gamma}|||g||_{q,\gamma}. \tag{3.179}$$

Proof. Let

$$\frac{1}{r} + \frac{1}{p_1} + \frac{1}{p_2} = 1. \tag{3.180}$$

We apply to the expression $|(f*g)_\gamma(x)|$ the Hölder inequality for three functions (see [438], p. 64):

$$|(f*g)_\gamma(x)| = \int\limits_{\mathbb{R}_+^n} f(y)({}^\gamma\mathbf{T}_x^y g)(x) y^\gamma dy$$

$$\le \int\limits_{\mathbb{R}_+^n} |f(y)|^{1-a}|({}^\gamma\mathbf{T}_x^y g)(x)|^{1-b}|f(y)|^a|({}^\gamma\mathbf{T}_x^y g)(x)|^b y^\gamma dy$$

$$\le \left(\int\limits_{\mathbb{R}_+^n} |f(y)|^{(1-a)r}|({}^\gamma\mathbf{T}_x^y g)(x)|^{(1-b)r} y^\gamma dy\right)^{1/r} \left(\int\limits_{\mathbb{R}_+^n} |f(y)|^{ap_1} y^\gamma dy\right)^{1/p_1}$$

$$\times \left(\int\limits_{\mathbb{R}_+^n} |({}^\gamma\mathbf{T}_x^y g)(x)|^{bp_2} y^\gamma dy\right)^{1/p_2}.$$

Consider the integral $\int\limits_{\mathbb{R}_+^n} |({}^\gamma\mathbf{T}_x^y g)(x)|^{bp_2} y^\gamma dy$. By producing in $({}^\gamma\mathbf{T}_x^y g)(x)$ the change of variables

$$z_{2i-1} = y_i \cos\alpha_i,\, z_{2i} = y_i \sin\alpha_i,\; 0 \le \alpha_i \le \pi,\; i = 1, ..., n,$$

and putting $\widetilde{\mathbb{R}}_+^{2n} = \{z = (z_1, ..., z_{2n}) \in \mathbb{R}^{2n} : z_{2i} > 0, i = 1, .., n\}$, we obtain

$$\int\limits_{\mathbb{R}_+^n} |({}^\gamma\mathbf{T}_x^y g)(x)|^{bp_2} y^\gamma dy$$

$$= \int\limits_{\widetilde{\mathbb{R}}_+^{2n}} |g(\sqrt{(z_1 - y_1)^2 + z_2^2}, ..., \sqrt{(z_{2n-1} - y_n)^2 + z_{2n}^2})|^{bp_2} \prod_{i=1}^n z_{2i}^{\gamma_i - 1} dz$$

$$= \{(z_{2i-1} - y_n) \to z_{2i-1}\} = \int\limits_{\widetilde{\mathbb{R}}_+^{2n}} |g(\sqrt{z_1^2 + z_2^2}, ..., \sqrt{z_{2n-1}^2 + z_{2n}^2})|^{bp_2} \prod_{i=1}^{n} z_{2i}^{\gamma_i - 1} dz$$

$$= \int\limits_{\mathbb{R}_+^n} |g(x)|^{bp_2} x^\gamma dx.$$

Therefore

$$|(f * g)_\gamma(x)| \leq ||f||_{ap_1}^a ||g||_{bp_2}^b \left(\int\limits_{\mathbb{R}_+^n} |f(y)|^{(1-a)r} |({}^\gamma \mathbf{T}_x^y g)(x)|^{(1-b)r} y^\gamma dy \right)^{1/r}.$$

We raise the last inequality to the power of r, multiply both parts by x^γ, and integrate over $\mathbb{R}_+^n$:

$$\begin{aligned}
&||(f * g)_\gamma||_{r,\gamma}^r \\
&\leq ||f||_{ap_1}^{ra} ||g||_{bp_2}^{rb} \int\limits_{\mathbb{R}_+^n} \int\limits_{\mathbb{R}_+^n} |f(y)|^{(1-a)r} |({}^\gamma \mathbf{T}_x^y g)(x)|^{(1-b)r} y^\gamma dy\, x^\gamma dx \\
&= ||f||_{ap_1}^{ra} ||g||_{bp_2}^{rb} \int\limits_{\mathbb{R}_+^n} |f(y)|^{(1-a)r} y^\gamma dy \int\limits_{\mathbb{R}_+^n} |({}^\gamma \mathbf{T}_x^y g)(x)|^{(1-b)r} x^\gamma dx \\
&= ||f||_{ap_1}^{ra} ||g||_{bp_2}^{rb} ||f||_{(1-a)r}^{(1-a)r} ||g||_{(1-b)r}^{(1-b)r}.
\end{aligned}$$

Choosing a and b so that $(1-a)r = ap_1$ and $(1-b)r = bp_2$, i.e., $a = \frac{r}{r+p_1}$ and $b = \frac{r}{r+p_2}$, we can write

$$||(f * g)_\gamma||_{r,\gamma}^r \leq ||f||_{ap_1}^{ra+ap_1} ||g||_{bp_2}^{rb+bp_2} = ||f||_{ap_1}^r ||g||_{bp_2}^r$$

or putting $ap_1 = p$ and $bp_2 = q$

$$||(f * g)_\gamma||_{r,\gamma} \leq ||f||_p ||g||_q.$$

It remains to show that with this choice of p_1 and p_2, (3.180) is valid:

$$\frac{1}{r} + \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r} + \frac{a}{p} + \frac{b}{q} = \frac{1}{r} + \frac{1}{p}\left(1 - \frac{p}{r}\right) + \frac{1}{q}\left(1 - \frac{q}{r}\right) =$$

$$\frac{1}{r} + \frac{1}{p} - \frac{1}{r} + \frac{1}{q} - \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1.$$

Inequality (3.179) is obtained from (3.178) by tending to the limit with $r \to \infty$ using (1.47) (p and q should be such that $1/p + 1/q = 1$). $\square$

3.4.4 Weighted spherical mean

In this subsection, we give the definition and properties of the spherical weighted mean, the main function of which is its effect on the operator Δ_γ.

Of great interest among various researchers is a generalization of the classical spherical mean of

$$M(x,r,u)=\frac{1}{|S_n(1)|}\int\limits_{S_n(1)} u(x+\beta r)dS. \tag{3.181}$$

So, in paper [597] a spherical mean in space with negative curvature was considered, and in [160] and [119] a generalization of the spherical mean generated by the Charles Dunkl transformation operator was studied. Here we consider the spherical weighted mean, which is the transformation operator intertwining the multi-dimensional operator $(\Delta_\gamma)_x=\sum\limits_{i=1}^n(B_{\gamma_i})_{x_i}$, $x\in\mathbb{R}^n_+$ and the one-dimensional Bessel operator $(B_{n+|\gamma|-1})_t$, $t>0$. Such spherical mean is closely related to the B-ultrahyperbolic equation of the form

$$\sum_{j=1}^n(B_\nu)_{x_j}u=\sum_{j=1}^n(B_\nu)_{y_j}u,\qquad u=u(x_1,...,x_n,y_1,...,y_n). \tag{3.182}$$

When constructing a weighted spherical mean, instead of the usual shift, a multi-dimensional generalized translation (3.169) is used.

Definition 28. *The* ***weighted spherical mean*** *of function $f(x)$, $x\in\overline{\mathbb{R}}^n_+$, for $n\geq 2$ is*

$$M_t^\gamma[f(x)]=(M_t^\gamma)_x[f(x)]=\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{S_1^+(n)} {}^\gamma\mathbf{T}_x^{t\theta}f(x)\theta^\gamma dS, \tag{3.183}$$

where $\theta^\gamma=\prod\limits_{i=1}^n\theta_i^{\gamma_i}$, $S_1^+(n)=\{\theta{:}|\theta|=1,\theta\in\mathbb{R}^n_+\}$ is a part of a sphere in $\mathbb{R}^n_+$, and $|S_1^+(n)|_\gamma$ is given by (1.107). For $n=1$ let $M_t^\gamma[f(x)]={}^\gamma T_x^t f(x)$.

Theorem 36. *The weighted spherical mean $M_t^\gamma[f(x)]$ is the transmutation operator intertwining $(\Delta_\gamma)_x$ and $(B_{n+|\gamma|-1})_t$ for $f\in C^2_{ev}$:*

$$(B_{n+|\gamma|-1})_t M_t^\gamma[f(x)]=M_t^\gamma[(\Delta_\gamma)_x f(x)]. \tag{3.184}$$

Proof. First of all we note that for the function $f\in C^2_{ev}$ (1.104) gives

$$|S_1^+(n)|_\gamma\int\limits_0^t\lambda^{n+|\gamma|-1}M_\lambda^\gamma[f(x)]d\lambda=\int\limits_0^t\lambda^{n+|\gamma|-1}d\lambda\int\limits_{S_1^+(n)}(T^{\lambda y}f)(x)\;y^\gamma dS_y$$
$$=\int\limits_{B_r^+(n)}(T^z f)(x)z^\gamma dz. \tag{3.185}$$

Let us apply the operator Δ_γ to both sides of the relation (3.185) with respect to x. Then we obtain

$$|S_1^+(n)|_\gamma \int\limits_0^t \lambda^{n+|\gamma|-1}(\Delta_\gamma)_x\, M_\lambda^\gamma[f(x)]d\lambda = \int\limits_{B_t^+(n)} (\Delta_\gamma)_x({}^\gamma\, T_x^z f)(x) z^\gamma dz$$

$$= \sum_{i=1}^n \int\limits_{B_t^+(n)} (B_{\gamma_i})_{z_i}({}^\gamma\, T_x^z f)(x) z^\gamma dz.$$

Formula (3.146) gives $(B_{\gamma_i})_{z_i}{}^{\gamma_i}T_{x_i}^{z_i} f(x) = (B_{\gamma_i})_{x_i}{}^{\gamma_i}T_{x_i}^{z_i} f(x)$ and therefore

$$(\Delta_\gamma)_x({}^\gamma T_x^z f)(x) = (\Delta_\gamma)_z({}^\gamma T_x^z f)(x).$$

Then

$$|S_1^+(n)|_\gamma \int\limits_0^t \lambda^{n+|\gamma|-1}(\Delta_\gamma)_x\, M_\lambda^\gamma[f(x)]d\lambda = \int\limits_{B_t^+(n)} \left[(\Delta_\gamma)_z({}^\gamma T_x^z f)(x)\right] z^\gamma dz. \tag{3.186}$$

By applying formula (1.101) to the right side of relation (3.186), we obtain

$$\sum_{i=1}^n \int\limits_{B_t^+(n)} (B_{\gamma_i})_{z_i}(T^z f)(x) z^\gamma dz = \sum_{i=1}^n \int\limits_{S_t^+(n)} \frac{\partial}{\partial z_i}(T^z f)(x)\, \cos(\vec{\nu},\vec{e}_i)\, z^\gamma\, dS_z,$$

where $\vec{e}_i$ is the direction of the axis Oz_i, $i = 1, ..., n$.

Now, by using the fact that the direction of the outward normal to the boundary of a ball with as center the origin coincides with the direction of the position vector of the point on the ball, we obtain the relation

$$\sum_{i=1}^n \int\limits_{B_t^+(n)} (B_{\gamma_i})_{z_i}(T^z f)(x) z^\gamma dz = t^{n+|\gamma|-1} \int\limits_{S_1^+(n)} \frac{\partial}{\partial t}(T^{t\theta} f)(x)\, \theta^\gamma dS_\theta$$

$$= |S_1^+(n)|_\gamma t^{n+|\gamma|-1} \frac{\partial}{\partial t} M_t^\gamma[f(x)].$$

Returning to (3.186), we obtain

$$\sum_{i=1}^n \int\limits_0^t \lambda^{n+|\gamma|-1} B_{\gamma_i} M_\lambda^\gamma[f(x)]d\lambda = t^{n+|\gamma|-1} \frac{\partial}{\partial t} M_t^\gamma[f(x)]. \tag{3.187}$$

By differentiating relation (3.187) with respect to t, we obtain

$$\sum_{i=1}^{n} t^{n+|\gamma|-1} B_{\gamma_i} M_t^{\gamma}[f(x)] = (n+|\gamma|-1)t^{n+|\gamma|-2}\frac{\partial}{\partial t}M_t^{\gamma}[f(x)] + t^{n+|\gamma|-1}\frac{\partial^2}{\partial t^2}M_t^{\gamma}[f(x)]$$

or

$$\sum_{i=1}^{n} B_{\gamma_i} M_t^{\gamma}[f(x)] = \frac{n+|\gamma|-1}{t}\frac{\partial}{\partial t}M_t^{\gamma}[f(x)] + \frac{\partial^2}{\partial t^2}M_t^{\gamma}[f(x)],$$

and so

$$(\Delta_\gamma)_x M_t^{\gamma}[f(x)] = \frac{n+|\gamma|-1}{t}\frac{\partial}{\partial t}M_t^{\gamma}[f(x)] + \frac{\partial^2}{\partial t^2}M_t^{\gamma}[f(x)]. \tag{3.188}$$

Now let us consider $(\Delta_\gamma)_x M_t^{\gamma}[f(x)]$. Using the commutativity of B_{γ_i} and $T_{x_i}^{t\theta_i}$ (see [242]) we obtain

$$(\Delta_\gamma)_x M_t^{\gamma}[f(x)] = \frac{1}{|S_1^+(n)|_\gamma}(\Delta_\gamma)_x \int\limits_{S_1^+(n)} {}^{\gamma}T_x^{t\theta} f(x)\theta^{\gamma}\, dS_\theta$$

$$= \frac{1}{|S_1^+(n)|_\gamma}\int\limits_{S_1^+(n)} {}^{\gamma}T_x^{t\theta}[(\Delta_\gamma)_x f(x)]\theta^{\gamma}\, dS_\theta = M_t^{\gamma}[(\Delta_\gamma)_x f(x)],$$

which with (3.188) gives (3.184). □

A similar proof can be found in [506].

We note the simplest properties of the weighted spherical mean.

1. Linearity and uniformity:

$$M_t^{\gamma}[af(x)+bg(x)] = aM_t^{\gamma}[f(x)] + bM_t^{\gamma}[g(x)], \qquad a,b\in\mathbb{R}.$$

2. Positivity: If $f(x)\geq 0$, then $M_t^{\gamma}[f(x)]\geq 0$.

3. $M_t^{\gamma}[1]=1$.

4. For $t=0$ equalities

$$M_t^{\gamma}[f(x)]|_{t=0} = f(x), \qquad \left.\frac{\partial}{\partial t}M_t^{\gamma}[f(x)]\right|_{t=0} = 0 \tag{3.189}$$

are valid.

5. If $f(x)\in C_{ev}^2$, then $M_t^{\gamma}\in C_{ev}^2(\mathbb{R}_+^n)$ by x and

$$(\Delta_\gamma)_x M_t^{\gamma}[f(x)] = M_t^{\gamma}[\Delta_\gamma f(x)].$$

6. From (3.140) and (3.142),

$$(M_t^\gamma)_x[\,\mathbf{j}_\gamma(x,\xi)] = \mathbf{j}_\gamma(x,\xi)\; j_{\frac{n+|\gamma|}{2}-1}(t|\xi|), \tag{3.190}$$

$$(M_t^\gamma)_x[\,\mathbf{i}_\gamma(x,\xi)] = \mathbf{i}_\gamma(x,\xi)\; i_{\frac{n+|\gamma|}{2}-1}(t|\xi|) \tag{3.191}$$

follow.

Weighted generalized functions generated by quadratic forms

4

In this chapter we consider certain types of weighted generalized functions associated with nondegenerate indefinite quadratic forms. Such functions and their derivatives are used for constructing fundamental solutions of iterated ultrahyperbolic equations with a Bessel operator and for constructing negative real powers of hyperbolic and ultrahyperbolic operators with a Bessel operator.

4.1 The weighted generalized function associated with a positive quadratic form and concentrated on a part of a cone

We consider the weighted generalized functions $\delta_\gamma(P)$ concentrated on a part of a cone and give formulas for its derivatives in this section. More precisely, a weighted generalized function is the function whose action on the test function is equal to the limit of the integral of a delta-like sequence of functions approximating a delta function on a part of a cone in $\mathbb{R}^n_+$ with a weight x^γ.

4.1.1 B-ultrahyperbolic operator

Here we discuss the B-ultrahyperbolic operator and a method of the weighted generalized function which is proposed to study this operator and obtain its fundamental solution for appropriate test functions.

I. M. Gelfand and G. E. Shilov in [177] proposed the idea of finding fundamental solutions of second order differential operators by studying coefficients of Laurent series of the weighted generalized function generated by the corresponding quadratic form of this operator. This method is convenient because if we have information about the residues of the generalized function, we can obtain a solution to the equation containing the iterated operator. Depending on the relation between the iteration order and the dimension of the space we obtain a fundamental solution or a solution of a homogeneous equation.

The weighted generalized function generated by the indefinite quadratic form

$$P = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$$

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is used for the construction of a fundamental solution to the ultrahyperbolic operators with Bessel operator, i.e.,

$$\Box_\gamma = \Box_{\gamma',\gamma''} = \frac{\partial^2}{\partial x_1^2} + \frac{\gamma_1}{x_1}\frac{\partial}{\partial x_1} + ... + \frac{\partial^2}{\partial x_p^2} + \frac{\gamma_p}{x_p}\frac{\partial}{\partial x_p}$$
$$- \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\gamma_{p+1}}{x_{p+1}}\frac{\partial}{\partial x_{p+1}} - ... - \frac{\partial^2}{\partial x_{p+q}^2} + \frac{\gamma_{p+q}}{x_{p+q}}\frac{\partial}{\partial x_{p+q}}, \tag{4.1}$$

where $\gamma_i > 0, i = 1, ..., p+q, \; p, q \in \mathbb{N}$. Operator $\Box_\gamma = \Box_{\gamma',\gamma''}$ is a *B-ultrahyperbolic operator*.

Let $p, q \in \mathbb{N}$, $n = p+q$, $x = (x_1, ..., x_n) = (x', x'') \in \mathbb{R}^n_+$, $x' = (x_1, ..., x_p)$, $x'' = (x_{p+1}, ..., x_{p+q})$. Weighted generalized functions used for solution to the general Euler–Poisson–Darboux equation and the construction of fractional powers of $\Box_\gamma$ are

$$P = |x'|^2 - |x''|^2, \qquad |x'|^2 = x_1^2 + ... + x_p^2, \qquad |x''|^2 = x_{p+1}^2 + ... + x_{p+q}^2 \tag{4.2}$$

and

$$\mathscr{P}(x) = \sum_{k=1}^{n} g_k x_k^2, \qquad g_k \in \mathbb{C}, \qquad k = 1, ..., n. \tag{4.3}$$

Next, we define and study the following weighted generalized functions associated with quadratic forms (4.2) and (4.3) for $\lambda \in \mathbb{C}$ (see [505]):

- $\delta_\gamma(P)$;
- $P^\lambda_{\gamma,+}$, $P^\lambda_{\gamma,-}$;
- $\mathscr{P}^\lambda_\gamma$, $\mathscr{P} = P_1 + iP_2$, where P_1 is an indefinite quadratic form with real coefficients and P_2 is a positive defined quadratic form;
- $(P + i0)^\lambda_\gamma$, $(P - i0)^\lambda_\gamma$;
- $(w^2 - |x|^2)^\lambda_\gamma$;
- $(c^2 + P + i0)^\lambda_\gamma$, $(c^2 + P - i0)^\lambda_\gamma$;
- $\mathscr{P}^\lambda_\gamma f(\mathscr{P}, \lambda)$, where $f(z, \lambda)$ is an entire function.

In Section 1.56 the Hankel transforms of $\mathscr{P}^\lambda_\gamma$, $(P \pm i0)^\lambda_\gamma$, $P^\lambda_{\gamma,\pm}$, $(w^2 - |x|^2)^\lambda_{+,\gamma}$, and $(c^2 + P \pm i0)^\lambda_\gamma$ will be found.

4.1.2 Weighted generalized function associated with a positive quadratic form

Here we consider the weighted generalized function r^λ, $r = |x|$, $\lambda \in \mathbb{C}$. This function is studied in [242] in the case when in the weighted functional $(r^\lambda_\gamma, \varphi)_\gamma$ weight was taken only by one variable.

The weighted generalized function r_γ^λ is defined by

$$(r_\gamma^\lambda, \psi)_\gamma = \int\limits_{\mathbb{R}^n_+} r^\lambda \psi(x) x^\gamma \, dx, \quad \psi \in S_{ev}.$$

This function is an analytic function of λ for $\mathrm{Re}\,\lambda > -(n+|\gamma|)$. For $\mathrm{Re}\,\lambda \leq -(n+|\gamma|)$ we may define the weighted generalized function r_γ^λ by analytic continuation.

For $\mathrm{Re}\,\lambda > -(n+|\gamma|)$, r_γ^λ can be differentiated by parameter λ, i.e.,

$$\frac{\partial}{\partial\lambda}(r^\lambda, \varphi)_\gamma = \int\limits_{\mathbb{R}^+_N} r^\lambda \ln r \; \varphi(x)(x')^\gamma \, dx.$$

Let us move on to spherical coordinates $x = r\Theta$, $r = |x|$, in $(r_\gamma^\lambda, \varphi)_\gamma$, writing it in the form

$$(r_\gamma^\lambda, \varphi)_\gamma = \int\limits_0^\infty r^{\lambda+n+|\gamma|-1} \int\limits_{S_1^+(n)} \varphi(r\Theta)\Theta^\gamma \, dS \, dr$$
$$= |S_1^+(n)|_\gamma \int\limits_0^\infty r^{\lambda+n+|\gamma|-1} M_\varphi^\gamma(r) \, dr,$$

where $|S_1^+(n)|_\gamma$ is given by (1.107) and

$$M_\varphi^\gamma(r) = M_0^\gamma[\varphi(r\Theta)] = \frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} \varphi(r\Theta)\theta^\gamma dS,$$

is the weighted spherical mean (3.183) at 0.

Theorem 37. *For $M_\varphi^\gamma(r)$, $\varphi \in S_{ev}$, the representation*

$$M_\varphi^\gamma(r) = \varphi(0) + \frac{1}{2!}(M_\varphi^\gamma)''(0)r^2 + ... + \frac{1}{(2p)!}(M_\varphi^\gamma)^{(2k)}(0)r^{2p} + ...$$
$$= |S_1^+(n)|_\gamma \sum_{p=0}^\infty \frac{(\triangle_\gamma)^p \varphi(0) r^{2p}}{2^p p!(n+|\gamma|)(n+|\gamma|+2)...(n+|\gamma|+2p-2)} \tag{4.4}$$

is valid.

Proof. In the beginning, we note that the function $M_\varphi^\gamma(r)$ is infinitely differentiable by r for $r > 0$ and decreases at $r \to \infty$ faster than any degree $\frac{1}{r}$, which follows from the similar properties of the function $\varphi(x)$.

It is easy to show that the function $M_\phi^\gamma(r)$ is infinitely differentiable at $r=0$, which is enough to expand the function $\phi(x)$ using the Taylor formula and make sure that all its odd derivatives are equal to zero when $r=0$. So the function $M_\phi^\gamma(r)\in S_{ev}$.

The regular nonweighted generalized function

$$x_+^\mu=\begin{cases}0 & x\leq 0,\\ x^\mu & x>0\end{cases}$$

acts on $\psi\in S_{ev}$ by the formula

$$(x_+^\mu,\psi)=\int\limits_0^\infty x^\mu\psi(x)\,dx.$$

The expression

$$(r_\gamma^\lambda,\varphi)_\gamma=|S_1^+(n)|_\gamma\int\limits_0^\infty r^{\lambda+n+|\gamma|-1}M_\varphi^\gamma(r)\,dr$$

can be considered as the result of applying the *nonweighted* generalized function $|S_1^+(n)|_\gamma\, x_+^\mu$, where $\mu=\lambda+n+|\gamma|-1$ to the test function $M_\varphi^\gamma(x)$:

$$(r^\lambda,\varphi(x))_\gamma=(|S_1^+(n)|_\gamma\, x_+^\mu, M_\varphi^\gamma(x)). \tag{4.5}$$

Therefore, we can use the results of the study of this generalized function given in the book [177].

The generalized function x_+^μ is an analytical function for $\operatorname{Re}\mu>-1$ or $\operatorname{Re}\lambda>-n-|\gamma|$. Its analytical extension to the set $\operatorname{Re}\mu>-n-1$, with excluded points $\mu=-1,-2,-3,...$, is determined by the equality

$$(x_+^\mu,\psi)=\int\limits_0^\infty x^\mu\psi(x)\,dx=\int\limits_0^1 x^\mu\Bigg[\psi(x)-\psi(0)-x\psi'(0)-...$$
$$...-\frac{x^{n-1}}{(N-1)!}\psi^{(n-1)}(0)\Bigg]\,dx+\int\limits_1^\infty x^\mu\psi(x)\,dx+\sum_{k=1}^n\frac{\psi^{(k-1)}(0)}{(k-1)!(\mu+k)}. \tag{4.6}$$

Note that in the strip of the complex plane $-n-1<\operatorname{Re}\mu<-n$ for $1\leq k\leq n$ the equality

$$-\frac{1}{\mu+k}=\int\limits_1^\infty x^{\mu+k-1}\,dx \tag{4.7}$$

holds. By virtue of equality (4.7), in the strip $-n-1<\operatorname{Re}\mu<-n$ formula (4.6) can be converted to a simpler one:

$$(x_+^\mu,\psi)=\int\limits_0^\infty x^\mu\left[\psi(x)-\psi(0)-x\psi'(0)-...-\frac{x^{n-1}}{(n-1)!}\psi^{(n-1)}(0)\right]dx. \quad (4.8)$$

The right side of expression (4.6) gives a regularized value of the integral to the left in this expression. The generalized function x_+^μ is defined for all $\mu\neq-1,-2,...$ and as a function of μ has first order poles at points $\mu=-1,-2,...$ ($\lambda=-(n+|\gamma|),-(n+|\gamma|+1),-(n+|\gamma|+2),...$). We calculate its residue at $\mu=-k$:

$$\operatorname*{res}_{\mu=-k}[(x_+^\mu,\psi)]=\frac{(-1)^{k-1}}{(k-1)!}(\delta^{(k-1)}(x),\psi(x)).$$

Therefore, the generalized function x_+^μ as a function of μ has at $\mu=-k$ a simple pole with the residue $\frac{(-1)^{k-1}}{(k-1)!}\ \delta^{(k-1)}(x)$, $k=1,2,....$

The residue of $(|S_1^+(n)|_\gamma\, x_+^\mu,M_\varphi^\gamma(x))$ at $\mu=-k$ ($\lambda=-(N+|\gamma|+k-1)$) is

$$\operatorname*{res}_{\mu=-k}[(|S_1^+(n)|_\gamma\, x_+^\mu,M_\varphi^\gamma(x))]=|S_1^+(n)|_\gamma\,\frac{(M_\varphi^\gamma)^{(k-1)}(0)}{(k-1)!}.$$

But since all the odd derivatives of the function $M_\varphi^\gamma(r)$ vanish at $r=0$, only a series of poles corresponding to odd values $\mu=-1,-3,...,-2p-1,...,$ $p=0,1,2,...,$ ($\lambda=-(n+|\gamma|),-(n+|\gamma|+2),...,-(n+|\gamma|+2p),...$) remains, and we can write

$$\operatorname*{res}_{\mu=-2p-1}[(|S_1^+(n)|_\gamma\, x_+^\mu,M_\varphi^\gamma(x))]=|S_1^+(n)|_\gamma\,\frac{(\delta^{(2p)}(x),M_\varphi^\gamma(x))}{(2p)!}$$
$$=|S_1^+(n)|_\gamma\,\frac{(M_\varphi^\gamma)^{(2p)}(0)}{(2p)!}.$$

Now let us return to weighted generalized functions. Considering formula (4.5) we note that the residue of $(r_\gamma^\lambda,\varphi)_\gamma$ as a function of λ at $\lambda=-(n+|\gamma|+2p)$, $p=0,1,2,...,$ is

$$\operatorname*{res}_{\lambda=-(n+|\gamma|+2p)}[(r_\gamma^\lambda,\varphi)_\gamma]=|S_1^+(n)|_\gamma\,\frac{(M_\varphi^\gamma)^{(2p)}(0)}{(2p)!}. \quad (4.9)$$

In particular, when $p=0$ at $\lambda=-(n+|\gamma|)$, the weighted generalized function r_γ^λ has a simple pole with residue

$$\operatorname*{res}_{\lambda=-(n+|\gamma|)}[(r_\gamma^\lambda,\varphi)_\gamma]=|S_1^+(n)|_\gamma\, M_\varphi^\gamma(0).$$

Using the first of condition (3.189) we obtain

$$\operatorname*{res}_{\lambda=-(n+|\gamma|)}[(r_\gamma^\lambda,\varphi)_\gamma]=|S_1^+(n)|_\gamma\,\varphi(0)=|S_1^+(n)|_\gamma\,(\delta(x),\varphi(x)). \quad (4.10)$$

It means that r_γ^λ as a function of λ at $\lambda=-(n+|\gamma|)$ has a simple pole with residue $|S_1^+(n)|_\gamma\,\delta(x)$.

Following P. Pizetti [440] (see also [177]) let us express the value $(M_\varphi^\gamma)^{(2p)}(0)$ directly through the function φ. To do this, we construct another expression for the residue of the weighted generalized function r_γ^λ. Prove that for $\operatorname{Re}\lambda>-n-|\lambda|$,

$$\triangle_\gamma(r_\gamma^{\lambda+2})=(\lambda+2)(\lambda+N+|\gamma|)r_\gamma^\lambda. \tag{4.11}$$

For $\operatorname{Re}\lambda>0$ formula (4.11) is proved by direct calculation of the left side. For other values of λ it is valid due to analytic continuation. Iterating formula (4.11), we get for any integer p the equality

$$r_\gamma^\lambda=\frac{\triangle_\gamma^p r_\gamma^{\lambda+2p}}{(\lambda+2)...(\lambda+2p)(\lambda+n+|\gamma|)...(\lambda+n+|\gamma|+2p-2)}. \tag{4.12}$$

Now the residue of r_γ^λ at $\lambda=-(n+|\gamma|+2p)$ is calculated as a residue of the right side of equality (4.12) at this λ. But for $\lambda=-(n+|\gamma|+2p)$ the denominator of expression (4.12) does not vanish; hence it suffices to find the residue of the numerator.

As for any test function $\varphi\in S_{ev}$, if the equality $(\triangle_\gamma^p r^{\lambda+2p},\varphi)_\gamma=(r^{\lambda+2p},\triangle_B^p\varphi)_\gamma$ is true, then it is needed to find a residue of $(r_\gamma^{\lambda+2p},\triangle_\gamma^p\varphi)_\gamma$ at $\lambda=-(n+|\gamma|+2p)$. Such residue was calculated in (4.10) and it equals

$$\operatorname*{res}_{\tau=-(n+|\gamma|)}[(r^\tau,\triangle_\gamma^p\varphi)_\gamma]=|S_1^+(n)|_\gamma\,\triangle_\gamma^p\varphi(0).$$

Then the residue of r_γ^λ as a function of λ at $\lambda=-(n+|\gamma|+2p)$ is

$$\begin{aligned}&\operatorname*{res}_{\lambda=-(n+|\gamma|+2p)}[(r_\gamma^\lambda,\varphi(x))_\gamma]\\&=\frac{|S_1^+(n)|_\gamma\,(\triangle_\gamma^p\delta(x),\varphi(x))}{2^p p!(n+|\gamma|)(n+|\gamma|+2)...(n+|\gamma|+2p-2)},\end{aligned} \tag{4.13}$$

or

$$\operatorname*{res}_{\lambda=-(n+|\gamma|+2p)}[r_\gamma^\lambda]=\frac{|S_1^+(n)|_\gamma\,\triangle_\gamma^p\delta(x)}{2^p p!(n+|\gamma|)(n+|\gamma|+2)...(n+|\gamma|+2p-2)}. \tag{4.14}$$

Comparing expression (4.13) with the previously obtained formula (4.9), we obtain an expression for the derivative with respect to r at zero of order $2p$, $p=0,1,2...$, for the weighted spherical mean:

$$(M_\varphi^\gamma)^{(2p)}(0)=\frac{(2p)!\triangle_\gamma^p\varphi(0)}{2^p p!(n+|\gamma|)(n+|\gamma|+2)...(n+|\gamma|+2p-2)}.$$

This makes it possible to write the decomposition of the function $M_\varphi^\gamma(r)$ in the Taylor series, which gives the statement. The proof is complete. $\square$

Formula (4.4) makes it possible to write a regularization of the weighted generalized function r_γ^λ for $\operatorname{Re}\lambda > -(n+|\gamma|+2p)$, $\lambda \neq -(n+|\gamma|), -(n+|\gamma|+2), \dots, -(n+|\gamma|+2p-2)$, in the form

$$
\begin{aligned}
(r^\lambda, \varphi)_\gamma &= |S_1^+(n)|_\gamma \int_0^\infty r^{\lambda+n+|\gamma|-1} M_\varphi^\gamma(r)\, dr = |S_1^+(n)|_\gamma (r^{\lambda+n+|\gamma|-1}, M_\varphi^\gamma(r)) \\
&= |S_1^+(n)|_\gamma \int_0^1 r^{\lambda+n+|\gamma|-1} [M_\varphi^\gamma(r) - \varphi(0) - \dots - \frac{1}{(2p)!} (M_\varphi^\gamma(0))^{(2p)} r^{2p}]\, dr \\
&+ |S_1^+(n)|_\gamma \int_1^\infty r^{\lambda+n+|\gamma|-1} M_\varphi^\gamma(r)\, dr + |S_1^+(n)|_\gamma \sum_{k=0}^{2p} \frac{(M_\varphi^\gamma(0))^{(2k)}}{(2k)!(\lambda+n+|\gamma|+2k)}.
\end{aligned}
\tag{4.15}
$$

Define the weighted functional $(r_\gamma^\lambda \ln^m r, \varphi)_\gamma$ by the formula

$$
(r_\gamma^\lambda \ln^m r, \varphi)_\gamma = \int_{\mathbb{R}_+^n} r_\gamma^\lambda \ln^m r \varphi(x) x^\gamma\, dx. \tag{4.16}
$$

We regularize $(r_\gamma^\lambda \ln^m r, \varphi)_\gamma$ at $\operatorname{Re}\lambda > -(n+|\gamma|+2p)$, $\lambda \neq -(n+|\gamma|), -(n+|\gamma|+2), \dots, -(n+|\gamma|+2p-2)$, using m-times differentiation by λ of formula (4.15):

$$
\begin{aligned}
(r_\gamma^\lambda \ln^m r, \varphi)_\gamma = |S_1^+(n)|_\gamma &\int_0^1 r^{\lambda+n+|\gamma|-1} \ln^m r [M_\varphi^\gamma(r) - \varphi(0) - \dots \\
&- \frac{1}{(2p)!} (M_\varphi^\gamma(0))^{(2p)} r^{2p}]\, dr \\
&+ |S_1^+(n)|_\gamma \int_1^\infty r^{\lambda+n+|\gamma|-1} \ln^m r M_\varphi^\gamma(r)\, dr \\
&+ |S_1^+(n)|_\gamma \sum_{k=0}^{2p} \frac{(-1)^m m! (M_\varphi^\gamma(0))^{(2k)}}{(2k)!(\lambda+n+|\gamma|+2k)^{m+1}}.
\end{aligned}
\tag{4.17}
$$

4.1.3 Weighted generalized function $\delta_\gamma(P)$

In this subsection we will study the singular generalized function δ_γ defined by the equality (see Section 1.2.3)

$$
(\delta_\gamma, \varphi)_\gamma = \varphi(0), \quad \varphi(x) \in S_{ev}.
$$

Let $p,q\in\mathbb{N}$, $n=p+q$, and

$$P=|x'|^2-|x''|^2=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2,$$

where $x=(x_1,...,x_n)=(x',x'')\in\mathbb{R}_n^+$, $x'=(x_1,...,x_p)$, $x''=(x_{p+1},...,x_{p+q})$.

Definition 29. *Let $\varphi\in S_{ev}$ vanish at the origin. For such φ we define the generalized function $\delta_\gamma(P)$ concentrated on the part of the cone $P=0$ belonging to $\mathbb{R}_n^+$ by the formula*

$$(\delta_\gamma(P),\varphi)_\gamma=\int\limits_{\mathbb{R}_+^n}\delta_\gamma(|x'|^2-|x''|^2)\varphi(x)x^\gamma dx. \tag{4.18}$$

If the function $\varphi\in S_{ev}$ does not vanish at the origin, then $(\delta_\gamma(P),\varphi)_\gamma$ is defined by regularizing the integral.

Lemma 12. *Let $\varphi\in S_{ev}$ vanish at the origin. For $(\delta_\gamma(P),\varphi(x))_\gamma$ when $p>1$, $q>1$, the representation*

$$(\delta_\gamma(P),\varphi(x))_\gamma=\frac{1}{2}\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\varphi(t\omega)t^{n+|\gamma|-3}\omega^\gamma dS'dS''dt, \tag{4.19}$$

where $\{|\omega'|=1\}^+=\{\omega'\in\mathbb{R}_+^p:|\omega'|=1\}$, $\{|\omega''|=1\}^+=\{\omega''\in\mathbb{R}_+^q:|\omega''|=1\}$, $\omega=(\omega',\omega'')$ is valid.

When $p=q=1$, $P=x^2-y^2$,

$$(\delta_\gamma(x^2-y^2),\varphi(x,y))_\gamma=\frac{1}{2}\int\limits_0^\infty\varphi(y,y)y^{|\gamma|-1}dy.$$

When $p=1$, $q=n-1>1$, $P=x_1^2-|x''|^2$, $\gamma''=(\gamma_2,...,\gamma_n)$,

$$(\delta_\gamma(P),\varphi(x))_\gamma=\frac{1}{2}\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\varphi(y,y\sigma)y^{n+|\gamma|-3}dy\,\sigma^{\gamma''}dS.$$

When $q=1$, $p=n-1>1$, $P=|x'|^2-x_n^2$, $\gamma'=(\gamma_1,...,\gamma_{n-1})$,

$$(\delta_\gamma(P),\varphi(x))_\gamma=\frac{1}{2}\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\varphi(x\sigma,x)x^{n+|\gamma|-3}dx\,\sigma^{\gamma'}dS.$$

Proof. Let first $p>1$, $q>1$. Turning to bipolar coordinates

$$x_1=r\omega_1,...,x_p=r\omega_p,\ x_{p+1}=s\omega_{p+1},...,x_{p+q}=s\omega_{p+q}, \tag{4.20}$$

where $r=\sqrt{x_1^2+...+x_p^2}$, $s=\sqrt{x_{p+1}^2+...+x_{p+q}^2}$, and putting $\omega'=(\omega_1,...,\omega_p)\in\mathbb{R}_+^p$, $\omega''=(\omega_{p+1},...,\omega_n)\in\mathbb{R}_+^q$, $|\omega'|=\sqrt{\omega_1^2+...+\omega_p^2}=1$, $|\omega''|=\sqrt{\omega_{p+1}^2+...+\omega_{p+q}^2}=1$, we obtain

$$(\delta_\gamma(P),\varphi(x))_\gamma=$$

$$\int\limits_0^\infty\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\delta(r^2-s^2)\varphi(r\omega',s\omega'')r^{p+|\gamma'|-1}s^{q+|\gamma''|-1}\omega^\gamma dS'dS''drds,$$

where

$$\{|\omega'|=1\}^+=\{\omega'\in\mathbb{R}_+^p:|\omega'|=1\},$$
$$\{|\omega''|=1\}^+=\{\omega''\in\mathbb{R}_+^q:|\omega''|=1\},$$

dS' is the surface element of $\{|\omega'|=1\}^+$ and dS'' is the surface element of $\{|\omega''|=1\}^+$. Replacing variables by formulas $r^2=u$, $s^2=v$, we obtain $dr=\frac{1}{2}u^{-\frac{1}{2}}du$, $ds=\frac{1}{2}v^{-\frac{1}{2}}dv$, and

$$(\delta_\gamma(P),\varphi(x))_\gamma=\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\delta(u-v)\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')$$
$$\times u^{\frac{p+|\gamma'|}{2}-1}v^{\frac{q+|\gamma''|}{2}-1}\omega^\gamma dS'dS''dudv$$
$$=\frac{1}{4}\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\varphi(\sqrt{v}\omega',\sqrt{v}\omega'')v^{\frac{p+|\gamma'|}{2}-1}v^{\frac{q+|\gamma''|}{2}-1}\omega^\gamma dS'dS''dv$$
$$=\frac{1}{4}\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\varphi(\sqrt{v}\omega)v^{\frac{n+|\gamma|}{2}-2}\omega^\gamma dS'dS''dv.$$

Going back to the variable s by the formula $v=s^2$ we get (4.19).

For $p=q=1$ the quadratic form P is x^2-y^2 and

$$(\delta_\gamma(x^2-y^2),\varphi(x,y))_\gamma$$
$$=\int\limits_0^\infty\int\limits_0^\infty\delta(x^2-y^2)\varphi(x,y)x^{\gamma_1}y^{\gamma_2}dxdy=\{x^2=u,y^2=v\}$$
$$=\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\delta(u-v)\varphi(\sqrt{u},\sqrt{v})u^{\frac{\gamma_1-1}{2}}v^{\frac{\gamma_2-1}{2}}dudv$$

$$=\frac{1}{4}\int\limits_0^\infty \varphi(\sqrt{v},\sqrt{v})v^{\frac{|\gamma|}{2}-1}dv=\{v=y^2\}=\frac{1}{2}\int\limits_0^\infty \varphi(y,y)y^{|\gamma|-1}dy.$$

Let us consider now $p=1$, $q=n-1>1$. We have

$$(\delta_\gamma(P),\varphi(x))_\gamma=(\delta(x_1^2-|x''|^2),\varphi(x))_\gamma=\int\limits_{\mathbb{R}^n_+}\delta(x_1^2-|x''|^2)\varphi(x)x^\gamma dx.$$

Passing to spherical coordinates $x''=\rho\sigma$, $\sigma\in\mathbb{R}^{n-1}_+$, we obtain

$$\begin{aligned}
&(\delta_\gamma(P),\varphi(x))_\gamma=(\delta(x_1^2-|x''|^2),\varphi(x))_\gamma\\
&=\int\limits_0^\infty\int\limits_0^\infty \delta(x_1^2-\rho^2)\rho^{n+|\gamma''|-2}x_1^{\gamma_1}dx_1d\rho\int\limits_{S_1^+(n-1)}\varphi(x_1,\rho\sigma)\sigma^{\gamma'}dS\\
&=\{x_1^2=u,\rho^2=v\}\\
&=\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\delta(u-v)\varphi(\sqrt{u},\sqrt{v}\sigma)u^{\frac{\gamma_1-1}{2}}v^{\frac{n+|\gamma''|-1}{2}-1}dudv\sigma^{\gamma'}dS\\
&=\frac{1}{4}\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\varphi(\sqrt{v},\sqrt{v}\sigma)v^{\frac{n+|\gamma|}{2}-2}dv\sigma^{\gamma'}dS=\{v=y^2\}\\
&=\frac{1}{2}\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\varphi(y,y\sigma)y^{n+|\gamma|-3}dy\sigma^{\gamma'}dS.
\end{aligned}$$

The case $q=1$, $p=n-1>1$ is considered similarly. □

Lemma 13. *The derivative of order k of function $\delta_\gamma(P)$ for $p>1$, $q>1$ has two representations denoted $\delta_{\gamma,1}^{(k)}(P)$ and $\delta_{\gamma,2}^{(k)}(P)$ of the form*

$$(\delta_{\gamma,1}^{(k)}(P),\varphi(x))_\gamma=\int\limits_0^\infty\left[\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^k\psi(r,s)s^{q+|\gamma''|-2}\right]\Bigg|_{s^2=r^2}r^{p+|\gamma'|-1}dr, \quad (4.21)$$

$$(\delta_{\gamma,2}^{(k)}(P),\varphi(x))_\gamma=(-1)^k\int\limits_0^\infty\left[\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k\psi(r,s)r^{p+|\gamma'|-2}\right]\Bigg|_{r^2=s^2}s^{q+|\gamma''|-1}ds, \quad (4.22)$$

where

$$\psi(r,s)=\frac{1}{2}\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\varphi(r\omega',s\omega'')\omega^\gamma\,dS'dS'',\qquad \varphi\in S_{ev}. \tag{4.23}$$

Integrals (4.21) *and* (4.22) *converge and coincide for* $k<\frac{n+|\gamma|-2}{2}$. *If* $k\geq\frac{n+|\gamma|-2}{2}$, *then these integrals need to be understood in the sense of regularized values.*

Proof. Let us find the derivative of the order k of $\delta_\gamma(P)$. After tending to bipolar coordinates (4.20) we obtain $P=r^2-s^2$ and

$$(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma=\int\limits_0^\infty\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\left(\frac{\partial^k}{\partial P^k}\delta(r^2-s^2)\right)$$
$$\times\,\varphi(r\omega',s\omega'')r^{p+|\gamma'|-1}s^{q+|\gamma''|-1}\omega^\gamma\,dS'dS''drds. \tag{4.24}$$

Replacing variables by formulas $r^2=u$, $s^2=v$, $dr=\frac{1}{2}u^{-\frac{1}{2}}du$, $ds=\frac{1}{2}v^{-\frac{1}{2}}dv$, we obtain $P=u-v$, $\frac{\partial}{\partial P}=\frac{\partial}{\partial u}$, and

$$(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma$$
$$=\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\left(\frac{\partial^k}{\partial u^k}\delta(u-v)\right)$$
$$\times\,\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')u^{\frac{p+|\gamma'|}{2}-1}v^{\frac{q+|\gamma''|}{2}-1}\omega^\gamma\,dS'dS''dudv$$
$$=(-1)^k\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\delta(u-v)\left(\frac{\partial^k}{\partial u^k}\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')u^{\frac{p+|\gamma'|}{2}-1}\right)$$
$$\times\,v^{\frac{q+|\gamma''|}{2}-1}\omega^\gamma\,dS'dS''dudv$$
$$=(-1)^k\frac{1}{4}\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\left.\left(\frac{\partial^k}{\partial u^k}\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')u^{\frac{p+|\gamma'|}{2}-1}\right)\right|_{u=v}$$
$$\times\,v^{\frac{q+|\gamma''|}{2}-1}\omega^\gamma\,dS'dS''dv.$$

Remembering that $u=r^2$, $v=s^2$ we can write

$$(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma$$
$$=(-1)^k\frac{1}{2}\int\limits_0^\infty\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\left.\left[\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k\varphi(r\omega',s\omega'')r^{p+|\gamma'|-2}\right]\right|_{r^2=s^2}$$
$$\times\,s^{q+|\gamma''|-1}\omega^\gamma\,dS'dS''ds.$$

Entering the designation (4.23), we obtain the following formula for $\delta_\gamma^{(k)}(P)$:

$$(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma=(-1)^k\int\limits_0^\infty\left[\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k\psi(r,s)r^{p+|\gamma'|-2}\right]\Bigg|_{r^2=s^2}s^{q+|\gamma''|-1}ds. \tag{4.25}$$

Returning to formula (4.24) and putting $r^2=-u$, $s^2=-v$, $u<0$, $v<0$, $dr=-\frac{1}{2}(-u)^{-\frac{1}{2}}du$, $ds=-\frac{1}{2}(-v)^{-\frac{1}{2}}dv$ $(u<0,\ v<0)$, we get $P=v-u$, $\frac{\partial}{\partial P}=\frac{\partial}{\partial v}$, and

$$\begin{aligned}
&(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma\\
&=\frac{1}{4}\int\limits_0^{-\infty}\int\limits_0^{-\infty}\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\left(\frac{\partial^k}{\partial v^k}\delta(v-u)\right)\\
&\quad\times\varphi(\sqrt{-u}\omega',\sqrt{-v}\omega'')(-u)^{\frac{p+|\gamma'|}{2}-1}(-v)^{\frac{q+|\gamma''|}{2}-1}\omega^\gamma dS'dS''dudv\\
&=(-1)^k\frac{1}{4}\int\limits_0^{-\infty}\int\limits_0^{-\infty}\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\delta(v-u)\frac{\partial^k}{\partial v^k}\\
&\quad\times\left[\varphi(\sqrt{-u}\omega',\sqrt{-v}\omega'')(-v)^{\frac{q+|\gamma''|}{2}-1}\right](-u)^{\frac{p+|\gamma'|}{2}-1}\omega^\gamma dS_1^p dS_1^q dudv\\
&=(-1)^k\frac{1}{4}\int\limits_0^{-\infty}\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\left[\frac{\partial^k}{\partial v^k}\varphi(\sqrt{-u}\omega',\sqrt{-v}\omega'')(-v)^{\frac{q+|\gamma''|}{2}-1}\right]\Bigg|_{v=u}\\
&\quad\times(-u)^{\frac{p+|\gamma'|}{2}-1}\omega^\gamma dS'dS''du.
\end{aligned}$$

Remembering that $-u=r^2$, $-v=s^2$, we return to the variables r and s:

$$\begin{aligned}
&(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma\\
&\quad=\frac{1}{2}\int\limits_0^{\infty}\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\left[\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^k\varphi(r\omega',s\omega'')s^{q+|\gamma''|-2}\right]\Bigg|_{s^2=r^2}\\
&\quad\quad\times r^{p+|\gamma'|-1}\omega^\gamma dS'dS''dr.
\end{aligned}$$

Using the designation (4.23), we write

$$(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma=\int\limits_0^\infty\left[\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^k\psi(r,s)s^{q+|\gamma''|-2}\right]\Bigg|_{s^2=r^2}r^{p+|\gamma'|-1}dr. \tag{4.26}$$

Further for functions (4.25) and (4.26) we will use notations $\delta^{(k)}_{\gamma,1}(P)$ and $\delta^{(k)}_{\gamma,2}(P)$, so

$$(\delta^{(k)}_{\gamma,1}(P),\varphi(x))_\gamma=\int\limits_0^\infty\left[\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^k\psi(r,s)s^{q+|\gamma''|-2}\right]\bigg|_{s^2=r^2}r^{p+|\gamma'|-1}dr,$$

$$(\delta^{(k)}_{\gamma,2}(P),\varphi(x))_\gamma=(-1)^k\int\limits_0^\infty\left[\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^k\psi(r,s)r^{p+|\gamma'|-2}\right]\bigg|_{r^2=s^2}s^{q+|\gamma''|-1}ds.$$

The integrals $\delta^{(k)}_{\gamma,1}(P)$ and $\delta^{(k)}_{\gamma,2}(P)$ converge and coincide for $k<\frac{n+|\gamma|-2}{2}$ for any $\varphi\in S_{ev}$. If, on the other hand, $k\geq\frac{n+|\gamma|-2}{2}$, these integrals must be understood in the sense of their regularizations. Specifically, let us make in $\delta^{(k)}_{\gamma,1}(P)$ and $\delta^{(k)}_{\gamma,2}(P)$ the formal change of variables $r^2=u$, $s^2=v$. Then we may write

$$(\delta^{(k)}_{\gamma,1}(P),\varphi(x))_\gamma=\frac{1}{4}\int\limits_0^\infty\left[\frac{\partial^k}{\partial v^k}\psi(\sqrt{u},\sqrt{v})v^{\frac{q+|\gamma''|}{2}-1}\right]\bigg|_{v=u}u^{\frac{p+|\gamma'|}{2}-1}du,$$

$$(\delta^{(k)}_{\gamma,2}(P),\varphi(x))_\gamma=\frac{(-1)^k}{4}\int\limits_0^\infty\left[\frac{\partial^k}{\partial u^k}\psi(\sqrt{u},\sqrt{v})u^{\frac{p+|\gamma'|}{2}-1}\right]\bigg|_{u=v}v^{\frac{q+|\gamma''|}{2}-1}dv.$$

The function $\psi(\sqrt{u},\sqrt{v})\in S_{ev}$ for u and v. Then

$$\left[\frac{\partial^k}{\partial v^k}\psi(\sqrt{u},\sqrt{v})v^{\frac{q+|\gamma''|}{2}-1}\right]\bigg|_{v=u}=u^{\frac{q+|\gamma''|}{2}-1-k}\Psi_1(u),$$

$$\left[\frac{\partial^k}{\partial u^k}\psi(\sqrt{u},\sqrt{v})u^{\frac{p+|\gamma'|}{2}-1}\right]\bigg|_{u=v}=v^{\frac{p+|\gamma'|}{2}-1-k}\Psi_2(v),$$

where $\Psi_1(u),\Psi_2(v)\in S_{ev}$, thus

$$(\delta^{(k)}_{\gamma,1}(P),\varphi(x))_\gamma=\frac{1}{4}\int\limits_0^\infty u^{\frac{n+|\gamma|-k}{2}-1}\Psi_1(u)du=\frac{1}{4}(u^\lambda_+,\Psi_1),$$

$$(\delta^{(k)}_{\gamma,2}(P),\varphi(x))_\gamma=\frac{(-1)^k}{4}\int\limits_0^\infty v^{\frac{n+|\gamma|-k}{2}-1}\Psi_2(v)dv=\frac{(-1)^k}{4}(v^\lambda_+,\Psi_2),$$

$$u^\lambda_+=\begin{cases}u^\lambda & u>0,\\ 0 & u\leq 0.\end{cases}$$

The regularization of this function is the generalized function u_+^λ, which for $\lambda \neq -1, -2, ...$ is obtained by analytic continuation u_+^λ from $\mathrm{Re}\,\lambda>0$. For $\lambda = -1, -2, ...$, $(n+|\gamma| = -1, -2, ...)$ this analytic generalized function has simple poles and the generalized function u_+^{-m}, $m \in \mathbb{N}$, is defined as the constant term in the Laurent expansion for u_+^λ about $\lambda = -m$ (see [177]). □

Remark 7. *Note that when $k = 0$ formulas (4.21) and (4.22) are equivalent to formula (4.19). We will use integrals (4.21) and (4.22) at $k\in\mathbb{N} \cup \{0\}$.*

We have been studying the case when $p > 1$ and $q > 1$. The cases in which either p or q is equal to unity are special cases, since in this case the transition to bipolar coordinates loses its meaning. Let us start from the case $p = q = 1$.

Lemma 14. *For $p = q = 1$ and $\varphi \in S_{ev}$ the derivative of order k of the weighted generalized function $\delta_\gamma(P)$ has two representations, denoted by $\delta^{(k)}_{\gamma,1}(P)$ and $\delta^{(k)}_{\gamma,2}(P)$, of the form*

$$(\delta^{(k)}_{\gamma,1}(x^2-y^2), \varphi(x,y))_\gamma = \frac{1}{2}\int\limits_0^\infty \left[\left(\frac{1}{2y}\frac{\partial}{\partial y}\right)^k \varphi(x,y)y^{\gamma_2-1}\right]\Bigg|_{y=x} x^{\gamma_1}dx \quad (4.27)$$

and

$$(\delta^{(k)}_{\gamma,2}(x^2-y^2), \varphi(x,y))_\gamma = (-1)^k\frac{1}{2}\int\limits_0^\infty \left[\left(\frac{1}{2x}\frac{\partial}{\partial x}\right)^k \varphi(x,y)x^{\gamma_1-1}\right]\Bigg|_{x=y} y^{\gamma_2}dy. \quad (4.28)$$

Proof. The quadratic form P for $p = q = 1$ is $P = x^2 - y^2$. Let us find the derivative of order k of $\delta_\gamma(x^2 - y^2)$:

$$(\delta^{(k)}_\gamma(P), \varphi(x,y))_\gamma = \int\limits_0^\infty\int\limits_0^\infty \left(\frac{\partial^k}{\partial P^k}\delta(x^2-y^2)\right)\varphi(x,y)x^{\gamma_1}y^{\gamma_2}dxdy. \quad (4.29)$$

Now let us choose the new variables u and v by formulas $x^2 = u$, $y^2 = v$. We obtain $dx = \frac{1}{2}u^{-\frac{1}{2}}du$, $dy = \frac{1}{2}v^{-\frac{1}{2}}dv$, $P = u - v$, $\frac{\partial}{\partial P} = \frac{\partial}{\partial u}$, and

$$\begin{aligned}
&(\delta^{(k)}_\gamma(P), \varphi(x,y))_\gamma \\
&= \frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty \left(\frac{\partial^k}{\partial u^k}\delta(u-v)\right)\varphi(\sqrt{u},\sqrt{v})u^{\frac{\gamma_1-1}{2}}v^{\frac{\gamma_2-1}{2}}dudv \\
&= (-1)^k\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty \delta(u-v)\left(\frac{\partial^k}{\partial u^k}\varphi(\sqrt{u},\sqrt{v})u^{\frac{\gamma_1-1}{2}}\right)v^{\frac{\gamma_2-1}{2}}dudv
\end{aligned}$$

$$= (-1)^k \frac{1}{4} \int\limits_0^\infty \left(\frac{\partial^k}{\partial u^k} \varphi(\sqrt{u}, \sqrt{v}) u^{\frac{\gamma_1 - 1}{2}} \right) \bigg|_{u=v} v^{\frac{\gamma_2 - 1}{2}} dv.$$

Remembering that $u = x^2$, $v = y^2$, we can return to the variables x and y. Denoting $\delta_\gamma^{(k)}(P)$ through $\delta_{\gamma,2}^{(k)}(P)$ we get (4.28) $\delta_\gamma^{(k)}(P)$. Similarly, making the change of variables $r^2 = -u$, $s^2 = -v$, $u<0$, $v<0$, we obtain (4.27). □

Lemma 15. *For $p = 1$, $q = n - 1$, and $\varphi \in S_{ev}$ the derivative of order k of the generalized weighted function $\delta_\gamma(P)$ has two representations of the forms*

$$(\delta_{\gamma,1}^{(k)}(x_1^2 - |x''|), \varphi(x))_\gamma$$
$$= \frac{1}{2} \int\limits_0^\infty \int\limits_{S_1^+(n-1)} \left(\left(\frac{1}{2\rho} \frac{\partial}{\partial \rho} \right)^k \varphi(x_1, \rho\sigma) \rho^{n+|\gamma''|-3} \right) \bigg|_{\rho = x_1} x_1^{\gamma_1} dx_1 \sigma^{\gamma'} dS, \tag{4.30}$$

$$(\delta_{\gamma,2}^{(k)}(x_1^2 - |x''|), \varphi(x))_\gamma$$
$$= (-1)^k \frac{1}{2} \int\limits_0^\infty \int\limits_{S_1^+(n-1)} \left(\left(\frac{1}{2x_1} \frac{\partial}{\partial x_1} \right)^k \varphi(x_1, \rho\sigma) x_1^{\gamma_1 - 1} \right) \bigg|_{x_1 = \rho} \rho^{n+|\gamma''|-2} d\rho \sigma^{\gamma'} dS, \tag{4.31}$$

where $x'' = (x_2, ..., x_n)$, $\gamma'' = (\gamma_2, ..., \gamma_n)$, $|\gamma''| = \gamma_2 + ... + \gamma_n$.

For $p = n - 1$, $q = 1$, and $\varphi \in S_{ev}$ the derivative of order k of the generalized weighted function $\delta_\gamma(|x'|^2 - x_n^2)$ has two representations of the forms

$$(\delta_{\gamma,1}^{(k)}(|x'|^2 - x_n^2), \varphi(x))_\gamma$$
$$= \frac{1}{2} \int\limits_0^\infty \int\limits_{S_1^+(n-1)} \left(\left(\frac{1}{2x_n} \frac{\partial}{\partial x_n} \right)^k \varphi(\rho\sigma, x_n) x_n^{\gamma_n - 1} \right) \bigg|_{x_n = \rho} \rho^{n+|\gamma'|-2} d\rho \sigma^{\gamma'} dS, \tag{4.32}$$

$$(\delta_{\gamma,2}^{(k)}(|x'|^2 - x_n^2), \varphi(x))_\gamma$$
$$= (-1)^k \frac{1}{2} \int\limits_0^\infty \int\limits_{S_1^+(n-1)} \left(\left(\frac{1}{2\rho} \frac{\partial}{\partial \rho} \right)^k \varphi(\rho\sigma, x_n) \rho^{n+|\gamma'|-3} \right) \bigg|_{\rho = x_n} x_n^{\gamma_n} dx_n \sigma^{\gamma'} dS, \tag{4.33}$$

where $x' = (x_1, ..., x_{n-1})$, $\gamma' = (\gamma_1, ..., \gamma_{n-1})$, $|\gamma'| = \gamma_1 + ... + \gamma_{n-1}$.

Integrals (4.30) and (4.31) as well as (4.32) and (4.33) converge and match for $k < \frac{n+|\gamma|-2}{2}$. More precisely, (4.30) matches to (4.31) and (4.32) matches to (4.33) for $k < \frac{n+|\gamma|-2}{2}$ for any $\varphi \in S_{ev}$. If $k \geq \frac{n+|\gamma|-2}{2}$, these integrals should be understood in the sense of their regularizations.

Proof. Let us find the derivative of order k of $\delta_\gamma(P)=\delta_\gamma(x_1^2-|x''|^2)$ when $p=1$, $q=n-1$. After passing to spherical coordinates $x''=\rho\sigma$, $\sigma\in\mathbb{R}_+^{n-1}$, we get $P=r^2-s^2$ and

$$
\begin{aligned}
&(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma=(\delta^{(k)}(x_1^2-|x''|^2),\varphi(x))_\gamma\\
&=\int\limits_0^\infty\int\limits_{S_1^+(n)}\left(\frac{\partial^k}{\partial P^k}\delta(x_1^2-\rho^2)\right)\rho^{n+|\gamma''|-2}x_1^{\gamma_1}dx_1d\rho\int\limits_{S_1^+(n-1)}\varphi(x_1,\rho\sigma)\sigma^{\gamma'}dS\\
&=\{x_1^2=u,\rho^2=v\}\\
&=\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\left(\frac{\partial^k}{\partial u^k}\delta(u-v)\right)\varphi(\sqrt{u},\sqrt{v}\sigma)u^{\frac{\gamma_1-1}{2}}v^{\frac{n+|\gamma''|-1}{2}-1}dudv\sigma^{\gamma'}dS\\
&=(-1)^k\frac{1}{4}\int\limits_0^\infty\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\delta(u-v)\left(\frac{\partial^k}{\partial u^k}\varphi(\sqrt{u},\sqrt{v}\sigma)u^{\frac{\gamma_1-1}{2}}\right)\\
&\quad\times v^{\frac{n+|\gamma''|-1}{2}-1}dudv\sigma^{\gamma'}dS\\
&=(-1)^k\frac{1}{4}\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\left(\frac{\partial^k}{\partial u^k}\varphi(\sqrt{u},\sqrt{v}\sigma)u^{\frac{\gamma_1-1}{2}}\right)\Big|_{u=v}v^{\frac{n+|\gamma''|-1}{2}-1}dv\sigma^{\gamma'}dS.
\end{aligned}
$$

Remembering that $u=x_1^2$, $v=\rho^2$ we go back to the variables x_1 and ρ:

$$
\begin{aligned}
&(\delta_\gamma^{(k)}(P),\varphi(x))_\gamma\\
&=(-1)^k\frac{1}{2}\int\limits_0^\infty\int\limits_{S_1^+(n-1)}\left(\left(\frac{1}{2x_1}\frac{\partial}{\partial x_1}\right)^k\varphi(x_1,\rho\sigma)x_1^{\gamma_1-1}\right)\Big|_{x_1=\rho}\rho^{n+|\gamma''|-2}d\rho\sigma^{\gamma'}dS,
\end{aligned}
$$

which gives (4.31). Similarly, we get representations (4.30), (4.32), and (4.33).

The resulting integral representations of $\delta_{\gamma,1}^{(k)}(P)$ and $\delta_{\gamma,2}^{(k)}(P)$ converge and coincide for $k<\frac{n+|\gamma|-2}{2}$ for any $\varphi\in S_{ev}$. If $k\geq\frac{n+|\gamma|-2}{2}$, then these integrals must be understood in the sense of their regularizations, as in Lemma 13. $\square$

4.2 Weighted generalized functions realized by the degrees of quadratic forms

In this section, we consider weighted generalized functions which are complex powers of indefinite quadratic forms. In addition to the obvious application to B-hyperbolic

and B-ultrahyperbolic equations, such functions are closely related to the Radon transform on manifolds.

4.2.1 Weighted generalized functions $P^{\lambda}_{\gamma,\pm}$

In this section we give the definitions of $P^{\lambda}_{\gamma,\pm}$ and find the singular points of this function.

Let $p\geq 1$, $q\geq 1$, $\gamma=(\gamma',\gamma'')$, $\gamma'=(\gamma_1,...,\gamma_p)$, $\gamma''=(\gamma_{p+1},...,\gamma_n)$, and

$$P(x)=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2, \qquad n=p+q.$$

Definition 30. *We define the weighted generalized functions $P^{\lambda}_{\gamma,+}$ and $P^{\lambda}_{\gamma,-}$ by*

$$(P^{\lambda}_{\gamma,+},\varphi)_\gamma=\int\limits_{\{P(x)>0\}^+} P^{\lambda}(x)\varphi(x)x^{\gamma}dx, \qquad \varphi\in S_{ev}, \tag{4.34}$$

and

$$(P^{\lambda}_{\gamma,-},\varphi)_\gamma=\int\limits_{\{P(x)<0\}^+} (-P(x))^{\lambda}\varphi(x)x^{\gamma}dx, \qquad \varphi\in S_{ev}, \tag{4.35}$$

where $\{P(x)>0\}^+=\{x\in\mathbb{R}^n_+:P(x)>0\}$, $\{P(x)<0\}^+=\{x\in\mathbb{R}^n_+:P(x)<0\}$, $\lambda\in\mathbb{C}$.

Let us find the singularities of $P^{\lambda}_{\gamma,\pm}$. We first put $p>1$, $q>1$. Transforming $P^{\lambda}_{\gamma,+}$ using bipolar coordinates (4.20) we obtain

$$(P^{\lambda}_{\gamma,+},\varphi)_\gamma=\int\limits_0^{\infty}\int\limits_0^{r}(r^2-s^2)^{\lambda}\psi(r,s)r^{p+|\gamma'|-1}s^{q+|\gamma''|-1}drds, \tag{4.36}$$

where $\psi(r,s)=\frac{1}{2}\int\limits_{\{|\omega'|=1\}^+}\int\limits_{\{|\omega''|=1\}^+}\varphi(r\omega',s\omega'')\omega^{\gamma}dS'dS''$ (see (4.23)). Now let us pass in (4.36) to variables $u=r^2$, $v=s^2$:

$$(P^{\lambda}_{\gamma,+},\varphi)_\gamma=\frac{1}{4}\int\limits_0^{\infty}\int\limits_0^{u}(u-v)^{\lambda}\psi_1(u,v)u^{\frac{p+|\gamma'|}{2}-1}s^{\frac{q+|\gamma''|}{2}-1}dudv,$$

where $\psi_1(u,v)=\psi(r,s)$ with $u=r^2$, $v=s^2$. Finally, we write $v=ut$, which transforms $P^{\lambda}_{\gamma,+}$ to the form

$$(P^{\lambda}_{\gamma,+},\varphi)_\gamma=\int\limits_0^{\infty}u^{\lambda+\frac{p+q+|\gamma|}{2}-1}\Phi(\lambda,u)du, \tag{4.37}$$

where

$$\Phi(\lambda,u)=\frac{1}{4}\int\limits_0^1(1-t)^{\lambda}t^{\frac{q+|\gamma''|}{2}-1}\psi_1(u,tu)dt. \tag{4.38}$$

Formula (4.37) shows that $P^{\lambda}_{\gamma,+}$ has two sets of poles. The first of these consists of the poles of function $\Phi(\lambda,u)$. Namely, for $t=1$ the function $\Phi(\lambda,u)$ has simple poles at

$$\lambda=-1,-2,...,-k,... \tag{4.39}$$

with residues

$$\operatorname*{res}_{\lambda=-k}\Phi(\lambda,u)=\frac{1}{4}\frac{(-1)^{k-1}}{(k-1)!}\frac{\partial^{k-1}}{\partial t^{k-1}}\left[t^{\frac{q+|\gamma''|-2}{2}}\psi_1(u,tu)\right]_{t=1}. \tag{4.40}$$

Besides, at regular points of $\Phi(\lambda,u)$ function (4.37) has poles at

$$\lambda=-\frac{n+|\gamma|}{2},\ -\frac{n+|\gamma|}{2}-1,\ ...,\ -\frac{n+|\gamma|}{2}-k,\ ..., \tag{4.41}$$

with residues

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}(P^{\lambda}_{\gamma,+},\varphi)_{\gamma}=\frac{1}{k!}\frac{\partial^k}{\partial u^k}\Phi\left(-\frac{n+|\gamma|}{2}-k,u\right)\Bigg|_{u=0}. \tag{4.42}$$

Sets of poles were obtained for $p>1$, $q>1$; however, the same sets are obtained when $p\geq 1$, $q\geq 1$. All following results concerning weighted generalized functions related to indefinite quadratic forms are valid for $p\geq 1$, $q\geq 1$.

So we have three cases.

The first case is when λ is a point in the first set (4.39) but does not belong to the second set (4.41). The second case is when λ belongs to the second set (4.41) but $\lambda\neq -k$, $k\in\mathbb{N}$. The third case is when λ is in both sets (4.39) and (4.41) simultaneously. We study these three cases separately, presenting the results in the form of the following three theorems. Residues of $P^{\lambda}_{\gamma,\pm}$ at poles are expressed through the weighted generalized function (4.18).

Theorem 38. *If $p\geq 1$, $q\geq 1$, $\lambda=-k$, $k\in\mathbb{N}$, and $n+|\gamma|\in\mathbb{R}\backslash\mathbb{N}$ or $n+|\gamma|\in\mathbb{N}$ and $n+|\gamma|=2k-1$, $k\in\mathbb{N}$, and besides, if $n+|\gamma|$ is even and $k<\frac{n+|\gamma|}{2}$, then the weighted generalized function $P^{\lambda}_{\gamma,+}$ has simple poles at such λ with residues*

$$\operatorname*{res}_{\lambda=-k}P^{\lambda}_{\gamma,+}=\frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}_{\gamma,1}(P). \tag{4.43}$$

Proof. Let us write $\Phi(\lambda,u)$ in the neighborhood of $\lambda=-k$ in the form

$$\Phi(\lambda,u)=\frac{\Phi_0(u)}{\lambda+k}+\Phi_1(\lambda,u),\qquad \Phi_0(u)=\operatorname*{res}_{\lambda=-k}\Phi(\lambda,u),$$

where $\Phi_1(\lambda,u)$ is regular at the $\lambda=-k$ weighted function. We obtain

$$(P^{\lambda}_{\gamma,+},\varphi)_\gamma=\frac{1}{\lambda+k}\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_0(u)du+\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_1(\lambda,u)du. \quad (4.44)$$

Integrals in (4.44) are regular at λ for $\lambda=-k$. So $(P^{\lambda}_{\gamma,+},\varphi)_\gamma$ has a simple pole at such a point and using (4.40) we can write

$$\operatorname*{res}_{\lambda=-k}(P^{\lambda}_{\gamma,+},\varphi)=\frac{(-1)^{k-1}}{4(k-1)!}\int\limits_0^\infty u^{\frac{n+|\gamma|}{2}-k-1}\frac{\partial^{k-1}}{\partial t^{k-1}}\left[t^{\frac{q+|\gamma''|}{2}-1}\psi_1(u,tu)\right]_{t=1}du. \quad (4.45)$$

If in (4.45) we put $tu=v$, then we obtain

$$\operatorname*{res}_{\lambda=-k}(P^{\lambda}_{\gamma,+},\varphi)=\frac{(-1)^{k-1}}{4(k-1)!}\int\limits_0^\infty \frac{\partial^{k-1}}{\partial v^{k-1}}\left[v^{\frac{q+|\gamma''|}{2}-1}\psi_1(u,v)\right]_{v=u}u^{\frac{p+|\gamma'|}{2}-1}du, \quad (4.46)$$

where integral is understood in the sense of its regularization at $k\geq\frac{n}{2}$. Note that if we write $u=r^2$ and $v=s^2$ in the formula for the $(k-1)$-th derivative of function $\delta_\gamma(P)$, defined by formula (4.21), we get

$$(\delta^{(k-1)}_{\gamma,1}(P),\varphi)_\gamma=\frac{1}{2}\int\limits_0^\infty\left[\frac{\partial^{k-1}}{\partial v^{k-1}}v^{\frac{q+|\gamma''|}{2}-1}\psi_1(u,v)\right]_{v=u}u^{\frac{p+|\gamma'|}{2}-1}du, \quad (4.47)$$

where

$$\psi_1(u,v)=\frac{1}{2}\int\limits_{S_p^+}\int\limits_{S_q^+}\varphi(\sqrt{u}\omega',\sqrt{v}\omega'')\omega^\gamma dS_p dS_q.$$

Formulas (4.46) and (4.47) give (4.42). For $k\geq\frac{n}{2}$ integral in (4.47) is to be understood in the sense of its regularization. In the case $n+|\gamma|\in\mathbb{R}\backslash\mathbb{N}$ or $n+|\gamma|\in\mathbb{N}$ and $n+|\gamma|=2k-1$, $k\in\mathbb{N}$, the regularization of integral in (4.47) is defined as its analytical continuation. □

Now let us consider the case when the singular point λ is in the second set (4.41), but not in the first (4.39). If $\lambda=-\frac{n+|\gamma|}{2}-k$, $k=0,1,2,...$, and $n+|\gamma|\in\mathbb{R}\backslash\mathbb{N}$ or $n+|\gamma|\in\mathbb{N}$ and $n+|\gamma|=2k-1$, $k\in\mathbb{N}$, then the weighted generalized function $\Phi(\lambda,u)$ is regular in the neighborhood of $\lambda=-\frac{n+|\gamma|}{2}-k$. So $(P^{\lambda}_{\gamma,+},\varphi)_\gamma$ will have a simple pole with residue given by (4.42). Thus the residue of $P^{\lambda}_{\gamma,+}$ at $\lambda=-\frac{n+|\gamma|}{2}-k$ is a weighted functional concentrated at the origin.

Before getting the expression for residue $\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\mathrm{res}}(P^\lambda_{\gamma,+},\varphi)$ through derivatives of function $\varphi(x)$ at the origin we get one useful formula.

Consider the B-ultrahyperbolic operator (4.1):

$$\Box_\gamma=\Box_{\gamma',\gamma''}=B_{\gamma_1'}+...+B_{\gamma_p'}-B_{\gamma''_{p+1}}-B_{\gamma''_{p+q}},\qquad B_{\gamma_i}=\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}.$$

Applying $\Box_\gamma$ to the $(\lambda+1)$-th power of quadratic form

$$P(x)=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2,\ n=p+q,\ p>1,\ q>1,$$

we obtain

$$\Box_\gamma P^{\lambda+1}(x)=4(\lambda+1)\left(\lambda+\frac{n+|\gamma|}{2}\right)P^\lambda(x). \tag{4.48}$$

Theorem 39. *Let $p\geq 1$, $q\geq 1$, $n+|\gamma|$ is not a natural number or $n+|\gamma|\in\mathbb{N}$ and $n+|\gamma|=2k-1$, $k\in\mathbb{N}$. Then either $p+|\gamma'|$ is not natural or $p+|\gamma'|\in\mathbb{N}$, $p+|\gamma'|=2m-1$, $m\in\mathbb{N}$, and $q+|\gamma''|$ is even. In this case $P^\lambda_{\gamma,+}$ has simple poles at $\lambda=-\frac{n+|\gamma|}{2}-k$, $k\in\mathbb{N}\cup\{0\}$, with residues*

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\mathrm{res}} P^\lambda_{\gamma,+}=\frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^{n+2k}k!}\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\Box_\gamma^k\delta_\gamma(x).$$

If $p+|\gamma'|$ is even, then $P^\lambda_{\gamma,+}$ is a regular generalized weighted function at $\lambda=-\frac{n+|\gamma|}{2}-k$, $k\in\mathbb{N}\cup\{0\}$.

Proof. Consider first $\lambda=-\frac{n+|\gamma|}{2}$. Using formula (4.42) we can write

$$\underset{\lambda=-\frac{n+|\gamma|}{2}}{\mathrm{res}}(P^\lambda_{\gamma,+},\varphi)_\gamma=\Phi\left(-\frac{n+|\gamma|}{2},0\right)=\frac{\psi_1(0,0)}{4}\int\limits_0^1(1-t)^{-\frac{n+|\gamma|}{2}}t^{\frac{q+|\gamma''|}{2}}dt$$
$$=\frac{1}{4}\,\psi_1(0,0)\,\frac{\Gamma\left(\frac{q+|\gamma''|}{2}\right)\Gamma\left(-\frac{n+|\gamma|}{2}+1\right)}{\Gamma\left(-\frac{p+|\gamma'|}{2}+1\right)}. \tag{4.49}$$

From the last formula it follows that if $p+|\gamma'|$ is even, then $\underset{\lambda=-\frac{n+|\gamma|}{2}}{\mathrm{res}}(P^\lambda_{\gamma,+},\varphi)=0$.

Now let $p+|\gamma'|$ not be a natural number or $p+|\gamma'|\in\mathbb{N}$, $p+|\gamma'|=2k-1$, $k\in\mathbb{N}$, and $q+|\gamma''|$ is even. We have

$$\psi_1(0,0)=\psi(0,0)=\varphi(0)\int\limits_{S_p^+}\int\limits_{S_q^+}\omega^\gamma\,dS_p dS_q=\varphi(0)|S_1^+(p)|_{\gamma'}|S_1^+(q)|_{\gamma''}, \tag{4.50}$$

where

$$|S_1^+(p)|_{\gamma'} = \frac{\prod\limits_{i=1}^{p} \Gamma\left(\frac{\gamma_i'+1}{2}\right)}{2^{p-1}\Gamma\left(\frac{p+|\gamma'|}{2}\right)}, \quad |S_1^+(q)|_{\gamma''} = \frac{\prod\limits_{i=1}^{q} \Gamma\left(\frac{\gamma_i''+1}{2}\right)}{2^{q-1}\Gamma\left(\frac{q+|\gamma''|}{2}\right)}. \tag{4.51}$$

After a simple calculation, we get

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}} (P_{\gamma,+}^{\lambda}, \varphi)_\gamma = \frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^n} \frac{\prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)} \varphi(0).$$

Besides,

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}} P_{\gamma,+}^{\lambda} = \frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^n} \frac{\prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)} \delta_\gamma(x). \tag{4.52}$$

Using the Green theorem and formula (4.48) we obtain

$$\int\limits_{\{P(x)>0\}^+} \Big(\varphi(x)[\Box_\gamma P^{\lambda+1}(x)] - P^{\lambda+1}(x)[\Box_\gamma \varphi(x)]\Big)\, x^\gamma\, dx = 0,$$

so

$$(P_{\gamma,+}^{\lambda}, \varphi)_\gamma = \frac{1}{2(\lambda+1)(2\lambda+n+|\gamma|)} (P_{\gamma,+}^{\lambda+1}, \Box_\gamma \varphi)_\gamma. \tag{4.53}$$

Applying formula (4.53) k times gives

$$(P_{\gamma,+}^{\lambda}, \varphi)_\gamma = \frac{(P_{\gamma,+}^{\lambda+k}, \Box_\gamma^k \varphi)_\gamma}{2^{2k}(\lambda+1)\ldots(\lambda+k)\left(\lambda+\frac{n+|\gamma|}{2}\right)\ldots\left(\lambda+\frac{n+|\gamma|}{2}+k-1\right)}. \tag{4.54}$$

Consequently,

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k} (P_{\gamma,+}^{\lambda}, \varphi)_\gamma = \operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k} (P_{\gamma,+}^{\lambda+k}, \Box_\gamma^k \varphi)_\gamma$$
$$\times \left.\frac{1}{2^{2k}(\lambda+1)\ldots(\lambda+k)\left(\lambda+\frac{n+|\gamma|}{2}\right)\ldots\left(\lambda+\frac{n+|\gamma|}{2}+k-1\right)}\right|_{\lambda=-\frac{n+|\gamma|}{2}-k}$$

and

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k} (P_{\gamma,+}^{\lambda+k}, \Box_\gamma^k \varphi)_\gamma = \operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}} (P_{\gamma,+}^{\lambda}, \Box_\gamma^k \varphi)_\gamma.$$

Therefore, if $p+|\gamma'|$ is even, then residues vanish. If $p+|\gamma'|$ is not natural or $p+|\gamma'|\in\mathbb{N}$, $p+|\gamma'|=2k-1$ and $k\in\mathbb{N}$, then (4.52) gives

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\mathrm{res}}(P^{\lambda}_{\gamma,+},\varphi)_\gamma=\frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^{n+2k}k!}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}(\square^k_\gamma\delta_\gamma(x),\varphi)_\gamma.$$

The proof is complete. □

Theorem 40. *Let $p\geq 1$, $q\geq 1$. If $n+|\gamma|$ is even, $p+|\gamma'|$ and $q+|\gamma''|$ are even, $k\in\mathbb{N}\cup\{0\}$, then $P^\lambda_{\gamma,+}$ has simple poles at $\lambda=-\frac{n+|\gamma|}{2}-k$ with residues*

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\mathrm{res}}P^{\lambda}_{\gamma,+}=\frac{1}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\left[(-1)^{\frac{n+|\gamma|}{2}+k-1}\delta_{\gamma,1}^{\left(\frac{n+|\gamma|}{2}+k-1\right)}(P)\right.$$
$$\left.+\frac{(-1)^{\frac{q+|\gamma''|}{2}}}{2^{2k}k!}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\square^k_\gamma\delta_\gamma(x)\right].$$

If $p+|\gamma'|$ and $q+|\gamma''|$ are not natural or $p+|\gamma'|,q+|\gamma''|\in\mathbb{N}$, and $p+|\gamma'|=2m-1$, $q+|\gamma''|=2k-1$, $m,k\in\mathbb{N}$, then $P^\lambda_{\gamma,+}$ has a second order pole at points $\lambda=-\frac{n+|\gamma|}{2}-k$. Coefficients $c^{(k)}_{-2}$ and $c^{(k)}_{-1}$ of decomposition $P^\lambda_{\gamma,+}$ in the Laurent series in the neighborhood of points $\lambda=-\frac{n+|\gamma|}{2}-k$ are

$$c^{(0)}_{-1}=\frac{1}{\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\left[(-1)^{\frac{n+|\gamma|}{2}+k-1}\delta_{\gamma,1}^{\left(\frac{n+|\gamma|}{2}+k-1\right)}(P)+\frac{(-1)^{\frac{n+|\gamma|}{2}-1}}{2^{2k}k!}\right.$$
$$\left.\times\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\sin\left(\frac{p+|\gamma'|}{2}\pi\right)\left(\psi\left(\frac{p+|\gamma'|}{2}\right)-\psi\left(\frac{n+|\gamma|}{2}\right)\right)\square^k_\gamma\delta_\gamma(x)\right],$$

$$c^{(k)}_{-2}=(-1)^{\frac{n+|\gamma|}{2}+1}\frac{\sin\frac{\pi(p+|\gamma'|)}{2}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n+2k}k!\pi\,\Gamma\left(\frac{n+|\gamma|+k}{2}\right)}\square^k_\gamma\delta_\gamma(x),$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$.

Proof. Let $n+|\gamma|$ be even and $\lambda=-\frac{n+|\gamma|}{2}-k$, $k\in\mathbb{N}\cup\{0\}$. We first write $(P^\lambda_{\gamma,+},\varphi)_\gamma$ in the form

$$(P^\lambda_{\gamma,+},\varphi)_\gamma=\frac{1}{\lambda+k}\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_0(u)du+\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_1(\lambda,u)du,\quad(4.55)$$

where $\Phi_0(u) = \underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\text{res}} \Phi(\lambda, u)$ and $\Phi_1(\lambda, u)$ is regular at $\lambda = -\frac{n+|\gamma|}{2} - k$. Each integral in (4.55) can have a simple pole at $\lambda = -\frac{n+|\gamma|}{2} - k$, so $(P^{\lambda}_{\gamma,+}, \varphi)_\gamma$ can have a second order pole at $\lambda = -\frac{n+|\gamma|}{2} - k$. In the neighborhood of such a point we may expand $P^{\lambda}_{\gamma,+}$ in the Laurent series

$$P^{\lambda}_{\gamma,+} = \frac{c^{(k)}_{-2}}{\left(\lambda + \frac{n+|\gamma|}{2} + k\right)^2} + \frac{c^{(k)}_{-1}}{\lambda + \frac{n+|\gamma|}{2} + k} + \ldots .$$

Let us find coefficients $c^{(k)}_{-1}$ and $c^{(k)}_{-2}$. We have

$$(c^{(k)}_{-2}, \varphi)_\gamma = \underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\text{res}} \int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1} \Phi_0(u) du = \frac{1}{k!} \Phi^{(k)}_0(0).$$

If $k = 0$, then $c^{(0)}_{-2} = \Phi_0(0)$. In accordance with (4.38) we get

$$\begin{aligned}\Phi_0(0) &= \frac{1}{4} \psi_1(0,0) \underset{\lambda=-\frac{n+|\gamma|}{2}}{\text{res}} \int\limits_0^1 (1-t)^\lambda t^{\frac{q+|\gamma''|-2}{2}} dt \\ &= \psi_1(0,0) \underset{\lambda=-\frac{n+|\gamma|}{2}}{\text{res}} \frac{\Gamma\left(\frac{q+|\gamma''|}{2}\right) \Gamma(\lambda+1)}{4\Gamma\left(\lambda + \frac{q+|\gamma''|}{2} + 1\right)}.\end{aligned}$$

Taking into account that $\psi_1(0,0) = \varphi(0) |S^+_1(p)|_{\gamma'} |S^+_1(q)|_{\gamma''}$, where $|S^+_1(p)|_{\gamma'}$ and $|S^+_1(q)|_{\gamma''}$ are given by (4.51), we can write

$$(c^{(0)}_{-2}, \varphi)_\gamma =$$

$$\frac{(-1)^{\frac{n+|\gamma|}{2}+1} B\left(\frac{p+|\gamma'|}{2}, \frac{q+|\gamma''|}{2}\right)}{4\pi} \sin \frac{\pi(p+|\gamma'|)}{2} |S^+_1(p)|_{\gamma'} |S^+_1(q)|_{\gamma''} \varphi(0).$$

Then $p + |\gamma'|$ is even (in this case $q + |\gamma''|$ is also even) and we have $c^{(k)}_{-2} = 0$. Thus $(P^{\lambda}_{\gamma,+}, \varphi)_\gamma$ has just a simple pole at $\lambda = -\frac{n+|\gamma|}{2}$. If, on the other hand, $p + |\gamma'|$ is not natural or $p + |\gamma'| \in \mathbb{N}$ and $p + |\gamma'| = 2k - 1$, $k \in \mathbb{N}$, then

$$c^{(0)}_{-2} = (-1)^{\frac{n+|\gamma|}{2}+1} \frac{\sin \frac{\pi(p+|\gamma'|)}{2} \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n \pi \Gamma\left(\frac{n+|\gamma|}{2}\right)} \delta_\gamma(x).$$

Using the same way as in the proof of Theorem 39, if $p + |\gamma'|$ and $q + |\gamma''|$ are even, then $P^{\lambda}_{\gamma,+}$ has a simple pole at $\lambda = -\frac{n+|\gamma|}{2} - k$. If $p + |\gamma'|$ and $q + |\gamma''|$ are not natural

or $p+|\gamma'|, q+|\gamma''|\in\mathbb{N}$ and $p+|\gamma'|=2m-1$, $q+|\gamma''|=2k-1$, $m,k\in\mathbb{N}$, then

$$c_{-2}^{(k)}=(-1)^{\frac{n+|\gamma|}{2}+1}\frac{\sin\frac{\pi(p+|\gamma'|)}{2}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n+2k}k!\pi\Gamma\left(\frac{n+|\gamma|+k}{2}\right)}\square_\gamma^k\delta_\gamma(x).$$

Let us find now $c_{-1}^{(k)}$. We have

$$(c_{-1}^{(k)},\varphi)=\int\limits_0^\infty u^{-k-1}\Phi_0(u)du+\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}$$
$$\times\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_1\left(-\frac{n+|\gamma|}{2}-k,u\right)du.$$

Since $\Phi_0(u)=\operatorname*{res}_{\lambda=-k}\Phi(\lambda,u)$, using formulas (4.40) and (4.47) we obtain

$$\int\limits_0^\infty u^{-k-1}\Phi_0(u)du=\frac{(-1)^{\frac{n+|\gamma|}{2}+k-1}}{\Gamma\left(\frac{n+|\gamma|}{2}+k-1\right)}\left(\delta_{\gamma,1}^{(\frac{n+|\gamma|}{2}+k-1)}(P),\varphi\right)_\gamma.$$

Therefore

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}\int\limits_0^\infty u^{\lambda+\frac{n+|\gamma|}{2}-1}\Phi_1\left(-\frac{n+|\gamma|}{2}-k,u\right)du$$
$$=\frac{1}{k!}\frac{\partial^k\Phi_1\left(-\frac{n+|\gamma|}{2}-k,u\right)}{\partial u^k}\bigg|_{u=0}=(\alpha_\gamma^{(k)},\varphi)_\gamma$$

and

$$c_{-1}^{(k)}=\frac{(-1)^{\frac{n+|\gamma|}{2}+k-1}}{\Gamma\left(\frac{n+|\gamma|}{2}+k-1\right)}\delta_{\gamma,1}^{(\frac{n+|\gamma|}{2}+k-1)}(P)+\alpha_\gamma^{(k)}.$$

For $k=0$ we get

$$(\alpha_\gamma^{(0)},\varphi)_\gamma=\Phi_1\left(-\frac{n+|\gamma|}{2},0\right).$$

To find $\Phi_1\left(-\frac{n+|\gamma|}{2},0\right)$ consider $\Phi(\lambda,0)$. Using (4.49)–(4.51), we can write

$$\Phi(\lambda,0)=\varphi(0)\frac{\Gamma(\lambda+1)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{p+|\gamma'|}{2}\right)\Gamma\left(\lambda+\frac{q+|\gamma''|}{2}+1\right)}.$$

The formula $\Gamma(1-x)\Gamma(x)=\frac{\pi}{\sin\pi x}$ gives

$$\Phi(\lambda,0)=\frac{\sin\pi\left(\lambda+\frac{q+|\gamma''|}{2}\right)}{\sin\pi\lambda}\frac{\Gamma\left(-\lambda-\frac{q+|\gamma''|}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{p+|\gamma'|}{2}\right)\Gamma(-\lambda)}\varphi(0).$$

If $p+|\gamma'|$ and $q+|\gamma''|$ are even, then

$$\lim_{\lambda\to-\frac{n+|\gamma|}{2}}\frac{\sin\pi\left(\lambda+\frac{q+|\gamma''|}{2}\right)}{\sin\pi\lambda}=(-1)^{\frac{q+|\gamma''|}{2}}.$$

Since $\Phi(\lambda,0)$ is regular at $\lambda=-\frac{n+|\gamma|}{2}$ and

$$\Phi_1\left(-\frac{n+|\gamma|}{2},0\right)=\Phi\left(-\frac{n+|\gamma|}{2}\right),$$

we have

$$(\alpha_\gamma^{(0)},\varphi)_\gamma=(-1)^{\frac{q+|\gamma''|}{2}}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\varphi(0).$$

If $p+|\gamma'|$ and $q+|\gamma''|$ are not natural or $p+|\gamma'|,q+|\gamma''|\in\mathbb{N}$ and $p+|\gamma'|=2m-1$, $q+|\gamma''|=2k-1$, $m,k\in\mathbb{N}$, then $\Phi(\lambda,0)$ has a pole at $\lambda=-\frac{n+|\gamma|}{2}$. In this case

$$(\alpha_\gamma^{(0)},\varphi)_\gamma=\Phi_1\left(-\frac{n+|\gamma|}{2},0\right)=(-1)^{\frac{n+|\gamma|}{2}-1}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)$$
$$\times\frac{\sin\left(\frac{p+|\gamma'|}{2}\pi\right)\left(\psi\left(\frac{p+|\gamma'|}{2}\right)-\psi\left(\frac{n+|\gamma|}{2}\right)\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\varphi(0),$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$. From this we obtain

$$c_{-1}^{(0)}=\frac{1}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\left[(-1)^{\frac{n+|\gamma|}{2}-1}\delta_{\gamma,1}^{\left(\frac{n+|\gamma|}{2}-1\right)}(P)+\theta\delta_\gamma(x)\right],$$

where

$$\theta=(-1)^{\frac{q+|\gamma''|}{2}}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)$$

if $p+|\gamma'|$ and $q+|\gamma''|$ are even. If $p+|\gamma'|$ and $q+|\gamma''|$ are not natural or $p+|\gamma'|, q+|\gamma''|\in\mathbb{N}$ and $p+|\gamma'|=2m-1, q+|\gamma''|=2k-1, m, k\in\mathbb{N}$, then

$$\theta=(-1)^{\frac{n+|\gamma|}{2}-1}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\sin\left(\frac{p+|\gamma'|}{2}\pi\right)$$
$$\times\left(\psi\left(\frac{p+|\gamma'|}{2}\right)-\psi\left(\frac{n+|\gamma|}{2}\right)\right).$$

Finally, in order to obtain $c_{-1}^{(k)}$ for any $k\geq 1$, we use formula (4.54) again. This completes the proof. □

One can likewise obtain results for the weighted generalized function $P_{\gamma,-}^{\lambda}$ of the form

$$(P_{\gamma,-}^{\lambda},\varphi)_\gamma=\int\limits_{\{P(x)<0\}^+}(-P(x))^\lambda\varphi(x)x^\gamma\,dx,\qquad \varphi\in S_{ev}$$

(see (4.35)). All that we have said above about $P_{\gamma,+}^{\lambda}$ remains true also for $P_{\gamma,-}^{\lambda}$ except that p and q must be interchanged, and in all formulas $\delta_{\gamma,1}^{(k)}$ must be replaced by $(-1)^k\delta_{\gamma,2}^{(k)}$, $k=0,1,2,\ldots$.

4.2.2 The weighted generalized function $\mathscr{P}_\gamma^\lambda$ and $(P\pm i0)_\gamma^\lambda$ associated with a quadratic form with complex coefficients

This chapter focuses on the residues at singular points of the weighted generalized function $\mathscr{P}_\gamma^\lambda$ and $(P\pm i0)_\gamma^\lambda$ associated with a quadratic form with complex coefficients.

Let us consider the space of all quadratic forms of diagonal type

$$\mathscr{P}(x)=\sum_{k=1}^{n}g_k x_k^2,$$

with coefficients $g_k\in\mathbb{C}$, $k=1,...,n$. We can write a quadratic form $\mathscr{P}$ as $\mathscr{P}=P_1+iP_2$, where P_1, P_2 are quadratic forms with real coefficients.

Definition 31. *Let the quadratic form $\mathscr{P}=P_1+iP_2$ have a positive definite imaginary part P_2. We define a single valued analytic function of λ by the formula*

$$\mathscr{P}^\lambda=e^{\lambda(\ln|\mathscr{P}|+i\arg\mathscr{P})},\qquad 0<\arg\mathscr{P}<\pi. \tag{4.56}$$

Now with $\mathscr{P}^\lambda$ we will associate the weighted generalized function $\mathscr{P}_\gamma^\lambda=(P_1+iP_2)_\gamma^\lambda$, defined as

$$(\mathscr{P}_\gamma^\lambda,\varphi)_\gamma=\int\limits_{\mathbb{R}_+^n}\mathscr{P}^\lambda(x)\varphi(x)x^\gamma\,dx.$$

Let us consider the weighted generalized function from Definition 31. Since we put $0<\arg\mathscr{P}<\pi$, $\mathscr{P}_\gamma^\lambda$ is in the upper complex half-plane. Further, the weighted generalized function $\mathscr{P}_\gamma^\lambda$ is analytic not only in λ but also in the coefficients g_r, $r=1,...,n$, of the quadratic form $\mathscr{P}$. This means that $\mathscr{P}_\gamma^\lambda$ is analytic on the upper complex half-plane of all quadratic forms $\mathscr{P}=P_1+iP_2$, where P_2 is positive definite. Since the analytic continuation is unique, the weighted generalized function $\mathscr{P}_\gamma^\lambda$ is unique defined by its values on the set of quadratic forms $\mathscr{P}=iP_2$. So instead of $\mathscr{P}=P_1+iP_2$ we will consider the form $\mathscr{P}=iP_2=\sum\limits_{k=1}^n g_k x_k^2$. This means that $g_k=ib_k$, $b_k\in\mathbb{R}$, $k=1,...,n$, and the form $P_2(x)=\sum\limits_{k=1}^n b_k x_k^2$ is positive definite. Then

$$(\mathscr{P}_\gamma^\lambda,\varphi)_\gamma=\int\limits_{\mathbb{R}_+^n}\left(\sum_{k=1}^n ib_k x_k^2\right)^\lambda\varphi(x)x^\gamma dx=e^{i\frac{\pi\lambda}{2}}\int\limits_{\mathbb{R}_+^n}\left(\sum_{k=1}^n b_k x_k^2\right)^\lambda\varphi(x)x^\gamma dx. \tag{4.57}$$

Taking into account that $b_k=-ig_k$, $k=1,...,n$, we can pass in equality (4.57) from $\sqrt{b_k}x_k$ to x_k and obtain

$$(\mathscr{P}_\gamma^\lambda,\varphi)_\gamma=\eta_\gamma(g)e^{i\frac{\pi\lambda}{2}}\int\limits_{\mathbb{R}_+^n} r^{2\lambda}\varphi_g(x)x^\gamma dx, \tag{4.58}$$

where $g=(g_1,...,g_n)$, g_k are coefficients of the form iP_2, $k=1,...,n$, $\eta_\gamma(g)=\prod\limits_{k=1}^n(-ig_k)^{-\frac{1+\gamma_k}{2}}$, $r^{2\lambda}=\left(\sum\limits_{k=1}^n x_k^2\right)^\lambda$, and $\varphi_g(x)=\varphi\left(\frac{x_1}{\sqrt{-ig_1}},...,\frac{x_n}{\sqrt{-ig_n}}\right)$.

The weighted generalized function $r_\gamma^{2\lambda}$ was studied in Section 4.1.2.

Formula (4.13) shows that at $\lambda=-\frac{n+|\gamma|+2p}{2}$, $p=0,1,2,...$, the weighted generalized function $r_\gamma^{2\lambda}$ has simple poles with residues

$$\underset{\lambda=-\frac{n+|\gamma|}{2}}{\operatorname{res}}[(r_\gamma^{2\lambda},\varphi)_\gamma]=|S_1^+(n)|_\gamma\,\delta_\gamma(x),$$

and $\lambda=-\frac{n+|\gamma|+2p}{2}$, $p=0,1,2,...$, with residues

$$\underset{\lambda=-\frac{n+|\gamma|}{2}}{\operatorname{res}}\mathscr{P}_\gamma^\lambda=\eta_\gamma(g)e^{-i\frac{\pi(n+|\gamma|)}{4}}|S_1^+(n)|_\gamma\,\delta_\gamma(x). \tag{4.59}$$

Let us consider the differential operator

$$\mathcal{B}_{\gamma,g}=\sum_{k=1}^n\frac{1}{g_k}\left(\frac{\partial^2}{\partial x_k^2}+\frac{\gamma_k}{x_k}\frac{\partial}{\partial x_k}\right),$$

where g_k are coefficients of the quadratic form iP_2. We have

$$\mathcal{B}_{\gamma,g}\mathscr{P}_\gamma^{\lambda+1}=4(\lambda+1)\left(\lambda+\frac{n+|\gamma|}{2}\right)\mathscr{P}_\gamma^{\lambda}. \tag{4.60}$$

Applying formula (4.60) k times we obtain

$$\mathcal{B}_{\gamma,g}^k(\partial)\mathscr{P}_\gamma^{\lambda+k}=4^k(\lambda+1)...(\lambda+k)\left(\lambda+\frac{n+|\gamma|}{2}\right)...\left(\lambda+\frac{n+|\gamma|}{2}+k-1\right)\mathscr{P}_\gamma^{\lambda}. \tag{4.61}$$

So

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}\mathscr{P}_\gamma^{\lambda}=$$

$$\left.\frac{1}{4^k(\lambda+1)...(\lambda+k)\left(\lambda+\frac{n+|\gamma|}{2}\right)...\left(\lambda+\frac{n+|\gamma|}{2}+k-1\right)}\right|_{\lambda=-\frac{n+|\gamma|}{2}-k}\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}}\mathcal{B}_{\gamma,g}^k\mathscr{P}_\gamma^{\lambda+k}.$$

Consequently, inserting (4.59) we find residues of $\mathscr{P}_\gamma^{\lambda}$ at $\lambda=-\frac{n+|\gamma|}{2}-k$:

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}\mathscr{P}_\gamma^{\lambda}=\frac{\eta_\gamma(g)e^{-i\frac{\pi(n+|\gamma|)}{4}}|S_1^+(n)|_\gamma\Gamma\left(\frac{n+|\gamma|}{2}\right)}{4^k k!\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\mathcal{B}_{\gamma,g}^k\delta_\gamma. \tag{4.62}$$

Formula (4.62) was obtained for the quadratic form $\mathscr{P}=iP_2=\sum\limits_{k=1}^n g_k x_k^2$ lying on the imaginary axis. We should now continue it analytically to the entire upper half-plane and obtain (4.62) for all quadratic forms $\mathscr{P}=P_1+iP_2$, $P_1=\sum\limits_{k=1}^n a_k x_k^2$, $P_2=\sum\limits_{k=1}^n b_k x_k^2$, where P_2 is positive definite. The coefficients of $\mathcal{B}_{\gamma,g}$ are expressed analytically through the coefficients of $\mathscr{P}$, or more precisely, they are equal to $\frac{1}{g_k}$. So the analytic continuation of $\mathcal{B}_{\gamma,g}$ is known. Analytic continuation of the function $\eta_\gamma(g)$ to the entire upper half-plane is

$$\eta_\gamma(g)=\prod_{k=1}^n(b_k(1-i\mu_k))^{-\frac{1+\gamma_k}{2}},\qquad \mu_k=\frac{a_k}{b_k}. \tag{4.63}$$

So if $\mathscr{P}(x)=P_1+iP_2=\sum\limits_{k=1}^n g_k x_k^2=\sum\limits_{k=1}^n(a_k+ib_k)x_k^2$, $a_k,b_k\in\mathbb{R}$, $k=1,..,n$, is a quadratic form with a positive definite imaginary part, then the weighted generalized function $\mathscr{P}_\gamma^{\lambda}$ is a regular analytic function of λ everywhere except at $\lambda=-\frac{n+|\gamma|}{2}-k$, $k=0,1,2,...$, and at these points $\mathscr{P}_\gamma^{\lambda}$ has simple poles. The lower half-plane of quadratic forms, that is, quadratic forms with negative definite imaginary

parts $\mathscr{P}(x)=P_1-iP_2=\sum\limits_{k=1}^{n} g_k x_k^2=\sum\limits_{k=1}^{n}(a_k-ib_k)x_k^2$, may be subjected to similar analysis. In this case the analytic continuation of $\eta_\gamma(g)$ to the lower half-plane is

$$\eta_\gamma(g)=\prod_{k=1}^{n}(b_k(1+i\mu_k))^{-\frac{1+\gamma_k}{2}},\qquad \mu_k=\frac{a_k}{b_k}. \tag{4.64}$$

Therefore from (4.62) we get two formulas. The first for residues of $(P_1+iP_2)_\gamma^\lambda$:

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}}(P_1+iP_2)_\gamma^\lambda=\frac{e^{-i\frac{\pi(n+|\gamma|)}{4}}|S_1^+(n)|_\gamma\Gamma\left(\frac{n+|\gamma|}{2}\right)}{4^k k!\prod\limits_{k=1}^{n}(b_k-ia_k))^{\frac{1+\gamma_k}{2}}\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\mathcal{B}_{\gamma,g}^k\delta_\gamma. \tag{4.65}$$

The second for residues of $(P_1-iP_2)_\gamma^\lambda$:

$$\underset{\lambda=-\frac{n+|\gamma|}{2}-k}{\operatorname{res}}(P_1-iP_2)_\gamma^\lambda=\frac{e^{i\frac{\pi(n+|\gamma|)}{4}}|S_1^+(n)|_\gamma\Gamma\left(\frac{n+|\gamma|}{2}\right)}{4^k k!\prod\limits_{k=1}^{n}(b_k+ia_k))^{\frac{1+\gamma_k}{2}}\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\mathcal{B}_{\gamma,g}^k\delta_\gamma. \tag{4.66}$$

In (4.65) and (4.66) a quadratic form $P_2=\sum\limits_{k=1}^{n} b_k x_k^2$ is positive definite.

Using weighted generalized functions $(P_1\pm iP_2)_\gamma^\lambda$ we construct weighted generalized functions $(P+i0)_\gamma^\lambda$ and $(P-i0)_\gamma^\lambda$. Functions $(P\pm i0)_\gamma^\lambda$ are used for writing fundamental solution to the iterated operator $\square_\gamma^k$, where $\square_\gamma=\sum\limits_{k=1}^{p}B_{\gamma_k}-\sum\limits_{j=p+1}^{n}B_{\gamma_j}$, $k=1,2,...$, and for the definition of real powers of the B-ultrahyperbolic operator $\square_\gamma$, in particular the B-hyperbolic operator when $p=1$ and $n=2,3,...$.

Definition 32. *Consider the nondegenerate quadratic form with real coefficients*

$$A(x)=\sum_{k=1}^{n}a_k x_k^2,\qquad a_k\in\mathbb{R}. \tag{4.67}$$

Form A has in the canonical representation p positive and q negative terms, where $p+q=n$. Let

$$\mathscr{P}=A+iP',$$

where P' is a positive definite quadratic form with real coefficients. Without loss of generality we may assume that P' is of the form

$$P'=\varepsilon(x_1^2+...+x_n^2),\qquad \varepsilon>0.$$

Let $\mathscr{P}^{\lambda}_{\gamma}=(A+iP')^{\lambda}_{\gamma}$. *We define the weighted generalized functions* $(A+i0)^{\lambda}_{\gamma}$ *and* $(A-i0)^{\lambda}_{\gamma}$ *for* $\operatorname{Re}\lambda>0$ *as*

$$(A+i0)^{\lambda}_{\gamma}=\lim_{\varepsilon\to 0}(A+iP')^{\lambda}_{\gamma},$$
$$(A-i0)^{\lambda}_{\gamma}=\lim_{\varepsilon\to 0}(A-iP')^{\lambda}_{\gamma},$$

where the limit can be taken under the integral sign in $\int\limits_{\mathbb{R}^n_+}\mathscr{P}^{\lambda}\varphi x^{\gamma}dx$. *For* $\operatorname{Re}\lambda<0$, $\lambda\neq -k$, $\lambda\neq -\frac{n+|\gamma|}{2}-k+1$, $k\in\mathbb{N}$, *expressions* $(A+i0)^{\lambda}_{\gamma}$ *and* $(A-i0)^{\lambda}_{\gamma}$ *are defined as analytic continuations and then the limit* $\varepsilon\to 0$ *is taken.*

Theorem 41. *Residues of* $(A+i0)^{\lambda}_{\gamma}$ *and* $(A-i0)^{\lambda}_{\gamma}$ *at* $\lambda=-\frac{n+|\gamma|}{2}-k$, $k=0,1,2,...$, *are weighted generalized functions concentrated on the vertex of the cone* $A(x)=0$*:*

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}(A+i0)^{\lambda}_{\gamma}=\frac{e^{-i\frac{\pi(q+|\gamma''|)}{2}}|S_1^+(n)|_{\gamma}\Gamma\left(\frac{n+|\gamma|}{2}\right)}{4^k k!\prod\limits_{k=1}^{n}|a_k|^{\frac{1+\gamma_k}{2}}\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\,\mathcal{B}^k_{\gamma,a}\delta_{\gamma}(x)\tag{4.68}$$

and

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}(A-i0)^{\lambda}_{\gamma}=\frac{e^{i\frac{\pi(q+|\gamma''|)}{2}}|S_1^+(n)|_{\gamma}\Gamma\left(\frac{n+|\gamma|}{2}\right)}{4^k k!\prod\limits_{k=1}^{n}|a_k|^{\frac{1+\gamma_k}{2}}\Gamma\left(\frac{n+|\gamma|}{2}+k\right)}\,\mathcal{B}^k_{\gamma,a}\delta_{\gamma}(x),\tag{4.69}$$

where

$$\mathcal{B}_{\gamma,a}=\sum_{k=1}^{n}\frac{1}{a_k}\left(\frac{\partial^2}{\partial x_k^2}+\frac{\gamma_k}{x_k}\frac{\partial}{\partial x_k}\right),$$

where $a_k\in\mathbb{R}$ *are coefficients of quadratic form* A.

Proof. The weighted generalized functions $(A+i0)^{\lambda}_{\gamma}$ and $(A-i0)^{\lambda}_{\gamma}$ can be expressed through the weighted generalized functions $A^{\lambda}_{\gamma,+}$ and $A^{\lambda}_{\gamma,-}$ of the form

$$(A^{\lambda}_{\gamma,+},\varphi)_{\gamma}=\int\limits_{\{A(x)>0\}^+}A^{\lambda}(x)\varphi(x)x^{\gamma}dx,\qquad \varphi\in S_{ev},$$

and

$$(A^{\lambda}_{\gamma,-},\varphi)_{\gamma}=\int\limits_{\{A(x)<0\}^+}(-A(x))^{\lambda}\varphi(x)x^{\gamma}dx,\qquad \varphi\in S_{ev},$$

where $\{A(x)>0\}^+ = \{x\in\mathbb{R}^n_+ : A(x)>0\}$, $\{A(x)<0\}^+ = \{x\in\mathbb{R}^n_+ : A(x)<0\}$, $\lambda\in\mathbb{C}$, by

$$(A+i0)^\lambda_\gamma = A^\lambda_{\gamma,+} + e^{\pi\lambda i} A^\lambda_{\gamma,-}, \tag{4.70}$$

$$(A-i0)^\lambda_\gamma = A^\lambda_{\gamma,+} + e^{-\pi\lambda i} A^\lambda_{\gamma,-}. \tag{4.71}$$

Indeed, for $\mathrm{Re}\lambda > 0$ the weighted functionals $(A^\lambda_{\gamma,+},\varphi)_\gamma$ and $(A^\lambda_{\gamma,-},\varphi)_\gamma$ correspond to the functions

$$A^\lambda_{\gamma,+} = \begin{cases} A^\lambda & A>0, \\ 0 & A\le 0, \end{cases}$$

$$A^\lambda_{\gamma,-} = \begin{cases} 0 & A\ge 0, \\ (-A)^\lambda & A<0, \end{cases}$$

where A is defined by (4.67). Then

$$(A\pm i0)^\lambda_\gamma = \lim_{\varepsilon\to 0}(A\pm i\varepsilon|x|)^\lambda_\gamma = \begin{cases} A^\lambda & A\ge 0, \\ e^{\pm\lambda\pi i}|A|^\lambda & A<0. \end{cases}$$

Since the analytic continuation is unique, formulas (4.70) and (4.71) can be used for $\mathrm{Re}\,\lambda\le 0$. For $\lambda=-k$, $k\in\mathbb{N}$, the weighted generalized function $\mathscr{P}^\lambda_\gamma$ does not have poles and $(A\pm i0)^\lambda_\gamma$ in this case is introduced by formulas (4.70) and (4.71). So we deduce that the weighted generalized functions $(A\pm i0)^\lambda_\gamma$ are analytic in $\lambda\in\mathbb{C}$ everywhere except at $\lambda=-\frac{n+|\gamma|}{2}-k$, $k=0,1,2,...$, where they have simple poles with residues

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}(A\pm i0)^\lambda_\gamma = \lim_{\varepsilon\to 0}\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k}(A\pm i\varepsilon|x|)^\lambda_\gamma.$$

Since the quadratic form A has in canonical representation p positive and q negative terms, from (4.63) and (4.64) we obtain

$$\lim_{\varepsilon\to 0}\prod_{k=1}^n(\varepsilon-ia_k)^{-\frac{1+\gamma_k}{2}} = \prod_{k=1}^n|a_k|^{-\frac{1+\gamma_k}{2}}(-i)^{-\frac{p+|\gamma'|}{2}}\,i^{-\frac{q+|\gamma''|}{2}}$$
$$= e^{\frac{\pi i}{4}(p+|\gamma'|-q-|\gamma''|)}\prod_{k=1}^n|a_k|^{-\frac{1+\gamma_k}{2}},$$

$$\lim_{\varepsilon\to 0}\prod_{k=1}^n(\varepsilon+ia_k)^{-\frac{1+\gamma_k}{2}} = \prod_{k=1}^n|a_k|^{-\frac{1+\gamma_k}{2}}\,i^{-\frac{p+|\gamma'|}{2}}(-i)^{-\frac{q+|\gamma''|}{2}}$$
$$= e^{\frac{\pi i}{4}(-p-|\gamma'|+q+|\gamma''|)}\prod_{k=1}^n|a_k|^{-\frac{1+\gamma_k}{2}},$$

where $|\gamma'| = \gamma_1 + ... + \gamma_p$, $|\gamma''| = \gamma_{p+1} + ... + \gamma_{p+q}$. Therefore, applying formulas (4.65) and (4.66) we get (4.68) and (4.69). $\square$

If in (4.67) all $a_k = 1$, then the quadratic form is $P = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2$, $n = p+q$, and for this form the weighted generalized functions $(P + i0)^\lambda_\gamma$ and $(P - i0)^\lambda_\gamma$ for $\operatorname{Re}\lambda > 0$ are defined by the formulas

$$(P + i0)^\lambda_\gamma = \lim_{\varepsilon\to 0}(P + i P')^\lambda_\gamma,$$
$$(P - i0)^\lambda_\gamma = \lim_{\varepsilon\to 0}(P - i P')^\lambda_\gamma,$$

where the limit can be taken under the integral sign in $\int\limits_{\mathbb{R}^n_+} \mathscr{P}^\lambda \varphi x^\gamma dx$. For $\operatorname{Re}\lambda < 0$, $\lambda \neq -k$, $\lambda \neq -\frac{n+|\gamma|}{2} - k + 1$, $k \in \mathbb{N}$, expressions $(P + i0)^\lambda_\gamma$ and $(P - i0)^\lambda_\gamma$ are defined by analytical continuation and then the limit $\varepsilon \to 0$ is taken.

We have

$$(P + i0)^\lambda_\gamma = P^\lambda_{\gamma,+} + e^{\pi\lambda i} P^\lambda_{\gamma,-}, \tag{4.72}$$
$$(P - i0)^\lambda_\gamma = P^\lambda_{\gamma,+} + e^{-\pi\lambda i} P^\lambda_{\gamma,-}. \tag{4.73}$$

From formulas (4.68) and (4.69) it follows that residues of $(P + i0)^\lambda_\gamma$ and $(P - i0)^\lambda_\gamma$ at $\lambda = -\frac{n+|\gamma|}{2} - k$, $k = 0, 1, 2, ...$, are weighted generalized functions concentrated on the vertex of the $P(x) = 0$ cone:

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k} (P + i0)^\lambda_\gamma = \frac{e^{-i\frac{\pi(q+|\gamma''|)}{2}} |S_1^+(n)|_\gamma \Gamma\left(\frac{n+|\gamma|}{2}\right)}{4^k k! \prod\limits_{k=1}^{n} \Gamma\left(\frac{n+|\gamma|}{2} + k\right)} \square^k_\gamma \delta_\gamma(x) \tag{4.74}$$

and

$$\operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}-k} (P - i0)^\lambda_\gamma = \frac{e^{i\frac{\pi(q+|\gamma''|)}{2}} |S_1^+(n)|_\gamma \Gamma\left(\frac{n+|\gamma|}{2}\right)}{4^k k! \prod\limits_{k=1}^{n} \Gamma\left(\frac{n+|\gamma|}{2} + k\right)} \square^k_\gamma \delta_\gamma(x). \tag{4.75}$$

By (4.72) and (4.73), formulas

$$P^\lambda_{+,\gamma} = -\frac{1}{2i\sin\lambda\pi}\left(e^{-\pi\lambda i}(P + i0)^\lambda_\gamma - e^{\pi\lambda i}(P - i0)^\lambda_\gamma\right) \tag{4.76}$$

and

$$P^\lambda_{-,\gamma} = \frac{1}{2i\sin\lambda\pi}\left((P + i0)^\lambda_\gamma - (P - i0)^\lambda_\gamma\right) \tag{4.77}$$

follow.

4.3 Other weighted generalized functions associated with a quadratic form

The introduced functions $P^{\lambda}_{\pm,\gamma}$ and $(P \pm i0)^{\lambda}_{\gamma}$ are used to obtain a fundamental solution of the B-ultrahyperbolic equation and to construct hyperbolic Riesz B-potentials. However, these functions are not enough to obtain solutions to the Cauchy problem for the general Euler–Poisson–Darboux equation. In this section, we consider the functions that will be required to solve this problem.

4.3.1 Functions $(w^2 - |x|^2)^{\lambda}_{+,\gamma}$ and $(c^2 + P \pm i0)^{\lambda}_{\gamma}$

Here we consider the weighted generalized function associated with the positive definite quadratic form $(w^2 - |x|^2)^{\lambda}_{+,\gamma}$ and the weighted generalized function associated with an indefinite quadratic form $(c^2 + P \pm i0)^{\lambda}_{\gamma}$, where c and w do not depend on $x \in \mathbb{R}^n_+$.

Definition 33. *Let $x \in \mathbb{R}^n_+$ and w does not depend on x. We define the weighted generalized function $(w^2-|x|^2)^{\lambda}_{+,\gamma}$ by the formula*

$$((w^2 - |x|^2)^{\lambda}_{+,\gamma}, \varphi)_{\gamma} = \int\limits_{\{|x|<w\}^+} (w^2 - |x|^2)^{\lambda} \varphi(x) x^{\gamma} dx, \quad \varphi \in S_{ev}, \quad \lambda \in C, \tag{4.78}$$

where $\{|x| < w\}^+ = \{x \in \mathbb{R}^n_+ : |x| < w\}$.

Definition 34. *Let $\varphi \in S_{ev}$,*

$$\mathscr{P} = P \pm iP', \qquad P = \sum_{k=1}^{n} a_k x_k^2, \qquad a_k \in \mathbb{R},$$

the quadratic form P has p positive and q negative terms, where $p+q=n$, and P' is a positive definite quadratic form with real coefficients. Without loss of generality we put $P' = \varepsilon(x_1^2 + ... + x_n^2)$, $\varepsilon > 0$, where c does not depend on x. We define the weighted generalized functions $(c^2 + P \pm iP')^{\lambda}_{\gamma}$ by

$$((c^2 + P + iP')^{\lambda}_{\gamma}, \varphi(x))_{\gamma} = \int\limits_{\mathbb{R}^n_+} (c^2 + P + iP')\varphi(x) x^{\gamma} dx$$

and

$$((c^2 + P - iP')^{\lambda}_{\gamma}, \varphi(x))_{\gamma} = \int\limits_{\mathbb{R}^n_+} (c^2 + P - iP')\varphi(x) x^{\gamma} dx.$$

The weighted generalized functions $(c^2+P+i0)_\gamma^\lambda$ *and* $(c^2+P-i0)_\gamma^\lambda$ *for* $\operatorname{Re}\lambda>0$ *are*

$$(c^2+P+i0)_\gamma^\lambda=\lim_{\varepsilon\to 0}(c^2+P+iP')_\gamma^\lambda,$$
$$(c^2+P-i0)_\gamma^\lambda=\lim_{\varepsilon\to 0}(c^2+P-iP')_\gamma^\lambda,$$

where the limit $\varepsilon\to 0$ *can be taken under the integral sign in* $\int\limits_{\mathbb{R}_+^n}\mathscr{P}^\lambda\varphi x^\gamma\,dx$.

4.3.2 General weighted generalized functions connected with quadratic form

In this subsection we introduce the new family of weighted generalized functions connected with the quadratic form $P(x)=x_1^2+...+x_p^2-x_{p+1}^2-...-x_{p+q}^2$, $n=p+q$. Functions of this family are generalizations of studied earlier functions.

Let P be a quadratic form with real coefficients, let P_1 be a positive definite quadratic form, and $\mathscr{P}=P\pm iP_1$.

Definition 35. *Let* $\varphi\in S_{ev}$, *and* $f(z,\lambda)$ *is the entire function of* z *and* λ. *The weighted generalized functions* $\mathscr{P}_\gamma^\lambda f(\mathscr{P},\lambda)$ *are given by*

$$(\mathscr{P}_\gamma^\lambda f(\mathscr{P},\lambda),\varphi(x))_\gamma=\int\limits_{\mathbb{R}_+^n}\mathscr{P}_\gamma^\lambda\cdot f(\mathscr{P},\lambda)\varphi(x)dx,$$

where $\lambda\in\mathbb{C}$, $\operatorname{Re}\lambda>-1$, *and* $\mathscr{P}$ *is a complex quadratic form with positive definite imaginary part. For* $\operatorname{Re}\lambda>-1$ *the function* $\mathscr{P}_\gamma^\lambda f(\mathscr{P},\lambda)$ *is analytic of* λ. *For other meanings of* λ *the weighted generalized function* $\mathscr{P}_\gamma^\lambda f(\mathscr{P},\lambda)$ *is defined as analytical continuation.*

From the decomposition of $f(z,\lambda)$ to power series by z it follows that for the quadratic form with real coefficients P limits

$$(P\pm i0)_\gamma^\lambda f(P\pm i0,\lambda)=\lim_{P_1\to 0}\mathscr{P}_\gamma^\lambda f(\mathscr{P},\lambda),\qquad \mathscr{P}=P\pm iP_1,$$

exist.

From formulas (4.72) and (4.73) we get

$$(P+i0)_\gamma^\lambda f(P,\lambda)=P_{\gamma,+}^\lambda f(P_+,\lambda)+e^{\pi\lambda i}P_{\gamma,-}^\lambda f(P_-,\lambda),\tag{4.79}$$
$$(P-i0)_\gamma^\lambda f(P,\lambda)=P_{\gamma,+}^\lambda f(P_+,\lambda)+e^{-\pi\lambda i}P_{\gamma,-}^\lambda f(P_-,\lambda).\tag{4.80}$$

The family of functions given by Definition 35 is quite wide. In particular weighted generalized functions generated by Bessel functions $J_{\frac{n+|\gamma|}{2}+\lambda}(\mathscr{P}^{1/2})$, $K_{\frac{n+|\gamma|}{2}+\lambda}(\mathscr{P}^{1/2})$, $H^{(1)}_{\frac{n+|\gamma|}{2}+\lambda}(\mathscr{P}^{1/2})$, $H^{(2)}_{\frac{n+|\gamma|}{2}+\lambda}(\mathscr{P}^{1/2})$, and $I_{\frac{n+|\gamma|}{2}+\lambda}(\mathscr{P}^{1/2})$ are from this family.

4.4 Hankel transform of weighted generalized functions generated by the quadratic form

The purpose of this section is to prove the formulas for the multi-dimensional Hankel transform of weighted generalized functions generated by the definite and indefinite quadratic form considered earlier.

4.4.1 Hankel transform of r_γ^λ

In this subsection we present a Hankel transform of r_γ^λ following [177,251,242]. In [251,242] an analogue of formula (4.83) was obtained in the case when instead of Hankel transform a mixed Fourier–Bessel transform is used and in the weighted functional $(r_\gamma^\lambda,\varphi)_\gamma$ weight was taken only by one variable.

Now, we perform the formula expressing Hankel transform of any radial function from $L_1^\gamma(\mathbb{R}^n_+)$.

Lemma 16. *Let $\varphi(s)$ be a function of one variable and $\varphi(|x|)\in L_1^\gamma(\mathbb{R}^n_+)$. The Hankel transform of the radial function $\varphi(|x|)$ is the radial function and the following formula is valid:*

$$\mathbf{F}_\gamma[\varphi(|x|)](\xi)=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty \varphi(r)\, j_{\frac{n+|\gamma|-2}{2}}(|\xi|r)\, r^{n+|\gamma|-1}\,dr. \tag{4.81}$$

Proof. In the Hankel transform of the radial function replacing $\mathbf{j}_\gamma(x,\xi)$ to $\mathbf{P}_\xi^\gamma[e^{-i\langle x,\xi\rangle}]$ by formula (3.138) and going over the spherical coordinates $x=r\sigma$ we get

$$\begin{aligned}\mathbf{F}_\gamma[\varphi(|x|)](\xi)&=\int\limits_{\mathbb{R}^n_+}\varphi(|x|)\mathbf{j}_\gamma(x,\xi)\,x^\gamma\,dx\\&=\int\limits_0^\infty\varphi(r)r^{n+|\gamma|-1}\,dr\int\limits_{S_1^+(n)}\mathbf{P}_\xi^\gamma[e^{-ir\langle\sigma,\xi\rangle}]\,\sigma^\gamma\,dS.\end{aligned}\tag{4.82}$$

Using formulas (3.143) and (3.141) we can write

$$\begin{aligned}&\int\limits_{S_1^+(n)}\mathbf{P}_\xi^\gamma[e^{-ir\langle\sigma,\xi\rangle}]\,\sigma^\gamma\,dS\\&=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}\,2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)}\int\limits_{-1}^{1}e^{-i|\xi|rp}(1-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp\end{aligned}$$

$$= \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)}\sqrt{\pi}\,2^{\frac{n+|\gamma|-2}{2}}\Gamma\left(\frac{n+|\gamma|-1}{2}\right)$$
$$\times (|\xi|r)^{\frac{2-n-|\gamma|}{2}} J_{\frac{n+|\gamma|-2}{2}}(|\xi|r)$$
$$= \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\, j_{\frac{n+|\gamma|-2}{2}}(|\xi|r).$$

Returning to (4.82) we get

$$\mathbf{F}_\gamma[\varphi(|x|)](\xi) = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty \varphi(r)\, j_{\frac{n+|\gamma|-2}{2}}(|\xi|r)\, r^{n+|\gamma|-1}\, dr. \qquad \square$$

Theorem 42. *The Hankel transform of r_γ^λ in the sense of (1.83) is*

$$\mathbf{F}_\gamma[r_\gamma^\lambda](\xi) = \mathcal{D}_{n,\gamma}(\lambda)\begin{cases} |\xi|^{-n-|\gamma|-\lambda} & \lambda\neq 2k,\ \lambda\neq-(n+|\gamma|+2k),\\ (-\triangle_\gamma)_\xi^{\lambda/2}\varphi)\delta_\gamma & \lambda=2k,\\ |\xi|^{-n-|\gamma|-\lambda}\ln|\xi| & \lambda=-(n+|\gamma|+2k),\end{cases} \tag{4.83}$$

where $\delta_\gamma=\delta_\gamma(\xi)$ is the weighted delta-function, $k=0,1,2,...,$ and

$$\mathcal{D}_{n,\gamma}(\lambda) = \begin{cases} \dfrac{2^{|\gamma|+\lambda}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} & \lambda\neq 2k,\ \lambda\neq-(n+|\gamma|+2k),\\ 1 & \lambda=2k,\\ \prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\dfrac{(-1)^{\frac{n+|\gamma|+\lambda}{2}}2^{|\gamma|+\lambda+1}}{\left[-\frac{n+|\gamma|+\lambda}{2}\right]!\,\Gamma\left(-\frac{\lambda}{2}\right)} & \lambda=-(n+|\gamma|+2k).\end{cases}$$

Proof. Let $\operatorname{Re}\lambda>-(n+|\gamma|)$. Then r_γ^λ is a locally summable function. Assuming that $-(n+|\gamma|)<\operatorname{Re}\lambda<-(n+|\gamma|)/2$, in this case formula (4.81) is valid and

$$\mathbf{F}_\gamma[r_\gamma^\lambda](\xi) = \frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty j_{\frac{n+|\gamma|-2}{2}}(|\xi|r)\, r^{n+|\gamma|-1+\lambda}\, dr$$
$$= \frac{2^{\frac{|\gamma|-n}{2}}}{|\xi|^{\frac{n+|\gamma|-2}{2}}}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\int\limits_0^\infty r^{\frac{n+|\gamma|}{2}+\lambda} J_{\frac{n+|\gamma|-2}{2}}(|\xi|r)\, dr.$$

To calculate the integral, using the formula for the Weber type integral, we get

$$\int\limits_0^\infty r_\gamma^{\frac{n+|\gamma|}{2}+\lambda} J_{\frac{n+|\gamma|-2}{2}}(|\xi|r)\,dr = \frac{2^{\frac{n+|\gamma|}{2}+\lambda}\Gamma\left(\frac{n+|\gamma|+\lambda}{2}\right)}{|\xi|^{\frac{n+|\gamma|}{2}+\lambda+1}\Gamma\left(-\frac{\lambda}{2}\right)}.$$

Then, we need to show that the Hankel transform $(\mathbf{F}_\gamma[r_\gamma^\lambda](\xi),\varphi)_\gamma$ according to (1.83) is equal to

$$\mathcal{G}_{n,\gamma}(\lambda)\left(\frac{1}{|\xi|^{n+|\gamma|+\lambda}},\varphi\right)_\gamma = (r_\gamma^\lambda,\mathbf{F}_\gamma\varphi)_\gamma, \tag{4.84}$$

where $\mathcal{G}_{n,\gamma}(\lambda) = \frac{2^{|\gamma|+\lambda}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)}$, $\lambda\neq 2k$, $\lambda\neq-(n+|\gamma|+2k)$, $k=0,1,2,\ldots$. The right side of equality (4.84) is defined and analytic for all $\lambda\in\mathbb{C}$, since $\mathbf{F}_\gamma\varphi\in S_{ev}$. The left part is analytic for all $\lambda\in\mathbb{C}$ except for $\lambda=2k$ and $\lambda=-(n+|\gamma|+2k)$, $k=0,1,2,\ldots$.

In the case $\lambda=2k$ the Hankel transform of r_γ^λ has the form

$$(\mathbf{F}_\gamma[r_\gamma^{2k}],\varphi)_\gamma = \int\limits_{\mathbb{R}_+^n}|x|^{2k}\int\limits_{\mathbb{R}_+^n}\mathbf{j}_\gamma(x,\xi)\varphi(\xi)\,\xi^\gamma\,d\xi\,x^\gamma\,dx.$$

Note that if

$$|x|^{2k}\mathbf{j}_\gamma(x,\xi) = (-\triangle_\gamma)_\xi^k\,\mathbf{j}_\gamma(x,\xi),$$

then we have

$$\begin{aligned}(\mathbf{F}_\gamma[r_\gamma^{2k}],\varphi)_\gamma &= \int\limits_{\mathbb{R}_+^n}\int\limits_{\mathbb{R}_+^n}(-\triangle_\gamma)_\xi^k\,\mathbf{j}_\gamma(x,\xi)\varphi(\xi)\,\xi^\gamma\,d\xi\,x^\gamma\,dx\\ &= \int\limits_{\mathbb{R}_+^n}\mathbf{F}_\gamma[1](\xi)(-\triangle_\gamma)_\xi^k\,\varphi(\xi))\,\xi^\gamma\,d\xi = (\delta_\gamma,(-\triangle_\gamma)_\xi^k\varphi))_\gamma.\end{aligned}$$

Consider the case $\lambda=\lambda_k=-(n+|\gamma|+2k)$. We can write equality (4.84) for λ in the neighborhood of λ_k in the following way:

$$(\lambda-\lambda_k)(r_\gamma^\lambda,\mathbf{F}_\gamma\varphi)_\gamma = \mathfrak{a}(\lambda)\left(\frac{1}{|\xi|^{n+|\gamma|+\lambda}},\varphi\right)_\gamma,$$

where $\mathfrak{a}(\lambda)=(\lambda-\lambda_k)\mathcal{G}_{n,\gamma}(\lambda)$. Differentiating this equality by λ we obtain

$$\frac{d}{d\lambda}((\lambda-\lambda_k)r_\gamma^\lambda,\mathbf{F}_\gamma\varphi)_\gamma = \left(\frac{\mathfrak{a}'(\lambda)+\mathfrak{a}(\lambda)\ln|\xi|}{|\xi|^{n+|\gamma|+\lambda}},\varphi\right)_\gamma. \tag{4.85}$$

Then from (4.85) for $\lambda \to \lambda_k$ it follows that

$$(r_\gamma^{\lambda_k}, \mathbf{F}_\gamma \varphi)_\gamma = \mathfrak{a}(\lambda_k)\left(\frac{\ln|\xi|}{|\xi|^{N+|\gamma|+\lambda_k}}, \varphi\right)_\gamma .$$

We calculate the constant $\mathfrak{a}(\lambda_k)$, $\lambda_k = -(n+|\gamma|+2k)$:

$$\mathfrak{a}(\lambda_k) = \lim_{\lambda\to\lambda_k} \mathfrak{a}(\lambda) = \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right) \frac{(-1)^k}{2^{2k+n-1} k! \Gamma\left(\frac{n+|\gamma|+2k}{2}\right)},$$

and since $\lambda = \lambda_k = -(n+|\gamma|+2k)$, $k = -\frac{n+|\gamma|+\lambda}{2}$ and

$$\mathfrak{a}(\lambda_k) = \prod_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right) \frac{(-1)^{\frac{n+|\gamma|+\lambda}{2}} 2^{|\gamma|+\lambda+1}}{\left[-\frac{n+|\gamma|+\lambda}{2}\right]! \, \Gamma\left(-\frac{\lambda}{2}\right)}.$$

The proof is complete. □

4.4.2 Hankel transforms of functions $\mathscr{P}_\gamma^\lambda$, $(P \pm i0)_\gamma^\lambda$, and $P_{\gamma,\pm}^\lambda$

In this subsection to find the multi-dimensional Hankel transform of weighted generalized functions related to the indefinite quadratic form, we will use formula (4.83) when $\lambda \neq -(n+|\gamma|+2k)$, $\lambda \neq 2k$, $k = 0, 1, 2, \ldots$:

$$\mathbf{F}_\gamma[r^\lambda](\xi) = \frac{2^{|\gamma|+\lambda} \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right) \Gamma\left(\frac{n+|\gamma|+\lambda}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)} |\xi|^{-n-|\gamma|-\lambda}. \tag{4.86}$$

Theorem 43. *The Hankel transform of $\mathscr{P}_\gamma^\lambda$ for $\lambda \neq k$, $\lambda \neq -\left(\frac{n+|\gamma|}{2}+k\right)$, $k = 0, 1, 2, \ldots$, is*

$$\mathbf{F}_\gamma[\mathscr{P}_\gamma^\lambda](\xi)$$
$$= \frac{2^{2\lambda+|\gamma|} e^{-\frac{n+|\gamma|}{4} i\pi} \prod\limits_{i=1}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right) \Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)}{\Gamma(-\lambda) \sqrt{(-i\alpha_1)^{1+\gamma_1}} \ldots \sqrt{(-i\alpha_n)^{1+\gamma_n}}} \left(\frac{\xi_1^2}{\alpha_1} + \ldots + \frac{\xi_n^2}{\alpha_n}\right)^{-\frac{n+|\gamma|}{2}-\lambda}. \tag{4.87}$$

Proof. If the weighted generalized function $\mathscr{P}_\gamma^\lambda$ is an analytic function of $\alpha_1, \ldots, \alpha_n$ in the region $\operatorname{Im}\alpha_k > 0$, $k = 1, 2, \ldots, n$, then the Hankel transform of $\mathscr{P}_\gamma^\lambda$ is also an analytic function in the same region. Therefore in order to find $\mathbf{F}_\gamma[\mathscr{P}_\gamma^\lambda]$ we need only treat the case in which all α_k are imaginary and then analytically continue the found Hankel transform to the whole complex plane. Putting $\alpha_k = ib_k$, $b_k > 0$, $k =$

$1, 2, ..., n$, we obtain

$$\mathbf{F}_\gamma[\mathscr{P}_\gamma^\lambda](\xi) = e^{\frac{\pi}{2}\lambda i} \int\limits_{\mathbb{R}_+^n} (b_1 x_1^2 + ... + b_n x_n^2)^\lambda \mathbf{j}(x,\xi) x^\gamma \, dx.$$

Change of variables by $x_i = \frac{y_i}{\sqrt{b_i}}$, $i = 1, ..., n$, transforms this to the form

$$\mathbf{F}_\gamma[\mathscr{P}_\gamma^\lambda](\xi) = e^{\frac{\pi}{2}\lambda i} b_1^{-\frac{1+\gamma_1}{2}} ....b_n^{-\frac{1+\gamma_n}{2}} \int\limits_{\mathbb{R}_+^n} r^{2\lambda} \mathbf{j}\left(\frac{y}{\sqrt{b}}, \xi\right) y^\gamma \, dy.$$

The Fourier transform of the weighted generalized function $r_\gamma^{2\lambda}$ has already been calculated in Section 4.4.1. Using (4.86) we have

$$\begin{aligned}\mathbf{F}_\gamma[\mathscr{P}_\gamma^\lambda](\xi) &= \frac{2^{2\lambda+|\gamma|} \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right) \Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)}{\Gamma(-\lambda)\sqrt{b_1^{1+\gamma_1}}...\sqrt{b_n^{1+\gamma_n}}} e^{\frac{\pi}{2}\lambda i} \\ &\quad \times \left(\frac{\xi_1^2}{b_1} + ... + \frac{\xi_n^2}{b_n}\right)^{-n-|\gamma|-2\lambda} \\ &= 2^{2\lambda+|\gamma|} e^{\frac{\pi}{2}\lambda i} \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right) \Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)}{\Gamma(-\lambda)\sqrt{(-i\alpha_1)^{1+\gamma_1}}...\sqrt{(-i\alpha_n)^{1+\gamma_n}}} \\ &\quad \times \left(\frac{\xi_1^2}{-i\alpha_1} + ... + \frac{\xi_n^2}{-i\alpha_n}\right)^{\frac{-n-|\gamma|}{2}-\lambda}.\end{aligned}$$

Taking the factor $-i$ out of the bracket $\left(\frac{\xi_1^2}{-i\alpha_1} + ... + \frac{\xi_n^2}{-i\alpha_n}\right)^{\frac{-n-|\gamma|}{2}-\lambda}$ we get (4.87). Now the uniqueness of analytic continuation implies that (4.87) remains valid also for any quadratic form whose imaginary part is positive definite. The square roots $\sqrt{(-i\alpha_1)^{1+\gamma_1}}...\sqrt{(-i\alpha_n)^{1+\gamma_n}}$ are calculated by $\sqrt{z} = |z|^{\frac{1}{2}} e^{\frac{1}{2}i \arg z}$. □

Theorem 44. *The Hankel transforms of* $(P \pm i0)_\gamma^\lambda$ *for* $\lambda \neq k$, $\lambda \neq -\left(\frac{n+|\gamma|}{2}+k\right)$, $k = 0, 1, 2, ...$, *are*

$$\mathbf{F}_\gamma[(P+i0)_\gamma^\lambda](\xi) = e^{-\frac{q+|\gamma''|}{2}i\pi} \beta_{n,\gamma}(\lambda)(Q-i0)^{-\frac{n+|\gamma|}{2}-\lambda}, \tag{4.88}$$

$$\mathbf{F}_\gamma[(P-i0)_\gamma^\lambda](\xi) = e^{\frac{q+|\gamma''|}{2}i\pi} \beta_{n,\gamma}(\lambda)(Q+i0)^{-\frac{n+|\gamma|}{2}-\lambda}, \tag{4.89}$$

where

$$Q = \xi_1^2 + ... + \xi_p^2 - \xi_{p+1}^2 - ... - \xi_{p+q}^2,$$

$$\beta_{n,\gamma}(\lambda)=2^{2\lambda+|\gamma|}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)}{\Gamma\left(-\lambda\right)}.$$

Proof. Let in formula (4.87) $\alpha_k=a_k+ib_k$, $k=1,...,n$. Then

$$\mathbf{F}_\gamma[\mathscr{P}_\gamma^\lambda](\xi)=2^{2\lambda+|\gamma|}e^{-\frac{n+|\gamma|}{4}i\pi}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)}{\Gamma\left(-\lambda\right)\sqrt{(b_1-ia_1)^{1+\gamma_1}}...\sqrt{(b_n-ia_k)^{1+\gamma_n}}}$$
$$\times\left(\frac{\xi_1^2}{a_1+ib_1}+...+\frac{\xi_n^2}{a_n+ib_n}\right)^{-\frac{n+|\gamma|}{2}-\lambda}. \tag{4.90}$$

Putting $a_1=1$,...,$a_p=1$, $a_{p+1}=-1$, ...,$a_{p+q}=-1$ in (4.90) and tending to limits $b_1{\to}0$,...,$b_n{\to}0$, we obtain

$$\mathbf{F}_\gamma[(P+i0)_\gamma^\lambda](\xi)$$
$$=2^{2\lambda+|\gamma|}e^{-\frac{n+|\gamma|}{4}i\pi}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)}{\Gamma\left(-\lambda\right)\sqrt{(-i)^{1+\gamma_1}}...\sqrt{(-i)^{1+\gamma_p}}\sqrt{i^{1+\gamma_{p+1}}\sqrt{i^{1+\gamma_n}}}}$$
$$\times\left(\frac{\xi_1^2}{1+i0}+...+\frac{\xi_p^2}{1+i0}+\frac{\xi_{p+1}^2}{-1+i0}+...+\frac{\xi_n^2}{-1+i0}\right)^{-\frac{n+|\gamma|}{2}-\lambda}$$
$$=2^{2\lambda+|\gamma|}e^{\frac{q+|\gamma''|}{2}i\pi}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)}{\Gamma\left(-\lambda\right)}$$
$$\times\Big(\xi_1^2(1-i0)+...+\xi_p^2(1-i0)+\xi_{p+1}^2(-1-i0)+...$$
$$+\xi_n^2(-1-i0)\Big)^{-\frac{n+|\gamma|}{2}-\lambda}$$
$$=2^{2\lambda+|\gamma|}e^{-\frac{q+|\gamma''|}{2}i\pi}\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)}{\Gamma\left(-\lambda\right)}(Q-i0)^{-\frac{n+|\gamma|}{2}-\lambda}.$$

That gives (4.88). Formula (4.89) is obtained similarly. □

Further for formulas (4.88) and (4.89) we will use a short notation:

$$\mathbf{F}_\gamma[(P\pm i0)_\gamma^\lambda]=e^{-\frac{q+|\gamma''|}{2}i\pi}\beta_{n,\gamma}(\lambda)(P\mp i0)^{-\frac{n+|\gamma|}{2}-\lambda}. \tag{4.91}$$

Theorem 45. *The Hankel transforms of $P^{\lambda}_{\gamma,\pm}$ for $\lambda \neq k$, $\lambda \neq -\left(\frac{n+|\gamma|}{2}+k\right)$, $k = 0,1,2,...$, are*

$$\mathbf{F}_{\gamma}[P^{\lambda}_{+}] = \frac{2^{2\lambda+|\gamma|-1}}{i\pi}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)\Gamma(1+\lambda)$$
$$\times\left(e^{-i\pi\left(\lambda+\frac{q+|\gamma''|}{2}\right)}(Q-i0)^{-\frac{n+|\gamma|}{2}-\lambda}-e^{i\pi\left(\lambda+\frac{q+|\gamma''|}{2}\right)}(Q+i0)^{-\frac{n+|\gamma|}{2}-\lambda}\right) \tag{4.92}$$

and

$$\mathbf{F}_{\gamma}[P^{\lambda}_{-}] = -\frac{2^{2\lambda+|\gamma|-1}}{i\pi}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}+\lambda\right)\Gamma(\lambda+1)$$
$$\times\left(e^{-\frac{q+|\gamma''|}{2}i\pi}(Q-i0)^{-\frac{n+|\gamma|}{2}-\lambda}-e^{\frac{q+|\gamma''|}{2}i\pi}(Q+i0)^{-\frac{n+|\gamma|}{2}-\lambda}\right), \tag{4.93}$$

where

$$Q = \xi_1^2+...+\xi_p^2-\xi_{p+1}^2-...-\xi_{p+q}^2.$$

Proof. Using formulas (4.76), (4.77), (4.88), (4.89), and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z},$$

after some elementary operations we obtain (4.92) and (4.93). □

4.4.3 Hankel transforms of functions $(w^2-|x|^2)^{\lambda}_{+,\gamma}$ and $(c^2+P\pm i0)^{\lambda}_{\gamma}$

Here we obtain the formulas of the Hankel transform of some functions from Section 4.3.1.

Theorem 46. *The following formula holds:*

$$(\mathbf{F}_{\gamma})_x\left[\frac{(w^2-|x|^2)^{\lambda}_{+,\gamma}}{\Gamma(\lambda+1)}\right](\xi) = \frac{w^{n+|\gamma|+2\lambda}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|}{2}+\lambda+1\right)}\ j_{\frac{n+|\gamma|}{2}+\lambda}(w|x|), \tag{4.94}$$

where $(w^2-|x|^2)^{\frac{k-n-|\gamma|-1}{2}}_{+,\gamma}$ is defined by formula (4.78), $w>0$.

Proof. Let first $\operatorname{Re}\lambda>-1$. We perform the integration in $\mathbf{F}_\gamma(w^2-|x|^2)^\lambda_{+,\gamma}$ by going to spherical coordinates and applying formula (3.140):

$$(\mathbf{F}_\gamma)_x(w^2-|x|^2)^\lambda_{+,\gamma}=\int\limits_{B^+_w(n)}\mathbf{j}_\gamma(x,\xi)(w^2-|x|^2)^\lambda x^\gamma\,dx=\{x=r\theta,r=|x|\}$$

$$=\int\limits_0^w(w^2-r^2)^\lambda r^{n+|\gamma|-1}dr\int\limits_{S^+_1(n)}\mathbf{j}_\gamma(r\theta,x)\theta^\gamma\,dS$$

$$=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^w(w^2-r^2)^\lambda j_{\frac{n+|\gamma|}{2}-1}(r|x|)r^{n+|\gamma|-1}dr$$

$$=|x|^{1-\frac{n+|\gamma|}{2}}2^{\frac{|\gamma|-n}{2}}\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\int\limits_0^w(w^2-r^2)^\lambda J_{\frac{n+|\gamma|}{2}-1}(r|x|)r^{\frac{n+|\gamma|}{2}}dr.$$

Using formula (2.12.4.6) from [456] of the form

$$\int\limits_0^w r^{\nu+1}(w^2-r^2)^{\beta-1}J_\nu(\mu r)dr=\frac{2^{\beta-1}w^{\beta+\nu}\Gamma(\beta)}{\mu^\beta}J_{\beta+\nu}(\mu w), \tag{4.95}$$

$$w>0,\qquad \operatorname{Re}\beta>0,\qquad \operatorname{Re}\nu>-1,$$

we obtain

$$\int\limits_0^w(w^2-r^2)^\lambda J_{\frac{n+|\gamma|}{2}-1}(r|x|)r^{\frac{n+|\gamma|}{2}}dr=\frac{2^\lambda w^{\frac{n+|\gamma|}{2}+\lambda}\Gamma(\lambda+1)}{|x|^{\lambda+1}}J_{\frac{n+|\gamma|}{2}+\lambda}(|x|\omega)$$

for $\operatorname{Re}\lambda>-1$ and

$$(\mathbf{F}_\gamma)_x(w^2-|x|^2)^\lambda_{+,\gamma}=\frac{w^{n+|\gamma|+2\lambda}\Gamma(\lambda+1)\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|}{2}+\lambda+1\right)}\,j_{\frac{n+|\gamma|}{2}+\lambda}(w|x|),$$

which coincides with (4.94). So we get (4.94) for $\operatorname{Re}\lambda>-1$. For other values of λ such that $\lambda\neq-1,-2,-3,...,$ equality (4.94) remains valid by analytic continuation in λ.

Residues of $\frac{(w^2-|x|^2)^\lambda_{+,\gamma}}{\Gamma(\lambda+1)}$ at $\lambda=-m$, $m\in N$, have forms (see Section 4.2.1)

$$\lim_{\lambda\to-m}\frac{(w^2-|x|^2)^\lambda_{+,\gamma}}{\Gamma(\lambda+1)}=\delta^{(m-1)}_\gamma(w^2-|x|^2).$$

Then for $\lambda = -m$ we get

$$(\mathbf{F}_\gamma)_x \left[\frac{(w^2 - |x|^2)^\lambda_{+,\gamma}}{\Gamma(\lambda+1)} \right](\xi) = \int\limits_{\mathbb{R}^n_+} \mathbf{j}_\gamma(x,\xi)\delta^{(m-1)}_\gamma(w^2-|x|^2)x^\gamma\, dx$$

$$= \frac{w^{n+|\gamma|-2m} \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n \Gamma\left(\frac{n+|\gamma|}{2} - m + 1\right)}\, j_{\frac{n+|\gamma|}{2}-m}(w|x|).$$

The proof is complete. □

Theorem 47. *The following formulas hold:*

$$\mathbf{F}_\gamma(w^2+P+i0)^\lambda_\gamma = \frac{2^{\frac{|\gamma|-n}{2}+\lambda+1} e^{-\frac{1}{2}q\pi i} w^{\frac{n+|\gamma|}{2}+\lambda} \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma(-\lambda)\sqrt{|\Delta|}}$$
$$\times \left[\frac{K_{\frac{n+|\gamma|}{2}+\lambda}(wQ^{\frac{1}{2}}_{\gamma,+})}{Q^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}_{\gamma,+}} + \frac{i\pi}{2} \frac{H^{(1)}_{-\frac{n+|\gamma|}{2}-\lambda}(wQ^{\frac{1}{2}}_{\gamma,-})}{Q^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}_{\gamma,-}} \right] \tag{4.96}$$

and

$$\mathbf{F}_\gamma(w^2+P-i0)^\lambda_\gamma = \frac{2^{\frac{|\gamma|-n}{2}+\lambda+1} e^{\frac{1}{2}q\pi i} w^{\frac{n+|\gamma|}{2}+\lambda} \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma(-\lambda)\sqrt{|\Delta|}}$$
$$\times \left[\frac{K_{\frac{n+|\gamma|}{2}+\lambda}(wQ^{\frac{1}{2}}_{\gamma,+})}{Q^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}_{\gamma,+}} - \frac{i\pi}{2} \frac{H^{(2)}_{-\frac{n+|\gamma|}{2}-\lambda}(wQ^{\frac{1}{2}}_{\gamma,-})}{Q^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}_{\gamma,-}} \right], \tag{4.97}$$

where $Q = \sum\limits_{i=1}^n \frac{1}{a_i}\xi_i^2$ is a quadratic form dual to $P = \sum\limits_{i=1}^n a_i x_i^2$, Δ is the coefficient matrix determinant P, $H^{(1)}_\alpha$ and $H^{(2)}_\alpha$ are Hankel functions of the first and second kind (1.14) *and* (1.15)*, respectively, and K_α is a modified Bessel function* (1.17).

Proof. We first consider the Hankel transform of the weighted generalized function $(w^2+P)^\lambda_\gamma$, where $P = |x|^2$ is a positive definite quadratic form and $\operatorname{Re}\lambda < -\frac{n+|\gamma|}{2}$.

Applying (3.140) we obtain

$$\mathbf{F}_\gamma[(w^2+P)_\gamma^\lambda](\xi)=\int\limits_{\mathbb{R}^n_+}\mathbf{j}_\gamma(x,\xi)(c^2+|x|^2)^\lambda x^\gamma\,dx=\{x=r\theta,r=|x|\}$$

$$=\int\limits_0^\infty(w^2+r^2)^\lambda r^{n+|\gamma|-1}dr\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(r\theta,x)\theta^\gamma\,dS$$

$$=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty(w^2+r^2)^\lambda j_{\frac{n+|\gamma|-2}{2}}(r|\xi|)r^{n+|\gamma|-1}dr$$

$$=|\xi|^{1-\frac{n+|\gamma|}{2}}2^{\frac{|\gamma|-n}{2}}\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\int\limits_0^\infty(w^2+r^2)^\lambda J_{\frac{n+|\gamma|}{2}-1}(r|\xi|)r^{\frac{n+|\gamma|}{2}}dr.$$

Using formula (2.12.4.28) from [456],

$$\int\limits_0^\infty x^{\nu+1}(x^2+z^2)^{-\rho}J_\nu(cx)dx=\frac{c^{\rho-1}z^{\nu-\rho+1}}{2^{\rho-1}\Gamma(\rho)}K_{\nu-\rho+1}(cz),$$

we get

$$\mathbf{F}_\gamma[(w^2+P)_\gamma^\lambda](\xi)=\frac{2^{\frac{|\gamma|-n}{2}+\lambda+1}w^{\frac{n+|\gamma|}{2}+\lambda}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{|\xi|^{\frac{n+|\gamma|}{2}+\lambda}\Gamma(-\lambda)}K_{\frac{n+|\gamma|}{2}+\lambda}(w|\xi|),\tag{4.98}$$

where $\lambda<\frac{1-n-|\gamma|}{4}$. For other values of λ the Hankel transform $\mathbf{F}_\gamma(w^2+P)_\gamma^\lambda$ remains valid by analytic continuation in λ.

Now let P be any real quadratic form. Let us consider the weighted generalized functions $(w^2+P+i0)_\gamma^\lambda$ and $(w^2+P-i0)_\gamma^\lambda$. In accordance with the uniqueness of the analytic continuation, (4.98) gives

$$\mathbf{F}_\gamma[(w^2+P\pm i0)_\gamma^\lambda](\xi)$$
$$=\frac{2^{\frac{|\gamma|-n}{2}+\lambda+1}e^{\mp\frac{1}{2}q\pi}w^{\frac{n+|\gamma|}{2}+\lambda}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{(Q\mp i0)^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}\Gamma(-\lambda)\sqrt{|\Delta|}}K_{\frac{n+|\gamma|}{2}+\lambda}(w(Q\mp i0)_\gamma^{\frac{1}{2}}),\tag{4.99}$$

where $Q=\sum\limits_{i=1}^n\frac{1}{a_i}\xi_i^2$ is a quadratic form dual to $P=\sum\limits_{i=1}^n a_ix_i^2$ and Δ is the coefficient matrix determinant of P. Taking into account the definitions of the modified Bessel

function of the first and second kind (1.16) and (1.17), we obtain

$$
\begin{aligned}
&\frac{K_{\frac{n+|\gamma|}{2}+\lambda}(w(Q\mp i0)_\gamma^{\frac{1}{2}})}{(Q\mp i0)^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}} = \frac{\pi}{2}\frac{I_{-\frac{n+|\gamma|}{2}-\lambda}(w(Q\mp i0)_\gamma^{\frac{1}{2}}) - I_{\frac{n+|\gamma|}{2}+\lambda}(w(Q\mp i0)_\gamma^{\frac{1}{2}})}{\sin\left(\left(\frac{n+|\gamma|}{2}+\lambda\right)\pi\right)(Q\mp i0)^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}}\\
&= \frac{\pi}{2\sin\left(\left(\frac{n+|\gamma|}{2}+\lambda\right)\pi\right)}\sum_{m=0}^{\infty}\frac{1}{m!}\\
&\quad\times\left(\frac{w^{2m-\frac{n+|\gamma|}{2}-\lambda}}{2^{2m-\frac{n+|\gamma|}{2}-\lambda}\Gamma\left(m-\frac{n+|\gamma|}{2}-\lambda+1\right)}(Q\mp i0)_\gamma^{m-\lambda-\frac{n+|\gamma|}{2}}\right.\\
&\quad\left.-\frac{w^{2m+\frac{n+|\gamma|}{2}+\lambda}}{2^{2m+\frac{n+|\gamma|}{2}+\lambda}\Gamma\left(m+\frac{n+|\gamma|}{2}+\lambda+1\right)}(Q\mp i0)_\gamma^{m}\right).
\end{aligned}
$$

The weighted generalized functions $(Q+i0)_\gamma^\lambda$ and $(Q-i0)_\gamma^\lambda$ are expressed through $Q_{\gamma,+}^\lambda$ and $Q_{\gamma,-}^\lambda$ by formulas

$$
(Q\mp i0)_\gamma^\mu = Q_{\gamma,+}^\mu + e^{\mp\pi\mu i}Q_{\gamma,-}^\mu.
$$

So

$$
\begin{aligned}
&\frac{K_{\frac{n+|\gamma|}{2}+\lambda}(w(Q\mp i0)_\gamma^{\frac{1}{2}})}{(Q\mp i0)^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}}\\
&= \frac{\pi}{2\sin\left(\left(\frac{n+|\gamma|}{2}+\lambda\right)\pi\right)}\sum_{m=0}^{\infty}\frac{1}{m!}\\
&\quad\times\left(\frac{w^{2m-\frac{n+|\gamma|}{2}-\lambda}}{2^{2m-\frac{n+|\gamma|}{2}-\lambda}\Gamma\left(m-\frac{n+|\gamma|}{2}-\lambda+1\right)}\right.\\
&\quad\times\left(Q_{\gamma,+}^{m-\lambda-\frac{n+|\gamma|}{2}}+e^{\mp\pi i\left(m-\lambda-\frac{n+|\gamma|}{2}\right)}Q_{\gamma,-}^{m-\lambda-\frac{n+|\gamma|}{2}}\right)\\
&\quad\left.-\frac{w^{2m+\frac{n+|\gamma|}{2}+\lambda}}{2^{2m+\frac{n+|\gamma|}{2}+\lambda}\Gamma\left(m+\frac{n+|\gamma|}{2}+\lambda+1\right)}\left(Q_{\gamma,+}^m+e^{\mp\pi i m}Q_{\gamma,-}^m\right)\right)\\
&= \frac{\pi}{2}\frac{I_{-\lambda-\frac{n+|\gamma|}{2}}(wQ_{\gamma,+}^{\frac{1}{2}}) - I_{\lambda+\frac{n+|\gamma|}{2}}(wQ_{\gamma,+}^{\frac{1}{2}})}{\sin\left(\left(\frac{n+|\gamma|}{2}+\lambda\right)\pi\right)(Q_{\gamma,+}^{\frac{1}{2}})^{\lambda+\frac{n+|\gamma|}{2}}}
\end{aligned}
$$

$$+\frac{i\pi}{2}\frac{J_{\lambda+\frac{n+|\gamma|}{2}}(wQ_{\gamma,-}^{\frac{1}{2}})-e^{\pm\left(\lambda+\frac{n+|\gamma|}{2}\right)\pi}J_{-\lambda-\frac{n+|\gamma|}{2}}(wQ_{\gamma,-}^{\frac{1}{2}})}{i\sin\left(-\left(\lambda+\frac{n+|\gamma|}{2}\right)\pi\right)(Q_{\gamma,-}^{\frac{1}{2}})^{\lambda+\frac{n+|\gamma|}{2}}}.$$

Noting that

$$K_\alpha(x)=\frac{\pi}{2}\frac{I_{-\alpha}(x)-I_\alpha(x)}{\sin(\alpha\pi)},$$

$$H_\alpha^{(1)}(x)=\frac{J_{-\alpha}(x)-e^{-\alpha\pi i}J_\alpha(x)}{i\sin(\alpha\pi)},\qquad H_\alpha^{(2)}(x)=\frac{J_{-\alpha}(x)-e^{\alpha\pi i}J_\alpha(x)}{-i\sin(\alpha\pi)},$$

we get

$$\frac{K_{\frac{n+|\gamma|}{2}+\lambda}(w(Q+i0)_\gamma^{\frac{1}{2}})}{(Q\mp i0)^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}}=\frac{K_{\lambda+\frac{n+|\gamma|}{2}}(wQ_{\gamma,+}^{\frac{1}{2}})}{(Q_{\gamma,+}^{\frac{1}{2}})^{\lambda+\frac{n+|\gamma|}{2}}}+\frac{i\pi}{2}\frac{H^{(1)}_{-\lambda-\frac{n+|\gamma|}{2}}(wQ_{\gamma,-}^{\frac{1}{2}})}{(Q_{\gamma,-}^{\frac{1}{2}})^{\lambda+\frac{n+|\gamma|}{2}}},$$

$$\frac{K_{\frac{n+|\gamma|}{2}+\lambda}(w(Q-i0)_\gamma^{\frac{1}{2}})}{(Q\mp i0)^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}}=\frac{K_{\lambda+\frac{n+|\gamma|}{2}}(wQ_{\gamma,+}^{\frac{1}{2}})}{(Q_{\gamma,+}^{\frac{1}{2}})^{\lambda+\frac{n+|\gamma|}{2}}}-\frac{i\pi}{2}\frac{H^{(2)}_{-\lambda-\frac{n+|\gamma|}{2}}(wQ_{\gamma,-}^{\frac{1}{2}})}{(Q_{\gamma,-}^{\frac{1}{2}})^{\lambda+\frac{n+|\gamma|}{2}}}.$$

Considering (4.99) finally we obtain (4.96) and (4.97). □

Corollary 4. *If P is a positive definite quadratic form, then*

$$\mathbf{F}_\gamma(w^2+P+i0)_\gamma^\lambda=\frac{2^{\frac{|\gamma|-n}{2}+\lambda+1}e^{-\frac{1}{2}q\pi i}w^{\frac{n+|\gamma|}{2}+\lambda}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma(-\lambda)\sqrt{|\Delta|}Q_{\gamma,+}^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}}K_{\frac{n+|\gamma|}{2}+\lambda}(wQ_{\gamma,+}^{\frac{1}{2}})\tag{4.100}$$

and

$$\mathbf{F}_\gamma(w^2+P-i0)_\gamma^\lambda=\frac{2^{\frac{|\gamma|-n}{2}+\lambda+1}e^{\frac{1}{2}q\pi i}w^{\frac{n+|\gamma|}{2}+\lambda}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma(-\lambda)\sqrt{|\Delta|}Q_{\gamma,+}^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}}K_{\frac{n+|\gamma|}{2}+\lambda}(wQ_{\gamma,+}^{\frac{1}{2}}).\tag{4.101}$$

If P is a negative definite quadratic form, then

$$\mathbf{F}_\gamma(w^2+P+i0)^\lambda_\gamma$$
$$=\frac{i\pi 2^{\frac{|\gamma|-n}{2}+\lambda}e^{-\frac{1}{2}q\pi i}w^{\frac{n+|\gamma|}{2}+\lambda}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma(-\lambda)\sqrt{|\Delta|}Q_{\gamma,-}^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}}H^{(1)}_{-\frac{n+|\gamma|}{2}-\lambda}(wQ_{\gamma,-}^{\frac{1}{2}}) \tag{4.102}$$

and

$$\mathbf{F}_\gamma(w^2+P-i0)^\lambda_\gamma$$
$$=-\frac{i\pi 2^{\frac{|\gamma|-n}{2}+\lambda}e^{\frac{1}{2}q\pi i}w^{\frac{n+|\gamma|}{2}+\lambda}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma(-\lambda)\sqrt{|\Delta|}Q_{\gamma,-}^{\frac{1}{2}\left(\frac{n+|\gamma|}{2}+\lambda\right)}}H^{(2)}_{-\frac{n+|\gamma|}{2}-\lambda}(wQ_{\gamma,-}^{\frac{1}{2}}), \tag{4.103}$$

where $Q=\sum\limits_{i=1}^{n}\frac{1}{a_i}\xi_i^2$ is dual to the $P=\sum\limits_{i=1}^{n}a_ix_i^2$ quadratic form, $H^{(1)}_\alpha$ and $H^{(2)}_\alpha$ are Hankel functions of the first and second kinds (1.14) *and* (1.15)*, respectively, and K_α is a modified Bessel function* (1.17).

Buschman–Erdélyi integral and transmutation operators

5

The term "Buschman–Erdélyi transmutations" was introduced by S. M. Sitnik and is now widely accepted. Integral equations with these operators were studied in the mid-1950s. S. M. Sitnik was the first to prove the transmutational nature of these operators. The classical Sonine and Poisson operators are special cases of the Buschman–Erdélyi transmutations, and Sonine–Dimovski and Poisson–Dimovski transmutations are their generalizations for hyper-Bessel equations and functions.

The Buschman–Erdélyi transmutations have many modifications. S. M. Sitnik introduced a convenient classification of them. Due to this classification we introduce Buschman–Erdélyi transmutations of the first kind; their kernels are expressed in terms of Legendre functions of the first kind. In the limiting case we define Buschman–Erdélyi transmutations of zero order smoothness being important in applications. The kernels of Buschman–Erdélyi transmutations of the second kind are expressed in terms of Legendre functions of the second kind. Some combination of operators of the first kind and the second kind leads to operators of the third kind. For the special choice of parameters they are unitary operators in the standard Lebesgue space. S. M. Sitnik proposed the terms "Sonine–Katrakhov" and "Poisson–Katrakhov" transmutations in honor of V. Katrakhov, who introduced and studied these operators.

The study of integral equations and invertibility for the Buschman–Erdélyi operators was started in the 1960s by P. Buschman and A. Erdélyi. These operators were also investigated by Higgins, Ta Li, Love, Habibullah, K. N. Srivastava, Ding Hoang An, Smirnov, Virchenko, Fedotova, Kilbas, Skoromnik, and others. During this period, for this class of operators only problems of solving integral equations, factorization, and invertibility were considered (cf. [494]).

The most detailed study of the Buschman–Erdélyi transmutations was started in [533,535] and continued in [230,234,534,535,537] and some other papers. Interesting and important results were proved by N. Virchenko and A. Kilbas and their disciples.

5.1 Buschman–Erdélyi transmutations of the first kind

5.1.1 Sonine–Poisson–Delsarte transmutations

Let us first consider the most well-known transmutations for the Bessel operator and the second derivative:

$$T\left(B_\nu\right) f=\left(D^2\right) T f,\ B_\nu=D^2+\frac{2 \nu+1}{x} D,\ D^2=\frac{d^2}{d x^2},\ \nu \in \mathbb{C}. \tag{5.1}$$

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The Poisson transmutation is defined by (see (3.120) where $\gamma = 2\nu + 1$)

$$\mathcal{P}_\nu f = \frac{1}{\Gamma(\nu+1)2^\nu x^{2\nu}} \int_0^x \left(x^2 - t^2\right)^{\nu - \frac{1}{2}} f(t)\, dt, \;\; \operatorname{Re} \nu > -\frac{1}{2}. \tag{5.2}$$

The Sonine transmutation is defined by

$$S_\nu f = \frac{2^{\nu+\frac{1}{2}}}{\Gamma(\frac{1}{2} - \nu)} \frac{d}{dx} \int_0^x \left(x^2 - t^2\right)^{-\nu - \frac{1}{2}} t^{2\nu+1} f(t)\, dt, \;\; \operatorname{Re} \nu < \frac{1}{2}. \tag{5.3}$$

The operators (5.2)–(5.3) intertwine by the formulas

$$S_\nu B_\nu = D^2 S_\nu, \quad P_\nu D^2 = B_\nu P_\nu. \tag{5.4}$$

The definition may be extended to $\nu \in \mathbb{C}$. We will use the historically more exact term Sonine–Poisson–Delsarte transmutations [532].

An important generalization for the Sonine–Poisson–Delsarte transmutations are the transmutations for the hyper-Bessel operators and functions. Such functions were first considered by Kummer and Delerue. The detailed study on these operators and hyper-Bessel functions was carried out by Dimovski, and further by Kiryakova. The corresponding transmutations were called Sonine–Dimovski and Poisson–Dimovski transmutations by Kiryakova [252]. In hyper-Bessel operators theory the leading role is for the Obrechkoff integral transform [252]. It is a transform with Meijer's G-function kernel which generalizes the Laplace, Meijer, and many other integral transforms introduced by different authors. Various results on the hyper-Bessel functions, connected equations, and transmutations were many times reviewed. The same is true for the Obrechkoff integral transform. In our opinion, the Obrechkoff transform and the Laplace, Fourier, Mellin, Stankovic transforms are essential basic elements from which many other transforms are constructed with corresponding applications.

5.1.2 Definition and main properties of Buschman–Erdélyi transmutations of the first kind

Let us define and study some main properties of the Buschman–Erdélyi transmutations of the first kind. This class of transmutations for some choice of parameters generalizes the Sonine–Poisson–Delsart transmutations, Riemann–Liouville and Erdélyi–Kober fractional integrals, and the Mehler–Fock transform.

Definition 36. *Define the Buschman–Erdélyi operators of the first kind by*

$$B_{0+}^{\nu,\mu} f = \int\limits_0^x \left(x^2 - t^2\right)^{-\frac{\mu}{2}} P_\nu^\mu \left(\frac{x}{t}\right) f(t) d\,t, \tag{5.5}$$

$$E_{0+}^{\nu,\mu} f = \int\limits_0^x \left(x^2 - t^2\right)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu \left(\frac{t}{x}\right) f(t) d\,t, \tag{5.6}$$

$$B_{-}^{\nu,\mu} f = \int_x^\infty \left(t^2 - x^2\right)^{-\frac{\mu}{2}} P_\nu^\mu \left(\frac{t}{x}\right) f(t) d t, \tag{5.7}$$

$$E_{-}^{\nu,\mu} f = \int_x^\infty \left(t^2 - x^2\right)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu \left(\frac{x}{t}\right) f(t) d t. \tag{5.8}$$

Here $P_\nu^\mu(z)$ *is the Legendre function of the first kind* (1.42), $\mathbb{P}_\nu^\mu(z)$ *is this function on the cut* $-1 \le t \le 1$ (1.44), *and* $f(x)$ *is a locally summable function with some growth conditions at* $x \to 0$, $x \to \infty$. *The parameters are* $\mu, \nu \in \mathbb{C}$, $\operatorname{Re} \mu < 1$, $\operatorname{Re} \nu \ge -1/2$.

Now we consider some main properties for this class of transmutations, following essentially [533,535], and also [230,234,532,537]. All following functions are defined on the positive semiaxis. So we use notations L_2 for the functional space $L_2(0, \infty)$ and $L_{2,k}$ for the power weighted space $L_{2,k}(0, \infty)$ equipped with the norm

$$\int_0^\infty |f(x)|^2 x^{2k+1}\, dx, \tag{5.9}$$

where $\mathbb{N}$ denotes the set of naturals, $\mathbb{N}_0$ positive integers, $\mathbb{Z}$ integers, and $\mathbb{R}$ real numbers.

First, add to Definition 36 the case of parameter $\mu = 1$. It defines a very important class of operators.

Definition 37. *Define for* $\mu = 1$ *the Buschman–Erdélyi operators of zero order smoothness by*

$$B_{0+}^{\nu,1} f = {}_1S_{0+}^\nu f = \frac{d}{dx} \int_0^x P_\nu \left(\frac{x}{t}\right) f(t)\, dt, \tag{5.10}$$

$$E_{0+}^{\nu,1} f = {}_1P_{-}^\nu f = \int_0^x P_\nu \left(\frac{t}{x}\right) \frac{df(t)}{dt}\, dt, \tag{5.11}$$

$$B_{-}^{\nu,1} f = {}_1S_{-}^\nu f = \int_x^\infty P_\nu \left(\frac{t}{x}\right) \left(-\frac{df(t)}{dt}\right) dt, \tag{5.12}$$

$$E_{-}^{\nu,1} f = {}_1P_{0+}^\nu f = \left(-\frac{d}{dx}\right) \int_x^\infty P_\nu \left(\frac{x}{t}\right) f(t)\, dt, \tag{5.13}$$

where $P_\nu(z) = P_\nu^0(z)$ *is the Legendre function.*

Theorem 48. *The next formulas hold true for factorizations of Buschman–Erdélyi transmutations for suitable functions via Riemann–Liouville fractional integrals and*

Buschman–Erdélyi operators of zero order smoothness:

$$B_{0+}^{\nu,\mu} f = I_{0+}^{1-\mu} \, {}_1S_{0+}^{\nu} f, \; B_{-}^{\nu,\mu} f = {}_1P_{-}^{\nu} \, I_{-}^{1-\mu} f, \tag{5.14}$$

$$E_{0+}^{\nu,\mu} f = {}_1P_{0+}^{\nu} \, I_{0+}^{1-\mu} f, \; E_{-}^{\nu,\mu} f = I_{-}^{1-\mu} \, {}_1S_{-}^{\nu} f. \tag{5.15}$$

These formulas allow to separate parameters ν and μ. We will prove soon that operators (5.10)–(5.13) are isomorphisms of $L_2(0,\infty)$ except for some special parameters. So, operators (5.5)–(5.8) roughly speaking are of the same smoothness in L_2 as integro-differentiations $I^{1-\mu}$ and they coincide with them for $\nu = 0$. It is also possible to define Buschman–Erdélyi operators for all $\mu \in \mathbb{C}$.

Definition 38. *Define the number $\rho = 1 - Re\,\mu$ as smoothness order for Buschman–Erdélyi operators* (5.5)–(5.8).

So for $\rho > 0$ (otherwise for $\operatorname{Re}\mu > 1$) the Buschman–Erdélyi operators are smoothing and for $\rho < 0$ (otherwise for $\operatorname{Re}\mu < 1$) they decrease smoothness in L_2-spaces. Operators (5.10)–(5.13) for which $\rho = 0$ due to Definition 5 are of zero smoothness order in accordance with their definition.

For some special parameters ν, μ the Buschman–Erdélyi operators of the first kind are reduced to other known operators. For $\mu = -\nu$ or $\mu = \nu + 2$ they reduce to Erdélyi–Kober operators, for $\nu = 0$ they reduce to fractional integro-differentiation $I_{0+}^{1-\mu}$ or $I_{-}^{1-\mu}$, for $\nu = -\frac{1}{2}$, $\mu = 0$, or $\mu = 1$ kernels reduce to elliptic integrals, and for $\mu = 0$, $x = 1$, $\nu = it - \frac{1}{2}$ the operator $B_{-}^{\nu,0}$ differs only by a constant from the Mehler–Fock transform.

As a pair for the Bessel operator consider a connected one

$$L_\nu = D^2 - \frac{\nu(\nu+1)}{x^2} = \left(\frac{d}{dx} - \frac{\nu}{x}\right)\left(\frac{d}{dx} + \frac{\nu}{x}\right), \tag{5.16}$$

which for $\nu \in \mathbb{N}$ is an angular momentum operator from quantum physics. Their transmutational relations are established in the next theorem.

Theorem 49. *For a given pair of transmutations X_ν, Y_ν,*

$$X_\nu L_\nu = D^2 X_\nu, \; Y_\nu D^2 = L_\nu Y_\nu, \tag{5.17}$$

define the new pair of transmutations by formulas

$$S_\nu = X_{\nu-1/2} x^{\nu+1/2}, \; P_\nu = x^{-(\nu+1/2)} Y_{\nu-1/2}. \tag{5.18}$$

Then for the new pair S_ν, P_ν the following formulas are valid:

$$S_\nu B_\nu = D^2 S_\nu, \; P_\nu D^2 = B_\nu P_\nu. \tag{5.19}$$

Theorem 50. *Let $Re\,\mu \le 1$. Then an operator $B_{0+}^{\nu,\mu}$ on proper functions is a Sonine type transmutation and* (5.17) *is valid.*

The same result holds true for other Buschman–Erdélyi operators, $E_{-}^{\nu,\mu}$ is Sonine type and $E_{0+}^{\nu,\mu}$, $B_{-}^{\nu,\mu}$ are Poisson type transmutations.

From these transmutation connections, we conclude that the Buschman–Erdélyi operators link the corresponding eigenfunctions for the two operators. They lead to formulas for the Bessel functions via exponents and trigonometric functions, and vice versa which generalize the classical Sonine and Poisson formulas.

5.1.3 Factorizations of the first kind Buschman–Erdélyi operators and the Mellin transform

Now consider factorizations of the Buschman–Erdélyi operators using standard fractional integrals. First let us list the main forms of fractional integro-differentiations: Riemann–Liouville (2.25) and (2.26), Erdélyi–Kober (2.34) and (2.35), and the fractional integral by function $g(x)$ (2.38) and (2.39).

Theorem 51. *The following factorization formulas are valid for the Buschman–Erdélyi operators of the first kind via the Riemann–Liouville and Erdélyi–Kober fractional integrals:*

$$B_{0+}^{\nu,\mu} = I_{0+}^{\nu+1-\mu} I_{0+;\,2,\,\nu+\frac{1}{2}}^{-(\nu+1)} \left(\frac{2}{x}\right)^{\nu+1}, \tag{5.20}$$

$$E_{0+}^{\nu,\mu} = \left(\frac{x}{2}\right)^{\nu+1} I_{0+;\,2,\,-\frac{1}{2}}^{\nu+1} I_{0+}^{-(\nu+\mu)}, \tag{5.21}$$

$$B_{-}^{\nu,\mu} = \left(\frac{2}{x}\right)^{\nu+1} I_{-;\,2,\,\nu+1}^{-(\nu+1)} I_{-}^{\nu-\mu+2}, \tag{5.22}$$

$$E_{-}^{\nu,\mu} = I_{-}^{-(\nu+\mu)} I_{-;\,2,\,0}^{\nu+1} \left(\frac{x}{2}\right)^{\nu+1}. \tag{5.23}$$

The Sonine–Poisson–Delsarte transmutations also are special cases for this class of operators.

Now let us study the properties of the Buschman–Erdélyi operators of zero order smoothness, defined by (5.10)–(5.13). A similar operator was introduced by Katrakhov by multiplying the Sonine operator with a fractional integral; his aim was to work with transmutation obeying good estimates in $L_2(0,\infty)$.

We use the Mellin transform presented in Definition 11. The Mellin convolution is defined by (1.63)

$$(f_1 * f_2)(x) = \int_0^\infty f_1\left(\frac{x}{y}\right) f_2(y)\,\frac{dy}{y}, \tag{5.24}$$

so the convolution operator with kernel K acts under the Mellin transform as a multiplication on multiplicator

$$M[Af](s) = M\left[\int_0^\infty K\left(\frac{x}{y}\right) f(y)\,\frac{dy}{y}\right](s) = M[K * f](s) = m_A(s)Mf(s), \tag{5.25}$$

$$m_A(s) = M[K](s).$$

We observe that the Mellin transform is a generalized Fourier transform on the semiaxis with Haar measure $\frac{dy}{y}$ [162]. It plays an important role for the theory of special functions; for example, the gamma function is a Mellin transform of the exponential. With the Mellin transform the important breakthrough in evaluating integrals was made in the 1970s when, mainly by O. Marichev, the famous Slater theorem was adapted for calculations. The Slater theorem taking the Mellin transform as input gives the function itself as output via hypergeometric functions (see [361]). This theorem turned out to be the milestone of powerful computer methods for calculating integrals for many problems in differential and integral equations. The package *Mathematica* of Wolfram Research is based on this theorem in calculating integrals.

Theorem 52. *The Buschman–Erdélyi operator of zero order smoothness ${}_1S^{\nu}_{0+}$ defined by (5.10) acts under the Mellin transform as convolution (5.25) with the multiplicator*

$$m(s) = \frac{\Gamma(-s/2 + \frac{\nu}{2} + 1)\Gamma(-s/2 - \frac{\nu}{2} + 1/2)}{\Gamma(1/2 - \frac{s}{2})\Gamma(1 - \frac{s}{2})} \tag{5.26}$$

for $\operatorname{Re} s < \min(2 + \operatorname{Re}\nu, 1 - \operatorname{Re}\nu)$. *Its norm is a periodic in ν and equals*

$$\|B^{\nu,1}_{0+}\|_{L_2} = \frac{1}{\min(1, \sqrt{1 - \sin\pi\nu})}. \tag{5.27}$$

This operator is bounded in $L_2(0,\infty)$ if $\nu \neq 2k + 1/2$, $k \in \mathbb{Z}$, and unbounded if $\nu = 2k + 1/2$, $k \in \mathbb{Z}$.

Proof. First let us prove formula (5.26) with a proper multiplicator. Using consequently formulas (7), p. 130, (2), p. 129, and (4), p. 130, from [361], we evaluate

$$\begin{aligned} M(B^{\nu,1}_{0+})(s) &= \frac{\Gamma(2-s)}{\Gamma(1-s)} M\left[\int_0^\infty \left\{H(\frac{x}{y} - 1)P_\nu(\frac{x}{y})\right\}\{yf(y)\}\,\frac{dy}{y}\right](s-1) \\ &= \frac{\Gamma(2-s)}{\Gamma(1-s)} M\left[(x^2-1)^0_+ P^0_\nu(x)\right](s-1)M\left[f\right](s), \end{aligned}$$

we use notations from [361] for Heaviside and cutting power functions

$$x^\alpha_+ = \begin{cases} x^\alpha & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad H(x) = x^0_+ = \begin{cases} 1 & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Further using formulas (14)(1), p. 234, and (4), p. 130, from [361], we evaluate

$$M\left[(x-1)_+^0 P_\nu^0(\sqrt{x})\right](s) = \frac{\Gamma(\frac{1}{2}+\frac{\nu}{2}-s)\Gamma(-\frac{\nu}{2}-s)}{\Gamma(1-s)\Gamma(\frac{1}{2}-s)},$$

$$\begin{aligned} M\left[(x^2-1)_+^0 P_\nu^0(x)\right](s-1) &= \frac{1}{2}\cdot\frac{\Gamma(\frac{1}{2}+\frac{\nu}{2}-\frac{s-1}{2})\Gamma(-\frac{\nu}{2}-\frac{s-1}{2})}{\Gamma(1-\frac{s-1}{2})\Gamma(\frac{1}{2}-\frac{s-1}{2})} \\ &= \frac{1}{2}\cdot\frac{\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(-\frac{s}{2}+\frac{3}{2})\Gamma(-\frac{s}{2}+1)} \end{aligned}$$

under conditions $\operatorname{Re} s < \min(2+\operatorname{Re}\nu, 1-\operatorname{Re}\nu)$. Now we evaluate the formula for

$$M(B_{0+}^{\nu,1})(s) = \frac{1}{2}\cdot\frac{\Gamma(2-s)}{\Gamma(1-s)}\cdot\Gamma(-\frac{s}{2}+\frac{3}{2})\Gamma(-\frac{s}{2}+1).$$

Applying to $\Gamma(2-s)$ the Legendre duplication formula (1.7) we evaluate

$$M(B_{0+}^{\nu,1})(s) = \frac{2^{-s}}{\sqrt{\pi}}\cdot\frac{\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(1-s)}.$$

We apply the Legendre duplication formula once more to $\Gamma(1-s)$ and the formula for the multiplicator (5.26) is proved. In the paper [535] it was shown that restrictions may be reduced to $0 < \operatorname{Re} s < 1$ for proper ν. These restrictions may be weakened because they were derived for the class of all hypergeometric functions but we need just one special case of the Legendre function for which specified restrictions may be easily verified directly.

Now we prove formula (5.27) for a norm. From the multiplicator value we just found and Theorem 4.7 from [524] on the line $\operatorname{Re} s = 1/2$, $s = iu + 1/2$, it follows that

$$|M(B_{0+}^{\nu,1})(iu+1/2)| = \frac{1}{\sqrt{2\pi}}\left|\frac{\Gamma(-i\frac{u}{2}-\frac{\nu}{2}+\frac{1}{4})\Gamma(-i\frac{u}{2}+\frac{\nu}{2}+\frac{3}{4})}{\Gamma(\frac{1}{2}-iu)}\right|.$$

Below the operator symbol in the multiplicator will be omitted. We use formulas for the modulus $|z| = \sqrt{z\bar{z}}$ and the gamma function $\overline{\Gamma(z)} = \Gamma(\bar{z})$ following from its definition as integral. The last property is true in general for the class of real analytic functions. So we derive

$$\begin{aligned} &|M(B_{0+}^{\nu,1})(iu+1/2)| \\ &\quad = \frac{1}{\sqrt{2\pi}}\left|\frac{\Gamma(-i\frac{u}{2}-\frac{\nu}{2}+\frac{1}{4})\Gamma(i\frac{u}{2}-\frac{\nu}{2}+\frac{1}{4})\Gamma(-i\frac{u}{2}+\frac{\nu}{2}+\frac{3}{4})\Gamma(i\frac{u}{2}+\frac{\nu}{2}+\frac{3}{4})}{\Gamma(\frac{1}{2}-iu)\Gamma(\frac{1}{2}+iu)}\right|. \end{aligned}$$

In the numerator we combine outer and inner terms and transform three pairs of gamma functions by formula (1.6). As a result we evaluate

$$
\begin{aligned}
|M(B_{0+}^{\nu,1})(iu+1/2)| &= \sqrt{\frac{\cos(\pi i u)}{2\cos\pi(\frac{\nu}{2}+\frac{1}{4}+i\frac{u}{2})\cos\pi(\frac{\nu}{2}+\frac{1}{4}-i\frac{u}{2})}}\\
&= \sqrt{\frac{\operatorname{ch}(\pi i u)}{\operatorname{ch}\pi u-\sin\pi\nu}}.
\end{aligned}
$$

We further denote $t=\operatorname{ch}\pi u$, $1\leqslant t<\infty$. So we derive once more applying Theorem 4.7 from [524]

$$
\sup_{u\in\mathbb{R}}|m(iu+\frac{1}{2})| = \sup_{1\leqslant t<\infty}\sqrt{\frac{t}{t-\sin\pi\nu}}.
$$

So if $\sin\pi\nu\geqslant 0$, then the supremum is achieved at $t=1$ and for the norm formula (5.27) we have

$$
\|B_{0+}^{\nu,1}\|_{L_2} = \frac{1}{\sqrt{1-\sin\pi\nu}}.
$$

Otherwise if $\sin\pi\nu\leqslant 0$, then the supremum is achieved at $t\to\infty$ and the following formula is valid:

$$
\|B_{0+}^{\nu,1}\|_{L_2} = 1.
$$

This part of the theorem is proved.

From the explicit values for norms and the above cited Theorem 4.7 from [524] follow conditions of boundedness or unboundedness and periodicity. The theorem is completely proved. □

Now we proceed to finding multiplicators for all Buschman–Erdélyi operators of zero order smoothness.

Theorem 53. *The Buschman–Erdélyi operator of zero order smoothness acts under the Mellin transform as convolutions (5.25). For their multiplicators the following formulas are valid:*

$$
\begin{aligned}
m_{{}_1S_{0+}^{\nu}}(s) &= \frac{\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{s}{2})\Gamma(1-\frac{s}{2})}\\
&= \frac{2^{-s}}{\sqrt{\pi}}\frac{\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)}{\Gamma(1-s)},\\
&\operatorname{Re}s<\min(2+\operatorname{Re}\nu,1-\operatorname{Re}\nu),
\end{aligned}
$$

$$
m_{{}_1P_{0+}^{\nu}}(s) = \frac{\Gamma(\frac{1}{2}-\frac{s}{2})\Gamma(1-\frac{s}{2})}{\Gamma(-\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(-\frac{s}{2}-\frac{\nu}{2}+\frac{1}{2})}, \qquad \operatorname{Re}s<1,
$$

$$m_{{}_1P_-^\nu}(s) = \frac{\Gamma(\frac{s}{2}+\frac{\nu}{2}+1)\Gamma(\frac{s}{2}-\frac{\nu}{2})}{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+\frac{1}{2})}, \qquad \operatorname{Re} s > \max(\operatorname{Re}\nu, -1-\operatorname{Re}\nu), \tag{5.28}$$

$$m_{{}_1S_-^\nu}(s) = \frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+\frac{1}{2})}{\Gamma(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{2})\Gamma(\frac{s}{2}-\frac{\nu}{2})}, \qquad \operatorname{Re} s > 0. \tag{5.29}$$

The following formulas are valid for norms of the Buschman–Erdélyi operator of zero order smoothness in L_2:

$$\|{}_1S_{0+}^\nu\| = \|{}_1P_-^\nu\| = 1/\min(1, \sqrt{1-\sin\pi\nu}),$$
$$\|{}_1P_{0+}^\nu\| = \|{}_1S_-^\nu\| = \max(1, \sqrt{1-\sin\pi\nu}).$$

Similar results are proved in [230] and [535] for power weight spaces.

Corollary 5. *The norms of operators* (5.10)–(5.13) *are periodic in ν with period 2 $\|X^\nu\| = \|X^{\nu+2}\|$, and X^ν is any of operators* (5.10)–(5.13).

Corollary 6. *The norms of the operators ${}_1S_{0+}^\nu$, ${}_1P_-^\nu$ are not bounded in general, every norm is greater than or equal to 1. The norms are equal to 1 if $\sin\pi\nu \le 0$. The operators ${}_1S_{0+}^\nu$, ${}_1P_-^\nu$ are unbounded in L_2 if and only if $\sin\pi\nu = 1$ (or $\nu = (2k)+1/2$, $k \in \mathbb{Z}$).*

Corollary 7. *The norms of the operators ${}_1P_{0+}^\nu$, ${}_1S_-^\nu$ are all bounded in ν, and every norm is not greater than $\sqrt{2}$. The norms are equal to 1 if $\sin\pi\nu \ge 0$. The operators ${}_1P_{0+}^\nu$, ${}_1S_-^\nu$ are bounded in L_2 for all ν. The norm maximum which equals to $\sqrt{2}$ is achieved if and only if $\sin\pi\nu = -1$ ($\nu = -1/2+(2k)$, $k \in \mathbb{Z}$).*

The most important property of the Buschman–Erdélyi operators of zero order smoothness is the unitarity for integer ν. It is just the case if we interpret for these parameters the operator L_ν as angular momentum operator in quantum mechanics.

Theorem 54. *The operators* (5.10)–(5.13) *are unitary in L_2 if and only if the parameter ν is an integer. In this case the pairs of operators $({}_1S_{0+}^\nu, {}_1P_-^\nu)$ and $({}_1S_-^\nu, {}_1P_{0+}^\nu)$ are mutually inverse.*

To formulate an interesting special case, let us suppose that operators (5.10)–(5.13) act on functions permitting outer or inner differentiation in integrals. It is enough to suppose that $xf(x) \to 0$ for $x \to 0$. Then for $\nu = 1$

$${}_1P_{0+}^1 f = (I - H_1)f, \ {}_1S_-^1 f = (I - H_2)f, \tag{5.30}$$

where H_1, H_2 are the famous Hardy operators,

$$H_1 f = \frac{1}{x}\int_0^x f(y)dy, \ H_2 f = \int_x^\infty \frac{f(y)}{y}dy, \tag{5.31}$$

and I is the identic operator.

Corollary 8. *The operators* (5.30) *are unitary in* L_2 *and mutually inverse. They are transmutations for the pair of differential operators* d^2/dx^2 *and* $d^2/dx^2 - 2/x^2$.

The unitarity of the shifted Hardy operators (5.30) in L_2 is a known fact [305].

Now we list some properties of the operators acting as convolutions by formula (5.25) and with some multiplicator under the Mellin transform and being transmutations for the second derivative and angular momentum operator in quantum mechanics.

Theorem 55. *Let an operator* S_ν *act by formulas* (5.25) *and* (5.17). *Then:*

(a) its multiplicator satisfies a functional equation

$$m(s) = m(s-2)\frac{(s-1)(s-2)}{(s-1)(s-2)-\nu(\nu+1)}; \tag{5.32}$$

(b) if any function $p(s)$ *is periodic with period 2* $(p(s) = p(s-2))$, *then a function* $p(s)m(s)$ *is a multiplicator for a new transmutation operator* S_2^ν *also acting by the rule* (5.17).

This theorem confirms the importance of studying transmutations in terms of the Mellin transform and multiplicator functions.

Define the Stieltjes transform by (cf. [494])

$$(Sf)(x) = \int\limits_0^\infty \frac{f(t)}{x+t}dt.$$

This operator also acts by formula (5.25) with multiplicator $p(s) = \pi/\sin(\pi s)$, it is bounded in L_2. Obviously $p(s) = p(s-2)$. So from Theorem 55 a convolution of the Stieltjes transform follows with bounded transmutations (5.10)–(5.13), and also transmutations of the same class bounded in L_2.

In this way many new classes of transmutations were introduced with special functions as kernels.

5.2 Buschman–Erdélyi transmutations of the second and third kind

5.2.1 Second kind Buschman–Erdélyi transmutation operators

Now we consider Buschman–Erdélyi transmutations of the second kind.

Definition 39. *Define a new pair of Buschman–Erdélyi transmutations of the second kind with Legendre functions of the second kind in kernels*

$$
{}_2S^{\nu} f = \frac{2}{\pi}\left(-\int_0^x (x^2-y^2)^{-\frac{1}{2}} Q_{\nu}^1\left(\frac{x}{y}\right) f(y)dy + \int_x^{\infty} (y^2-x^2)^{-\frac{1}{2}} \mathbb{Q}_{\nu}^1\left(\frac{x}{y}\right) f(y)dy\right), \tag{5.33}
$$

$$
{}_2P^{\nu} f = \frac{2}{\pi}\left(-\int_0^x (x^2-y^2)^{-\frac{1}{2}} \mathbb{Q}_{\nu}^1\left(\frac{y}{x}\right) f(y)dy - \int_x^{\infty} (y^2-x^2)^{-\frac{1}{2}} Q_{\nu}^1\left(\frac{y}{x}\right) f(y)dy\right), \tag{5.34}
$$

where Q_ν^μ is the Legendre functions of the second kind (1.43) *and $\mathbb{Q}_\nu^\mu$ is the Legendre function of the second kind on the cut* (1.45).

These operators are analogues of Buschman–Erdélyi transmutations of zero order smoothness. If $y \to x \pm 0$, then integrals are defined by principal values. It is proved that they are transmutations of Sonine type for (5.33) and of Poisson type for (5.34).

Theorem 56. *Operators* (5.33) *and* (5.34) *are of the form* (5.25) *with multiplicators*

$$
m_{{}_2S^\nu}(s) = p(s)\, m_{{}_1S_-^\nu}(s), \tag{5.35}
$$

$$
m_{{}_2P^\nu}(s) = \frac{1}{p(s)}\, m_{{}_1P_-^\nu}(s), \tag{5.36}
$$

with multiplicators of operators ${}_1S_-^\nu, {}_1P_-^\nu$ defined by (5.28) *and* (5.29)*, where the period 2 function $p(s)$ equals*

$$
p(s) = \frac{\sin\pi\nu + \cos\pi s}{\sin\pi\nu - \sin\pi s}. \tag{5.37}
$$

Theorem 57. *The following formulas for norms are valid:*

$$
\|{}_2S^\nu\|_{L_2} = \max(1, \sqrt{1+\sin\pi\nu}), \tag{5.38}
$$

$$
\|{}_2P^\nu\|_{L_2} = 1/\min(1, \sqrt{1+\sin\pi\nu}). \tag{5.39}
$$

Corollary 9. *Operator ${}_2S^\nu$ is bounded for all ν. Operator ${}_2P^\nu$ is not bounded if and only if* $\sin\pi\nu = -1$.

Theorem 58. *Operators ${}_2S^\nu$ and ${}_2P^\nu$ are unitary in L_2 if and only if $\nu \in \mathbb{Z}$.*

Theorem 59. *Let $\nu = i\beta + 1/2$, $\beta \in \mathbb{R}$. Then*

$$
\|{}_2S^\nu\|_{L_2} = \sqrt{1+\operatorname{ch}\pi\beta},\ \|{}_2P^\nu\|_{L_2} = 1. \tag{5.40}
$$

Theorem 60. *The following formulas are valid:*

$$
{}_2S^0 f = \frac{2}{\pi}\int_0^\infty \frac{y}{x^2-y^2} f(y)\,dy, \tag{5.41}
$$

$$
{}_2S^{-1} f = \frac{2}{\pi}\int_0^\infty \frac{x}{x^2-y^2} f(y)\,dy. \tag{5.42}
$$

So in this case the operator ${}_2S^\nu$ reduces to a pair of semiaxis Hilbert transforms [494].

For operators of the second kind we also introduce more general ones with two parameters analogously to Buschman–Erdélyi transmutations of the first kind by formulas

$$
\begin{aligned}
{}_2S^{\nu,\mu} f = \frac{2}{\pi}\Bigg(&\int_0^x (x^2+y^2)^{-\frac{\mu}{2}} e^{-\mu\pi i} Q_\nu^\mu\left(\frac{x}{y}\right) f(y)\,dy \\
&+ \int_x^\infty (y^2+x^2)^{-\frac{\mu}{2}} \mathbb{Q}_\nu^\mu\left(\frac{x}{y}\right) f(y)\,dy\Bigg),
\end{aligned} \tag{5.43}
$$

where $Q_\nu^\mu(z)$ is the Legendre function of the second kind (1.43), $\mathbb{Q}_\nu^\mu(z)$ is this function on the cut (1.45), and $\mathrm{Re}\,\nu < 1$. The second operator may be defined as formally conjugate in $L_2(0,\infty)$ to (5.43).

Theorem 61. *The operator* (5.43) *on* $C_0^\infty(0,\infty)$ *is well defined and acts by*

$$
M[{}_2S^\nu](s) = m(s)\cdot M[x^{1-\mu} f](s),
$$

$$
m(s) = 2^{\mu-1}\left(\frac{\cos\pi(\mu-s)-\cos\pi\nu}{\sin\pi(\mu-s)-\sin\pi\nu}\right)\left(\frac{\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+\frac{1}{2}))}{\Gamma(\frac{s}{2}+\frac{1-\nu-\mu}{2})\Gamma(\frac{s}{2}+1+\frac{\nu-\mu}{2})}\right).
$$

5.2.2 Sonine–Katrakhov and Poisson–Katrakhov transmutations

Now we construct transmutations which are unitary for all ν. They are defined by formulas

$$
S_U^\nu f = -\sin\frac{\pi\nu}{2}\,{}_2S^\nu f + \cos\frac{\pi\nu}{2}\,{}_1S_-^\nu f, \tag{5.44}
$$

$$
P_U^\nu f = -\sin\frac{\pi\nu}{2}\,{}_2P^\nu f + \cos\frac{\pi\nu}{2}\,{}_1P_-^\nu f. \tag{5.45}
$$

For all values $\nu\in\mathbb{R}$ they are linear combinations of Buschman–Erdélyi transmutations of the first and second kinds of zero order smoothness. Also they are in the defined

below class of Buschman–Erdélyi transmutations of the third kind. The following integral representations are valid:

$$S_U^\nu f = \cos\frac{\pi\nu}{2}\left(-\frac{d}{dx}\right)\int_x^\infty P_\nu\left(\frac{x}{y}\right) f(y)\,dy$$
$$+\frac{2}{\pi}\sin\frac{\pi\nu}{2}\left(\int_0^x (x^2-y^2)^{-\frac{1}{2}} Q_\nu^1\left(\frac{x}{y}\right) f(y)\,dy\right.$$
$$\left.-\int_x^\infty (y^2-x^2)^{-\frac{1}{2}} \mathbb{Q}_\nu^1\left(\frac{x}{y}\right) f(y)\,dy\right), \tag{5.46}$$

$$P_U^\nu f = \cos\frac{\pi\nu}{2}\int_0^x P_\nu\left(\frac{y}{x}\right)\left(\frac{d}{dy}\right) f(y)\,dy$$
$$-\frac{2}{\pi}\sin\frac{\pi\nu}{2}\left(-\int_0^x (x^2-y^2)^{-\frac{1}{2}} \mathbb{Q}_\nu^1\left(\frac{y}{x}\right) f(y)\,dy\right.$$
$$\left.-\int_x^\infty (y^2-x^2)^{-\frac{1}{2}} Q_\nu^1\left(\frac{y}{x}\right) f(y)\,dy\right). \tag{5.47}$$

Theorem 62. *Operators (5.44)–(5.45) and (5.46)–(5.47) for all $\nu \in \mathbb{R}$ are unitary, mutually inverse, and conjugate in L_2. They are transmutations acting by (5.16); S_U^ν is a Sonine type transmutation and P_U^ν is a Poisson type one.*

Transmutations like (5.46)–(5.47) but with kernels into more complicated form with hypergeometric functions were first introduced by Katrakhov in 1980. For this reason S. M. Sitnik termed this class of operators Sonine–Katrakhov and Poisson–Katrakhov. In S. M. Sitnik's papers these operators were reduced to more simple forms of Buschman–Erdélyi ones. This made it possible to include this class of operators in general composition (or factorization) methods [146,229,234,525,537].

5.2.3 Buschman–Erdélyi transmutations of the third kind with arbitrary weight function

Define sine and cosine Fourier transforms with inverses

$$F_c f = \sqrt{\frac{2}{\pi}}\int_0^\infty f(y)\cos(ty)\,dy, \quad F_c^{-1} = F_c, \tag{5.48}$$

$$F_s f = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(y) \sin(ty)\, dy, \quad F_s^{-1} = F_s. \tag{5.49}$$

We define the Hankel (Fourier–Bessel) transform here a little different and its inverse in (1.56)–(1.57) by

$$\begin{aligned} F_\nu f &= \frac{1}{2^\nu \Gamma(\nu+1)} \int\limits_0^\infty f(y)\, j_\nu(ty)\, y^{2\nu+1}\, dy \\ &= \int\limits_0^\infty f(y) \frac{J_\nu(ty)}{(ty)^\nu}\, y^{2\nu+1}\, dy = \frac{1}{t^\nu} \int\limits_0^\infty f(y) J_\nu(ty)\, y^{\nu+1}\, dy, \end{aligned} \tag{5.50}$$

$$F_\nu^{-1} f = \frac{1}{(y)^\nu} \int\limits_0^\infty f(t) J_\nu(yt)\, t^{\nu+1}\, dt. \tag{5.51}$$

Here $J_\nu(\cdot)$ is the Bessel function (1.13) and $j_\nu(\cdot)$ is the normalized Bessel function (1.19). Operators (5.48) and (5.49) are unitary self-conjugate in $L_2(0,\infty)$. Operators (5.50) and (5.51) are unitary self-conjugate in the power weighted space $L_{2,\,\nu}(0,\infty)$.

Now define on proper functions the first pair of Buschman–Erdélyi transmutations of the third kind

$$S_{\nu,\,c}^{(\varphi)} = F_c^{-1} \left(\frac{1}{\varphi(t)} F_\nu \right), \tag{5.52}$$

$$P_{\nu,\,c}^{(\varphi)} = F_\nu^{-1} \left(\varphi(t) F_c \right), \tag{5.53}$$

and the second pair by

$$S_{\nu,\,s}^{(\varphi)} = F_s^{-1} \left(\frac{1}{\varphi(t)} F_\nu \right), \tag{5.54}$$

$$P_{\nu,\,s}^{(\varphi)} = F_\nu^{-1} \left(\varphi(t) F_s \right), \tag{5.55}$$

with $\varphi(t)$ being an arbitrary weight function.

The operators defined on proper functions are transmutations for B_ν and D^2. They may be expressed in the integral form.

Theorem 63. *Define transmutations for B_ν and D^2 by formulas*

$$S_{\nu,\left\{\substack{s\\c}\right\}}^{(\varphi)} = F_{\left\{\substack{s\\c}\right\}}^{-1} \left(\frac{1}{\varphi(t)} F_\nu \right),$$

$$P_{\nu,\left\{\substack{s\\c}\right\}}^{(\varphi)} = F_\nu^{-1} \left(\varphi(t) F_{\left\{\substack{s\\c}\right\}} \right).$$

Then for the Sonine type transmutation an integral form is valid,

$$\left(S^{(\varphi)}_{\nu,\left\{\substack{s\\c}\right\}} f\right)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty K(x,y) f(y)\, dy, \tag{5.56}$$

where

$$K(x,y) = y^{\nu+1} \int_0^\infty \frac{\begin{Bmatrix} \sin(xt) \\ \cos(xt) \end{Bmatrix}}{\varphi(t)\, t^\nu} J_\nu(yt) dt.$$

For the Poisson type transmutation an integral form is valid,

$$\left(P^{(\varphi)}_{\nu,\left\{\substack{s\\c}\right\}} f\right)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty G(x,y) f(y)\, dy, \tag{5.57}$$

where

$$G(x,y) = \frac{1}{x^\nu} \int_0^\infty \varphi(t)\, t^{\nu+1} \begin{Bmatrix} \sin(yt) \\ \cos(yt) \end{Bmatrix} J_\nu(xt) dt.$$

The above introduced unitary transmutations of Sonine–Katrakhov and Poisson–Katrakhov are special cases of this class of operators. For this case we must choose a weight function $\varphi(t)$ as a power function depending on the parameter ν.

5.2.4 Some applications of Buschman–Erdélyi transmutations

In this subsection we briefly discuss some applications of Buschman–Erdélyi operators. Section 8.1 contains other applications in expanded form.

The above classes of transmutations may be used for deriving explicit formulas for solutions of partial differential equations with Bessel operators via unperturbed equation solutions. An example is the B-elliptic equation of the form

$$\sum_{k=1}^n B_{\nu,x_k} u(x_1,\dots,x_n) = f, \tag{5.58}$$

and also similar B-hyperbolic and B-parabolic equations. This idea works by Sonine–Poisson–Delsarte transmutations (cf. [51–53,56,242]). New results follow automatically for new classes of transmutations.

Now let us consider the Euler–Poisson–Darboux equation in a half-space

$$B_{\alpha,\,t} u(t,x) = \frac{\partial^2 u}{\partial t^2} + \frac{2\alpha+1}{t}\frac{\partial u}{\partial t} = \Delta_x u + F(t,x),$$

with $t > 0$, $x \in \mathbb{R}^n$. Let us consider a general plan for finding different initial and boundary conditions at $t = 0$ with guaranteed existence of solutions. Define any transmutations $X_{\alpha,\,t}$ and $Y_{\alpha,\,t}$ satisfying (5.16). Suppose that functions

$$X_{\alpha,\,t} u = v(t,x), \qquad X_{\alpha,\,t} F = G(t,x)$$

exist. Suppose that the unperturbed Cauchy problem

$$\frac{\partial^2 v}{\partial t^2} = \Delta_x v + G,\ v|_{t=0} = \varphi(x),\ v'_t|_{t=0} = \psi(x) \tag{5.59}$$

is correctly solvable in a half-space. Then if $Y_{\alpha,\,t} = X^{-1}_{\alpha,\,t}$, then we receive the following initial conditions:

$$X_\alpha u|_{t=0} = a(x),\ (X_\alpha u)'|_{t=0} = b(x). \tag{5.60}$$

By this method the choice of different classes of transmutations (Sonine–Poisson–Delsarte, Buschman–Erdélyi of the first, second, and third kinds, Buschman–Erdélyi of the zero order smoothness, unitary transmutations of Sonine–Katrakhov and Poisson–Katrakhov, transmutations with general kernels) will correspond to different kinds of initial conditions [535].

In the monograph of Pskhu [459] this method is applied for solving an equation with fractional derivatives with the usage of the Stankovic transform. Glushak applied Buschman–Erdélyi operators in [185].

This class of operators was thoroughly studied by Levitan [316,327]. It has many applications to partial differential equations, including Bessel operators [317]. Generalized translations are used for moving singular points from the origin to any location. They are explicitly expressed via transmutations [317]. Due to this fact new classes of transmutations lead to new classes of generalized translations.

In recent years Dunkl operators were thoroughly studied. These are difference–differentiation operators consisting of combinations of classical derivatives and finite differences. In higher dimensions Dunkl operators are defined by symmetry and reflection groups. For this class there are many results on transmutations which are of Sonine–Poisson–Delsarte and Buschman–Erdélyi types (cf. [560] and references therein).

It has been known for many years that a problem of describing polynomial solutions for the B-elliptic equation does not need the new theory. The answer is in the transmutation theory. The simple fact that Sonine–Poisson–Delsarte transmutations transform power functions into other power functions means that they also transform explicitly so-called B-harmonic polynomials into classical harmonic polynomials and vice versa. The same is true for generalized B-harmonics because they are restrictions of B-harmonic polynomials onto the unit sphere. This approach is thoroughly applied by Rubin [485]. Usage of Buschman–Erdélyi operators refreshes this theory with new possibilities.

Now let us construct integral operators of Buschman–Erdélyi with more general functions as kernels. Consider an operator ${}_1S^{\nu}_{0+}$. It has the form

$$ {}_1S^{\nu}_{0+} = \frac{d}{dx}\int_0^x K\left(\frac{x}{y}\right) f(y)\,dy, \tag{5.61}$$

with kernel K expressed by $K(z) = P_\nu(z)$. Simple properties of special functions lead to the fact that ${}_1S^{\nu}_{0+}$ is a special case of (5.61) with the Gegenbauer function kernel

$$ K(z) = \frac{\Gamma(\alpha+1)\,\Gamma(2\beta)}{2^{p-\frac{1}{2}}\Gamma(\alpha+2\beta)\,\Gamma(\beta+\frac{1}{2})}(z^\alpha-1)^{\beta-\frac{1}{2}}C^{\beta}_{\alpha}(z) \tag{5.62}$$

with $\alpha=\nu$, $\beta=\frac{1}{2}$, or with the Jacobi function kernel

$$ K(z) = \frac{\Gamma(\alpha+1)}{2^\rho\Gamma(\alpha+\rho+1)}(z-1)^\rho(z+1)^\sigma P^{(\rho,\sigma)}_{\alpha}(z) \tag{5.63}$$

with $\alpha=\nu$, $\rho=\sigma=0$. More general are operators with the Gauss hypergeometric function kernel ${}_2F_1$ or Meijer G-function $G^{m,n}_{p,q}$ (1.41) or Fox–Wright $E_{\alpha,\beta}$ function (1.40) kernels (cf. [494]). For studying such operators inequalities for kernel functions are very useful (e.g., [228,528]).

Define the first class of generalized operators.

Definition 40. *Define Gauss–Buschman–Erdélyi operators by formulas*

$$ {}_1F_{0+}(a,b,c)[f] = \frac{1}{2^{c-1}\Gamma(c)}, \tag{5.64}$$

$$ \int_0^x \left(\frac{x}{y}-1\right)^{c-1}\left(\frac{x}{y}+1\right)^{a+b-c} {}_2F_1\left({}^{a,b}_{c}|\frac{1}{2}-\frac{1}{2}\frac{x}{y}\right) f(y)\,dy,$$

$$ {}_2F_{0+}(a,b,c)[f] = \frac{1}{2^{c-1}\Gamma(c)}, \tag{5.65}$$

$$ \int_0^x \left(\frac{y}{x}-1\right)^{c-1}\left(\frac{y}{x}+1\right)^{a+b-c} {}_2F_1\left({}^{a,b}_{c}|\frac{1}{2}-\frac{1}{2}\frac{y}{x}\right) f(y)\,dy,$$

$$ {}_1F_{-}(a,b,c)[f] = \frac{1}{2^{c-1}\Gamma(c)}, \tag{5.66}$$

$$ \int_0^x \left(\frac{y}{x}-1\right)^{c-1}\left(\frac{y}{x}+1\right)^{a+b-c} {}_2F_1\left({}^{a,b}_{c}|\frac{1}{2}-\frac{1}{2}\frac{y}{x}\right) f(y)\,dy,$$

$$ {}_2F_-(a,b,c)[f] = \frac{1}{2^{c-1}\Gamma(c)}, \tag{5.67}$$

$$\int\limits_0^x \left(\frac{x}{y}-1\right)^{c-1}\left(\frac{x}{y}+1\right)^{a+b-c} {}_2F_1\left({}^{a,b}_{c}\Big|\frac{1}{2}-\frac{1}{2}\frac{x}{y}\right) f(y)\,dy,$$

$$ {}_3F_{0+}[f] = \frac{d}{dx}\,{}_1F_{0+}[f], \quad {}_4F_{0+}[f] = {}_2F_{0+}\frac{d}{dx}[f], \tag{5.68}$$

$$ {}_3F_-[f] = {}_1F_-\left(-\frac{d}{dx}\right)[f], \quad {}_4F_-[f] = \left(-\frac{d}{dx}\right){}_2F_-[f]. \tag{5.69}$$

The symbol ${}_2F_1$ in definitions (5.65) and (5.67) means the Gauss hypergeometric function on the natural domain and in (5.64) and (5.66) the main branch of its analytical continuation.

Operators (5.64)–(5.67) generalize Buschman–Erdélyi ones (5.5)–(5.8), respectively. They reduce to Buschman–Erdélyi for the choice of parameters $a = -(\nu+\mu)$, $b = 1+\nu-\mu$, $c = 1-\mu$. For operators (5.64)–(5.67) the above results are generalized with necessary changes. For example they are factorized via more simple operators (5.68) and (5.69) with a special choice of parameters.

Operators (5.68) and (5.69) are generalizations of (5.14) and (5.15). For them the following result is true.

Theorem 64. *Operators* (5.68) *and* (5.69) *may be extended to isometric in* $L_2(0,\infty)$ *if and only if they coincide with Buschman–Erdélyi operators of zero order smoothness* (5.14) *and* (5.15) *for integer values of* $\nu = \frac{1}{2}(b-a-1)$.

This theorem singles out Buschman–Erdélyi operators of zero order smoothness at least in the class (5.64)–(5.69). Operators (5.64)–(5.67) are generalizations of fractional integrals. Analogously may be studied generalizations to (5.33)–(5.34), (5.43), and (5.46)–(5.47).

More general are operators with G-function kernel,

$$ {}_1G_{0+}(\alpha,\beta,\delta,\gamma)[f] = \frac{2^\delta}{\Gamma(1-\alpha)\Gamma(1-\beta)}$$

$$\times \int\limits_0^x \left(\frac{x}{y}-1\right)^{-\delta}\left(\frac{x}{y}+1\right)^{1+\delta-\alpha-\beta} G^{1\,2}_{2\,2}\left(\frac{x}{2y}-\frac{1}{2}\Big|^{\alpha,\,\beta}_{\gamma,\,\delta}\right) f(y)\,dy. \tag{5.70}$$

Other operators are with different intervals of integration and parameters of the G-function. For $\alpha = 1-a$, $\beta = 1-b$, $\delta = 1-c$, $\gamma = 0$ (5.70) reduces to (5.64), for $\alpha = 1+\nu$, $\beta = -\nu$, $\delta = \gamma = 0$ (5.70) reduces to Buschman–Erdélyi operators of zero order smoothness ${}_1S^{\nu}_{0+}$.

Further generalizations are in terms of Wright or Fox functions. They lead to Wright–Buschman–Erdélyi and Fox–Buschman–Erdélyi operators. These classes are

connected with Sonine–Dimovski and Poisson–Dimovski transmutations [89,93], and also with generalized fractional integrals introduced by Kiryakova [252].

V. Katrakhov found a new approach for boundary value problems for elliptic equations with strong singularities of infinite order. For example for the Poisson equation he studied problems with solutions of arbitrary growth. At singular points he proposed a new kind of boundary condition: the K-trace. His results are based on constant usage of Buschman–Erdélyi transmutations of the first kind for definition of norms, solution estimates, and correctness proofs [225,227].

Moreover in joint papers with I. Kipriyanov he introduced and studied new classes of pseudodifferential operators based on transmutational techniques [248–250]. These results were paraphrased in a reorganized manner in [52].

5.3 Multi-dimensional integral transforms of Buschman–Erdélyi type with Legendre functions in kernels

In this section we consider generalizations of Buschman–Erdélyi operators for the multi-dimensional case. This case was studied by O. V. Skoromnik and S. M. Sitnik (see [538]).

5.3.1 Basic definitions

Let $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n_+$, $t=(t_1,t_2,...,t_n)\in\mathbb{R}^n_+$, $x\cdot t=\sum\limits_{n=1}^{n} x_n t_n$ denotes their scalar product, and in particular, $x\cdot 1=\sum\limits_{n=1}^{n} x_n$. The expression $x>t$ means that $x_1>t_1,...,x_n>t_n$, the nonstrict inequality $\geq$ has a similar meaning, by $\mathbb{N}=\{1,2,...\}$ we denote the set of positive integers, $\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$, and $\mathbb{N}_0^n=\mathbb{N}_0\times\mathbb{N}_0\times...\times\mathbb{N}_0$. We denote

$\mathbf{m}=(m_1,m_2,...,m_n)\in\mathbb{N}_0^n$, $m_1=m_2=...=m_n$;
$\mathbf{n}=(\overline{n}_1,\overline{n}_2,...,\overline{n}_n)\in\mathbb{N}_0^n$, $\overline{n}_1=\overline{n}_2=...=\overline{n}_n$;
$\mathbf{p}=(p_1,p_2,...,p_n)\in\mathbb{N}_0^n$, $p_1=p_2=...=p_n$;
$\mathbf{q}=(q_1,q_2,...,q_n)\in\mathbb{N}_0^n$, $q_1=q_2=...=q_n$, $(0\leq\mathbf{m}\leq\mathbf{q},\ 0\leq\mathbf{m}\leq\mathbf{p})$.

We put

$\sigma=(\sigma_1,\sigma_2,...,\sigma_n)\in\mathbb{C}^n$;
$\kappa=(\kappa_1,\kappa_2,...,\kappa_n)\in\mathbb{C}^n$;
$\delta=(\delta_1,\delta_2,...,\delta_n)\in\mathbb{R}^n$;
$\gamma=(\gamma_1,\gamma_2,...,\gamma_n)\in\mathbb{R}^n$, $0<\gamma<1$;
$a_i=(a_{i_1},a_{i_2},...,a_{i_n})$, $1\leq i\leq p$, $a_{i_1},a_{i_2},...,a_{i_n}\in\mathbb{C}$, $1\leq i_1\leq p_1,...,1\leq i_n\leq p_n$;
$b_j=(b_{j_1},b_{j_2},...,b_{j_n})$, $1\leq j\leq q$, $b_{j_1},b_{j_2},...,b_{j_n}\in\mathbb{C}$, $1\leq j_1\leq q_1,...,1\leq j_n\leq q_n$;
$\alpha_i=(\alpha_{i_1},\alpha_{i_2},...,\alpha_{i_n})$, $1\leq i\leq p$, $\alpha_{i_1},\alpha_{i_2},...,\alpha_{i_n}\in\mathbb{R}^1_+$, $1\leq i_1\leq p_1,...,1\leq i_n\leq p_n$;

$\beta_j = (\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_n})$, $1 \le j \le q$, $\beta_{j_1}, \beta_{j_2}, ..., \beta_{j_n} \in \mathbb{R}^1_+$, $1 \le j_1 \le q_1, ..., 1 \le j_n \le q_n$;
$\mathbf{k} = (k_1, k_2, ..., k_n) \in \mathbb{N}_0^n = \mathbb{N}_0 \times ... \times \mathbb{N}_0$, $k_i \in \mathbb{N}_0$, $i = 1, 2, ..., n$, is a multi-index with $\mathbf{k}! = k_1! \cdots k_n!$ and $|\mathbf{k}| = k_1 + k_2 + ... + k_n$;
for $l = (l_1, l_2, ..., l_n) \in \mathbb{R}^n_+$,

$$D^l = \frac{\partial^{|l|}}{(\partial x_1)^{l_1} \cdots (\partial x_n)^{l_n}},$$

$dt = dt_1 \cdot dt_2 \cdots dt_n$, $t^l = t^{l_1} \cdots t^{l_n}$;
$x^2 - t^2 = (x_1^2 - t_1^2) \cdots (x_n^2 - t_n^2)$; $f(t) = f(t_1, t_2, ..., t_n)$.

We introduce the function

$$\mathrm{H}^{\mathbf{m},\,\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}\left[\frac{x}{t}\,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] = \prod_{k=1}^{n} \mathrm{H}^{m_k,\,\overline{n}_k}_{p_k,\,q_k}\left[\frac{x_k}{t_k}\,\middle|\, \begin{matrix} (a_{i_k},\alpha_{i_k})_{1,p_k} \\ (b_{j_k},\beta_{j_k})_{1,q_k} \end{matrix}\right], \tag{5.71}$$

which is the product of the H-functions $\mathrm{H}^{m,\,n}_{p,\,q}[z]$. Such a function is defined by

$$\mathrm{H}^{m,n}_{p,\,q}[z] \equiv \mathrm{H}^{m,n}_{p,q}\left[z\,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] = \frac{1}{2\pi i}\int_L \mathcal{H}^{m,n}_{p,q}(s) z^{-s} ds, \; z \neq 0,$$

where

$$\mathcal{H}^{m,n}_{p,\,q}(s) \equiv \mathcal{H}^{m,n}_{p,\,q}\left[\begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\,\middle|\, s\right] = \frac{\prod\limits_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod\limits_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod\limits_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod\limits_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}. \tag{5.72}$$

Here L is a specially chosen infinite contour and empty product, if it occurs, being taken to be one. Note that most of the elementary and special functions are special cases of the H-function and one may find its properties in the books by Mathai and Saxena ([362], Chapter 2), Srivastava, Gupta, and Goyal ([499], Chapter 1), Prudnikov, Brychcov, and Marichev ([457], Section 8.3), and Kilbas and Saigo ([239], Chapters 1 and 2).

We introduce the function

$$\mathrm{P}^{\gamma}_{\delta}[z] = \prod_{k=1}^{n} \mathrm{P}^{\gamma_k}_{\delta_k}[z_k],$$

which is the product of the Legendre functions $\mathrm{P}^{\overline{\gamma}}_{\overline{\delta}}(z)$ of the first kind. For complex $\overline{\gamma}$, $Re(\overline{\gamma}) < 1$, and $\overline{\delta}, z \in \mathrm{C}$ this function is defined by

$$\mathrm{P}^{\overline{\gamma}}_{\overline{\delta}}(z) = \frac{1}{\Gamma(1-\overline{\gamma})}\left(\frac{z+1}{z-1}\right)^{\frac{\overline{\gamma}}{2}} {}_2\mathrm{F}_1\left(-\overline{\delta},\, 1+\overline{\delta};\, 1-\overline{\gamma};\, \frac{1-z}{2}\right),$$

$$|\arg(z-1)| < \pi,$$
$$\mathrm{P}_{\overline{\delta}}^{\overline{\gamma}}(x) = \frac{1}{\overline{\Gamma}(1-\overline{\gamma})}\left(\frac{1+x}{1-x}\right)^{\frac{\overline{\gamma}}{2}} {}_2\mathrm{F}_1\left(-\overline{\delta},\, 1+\overline{\delta};\, 1-\overline{\gamma};\, \frac{1-x}{2}\right),$$
$$-1 < x < 1$$

(see [121], formulas (3.2)(3) and (3.4)(6), [457], Section 11.18), where ${}_2\mathrm{F}_1(-\overline{\delta},\, 1+\overline{\delta};\, 1-\overline{\gamma};\, z)$ is the Gauss hypergeometric function (1.33).

We introduce integral transforms:

$$\left(\mathscr{H}^1_{\sigma,\kappa} f\right)(x) = x^{\sigma} \int\limits_{\mathbb{R}^n_+} \mathrm{H}^{\mathbf{m},\,\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}\left[\frac{x}{t}\,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] t^{\kappa} f(t)\frac{dt}{t} (x>0), \tag{5.73}$$

$$\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(x) = \int\limits_{\mathbb{R}^n_+} \left(x^2-t^2\right)^{-\gamma/2} \mathrm{P}^{\gamma}_{\delta}\left(\frac{x}{t}\right) f(t)dt = g(x) \ \ (x>0), \tag{5.74}$$

$$\left(\mathrm{P}^{\gamma}_{\delta,2} f\right)(x) = \int\limits_{\mathbb{R}^n_+} \left(x^2-t^2\right)^{-\gamma/2} \mathrm{P}^{\gamma}_{\delta}\left(\frac{t}{x}\right) f(t)dt = g(x) \ \ (x>0). \tag{5.75}$$

Definition 41. *The weighted space* $\mathfrak{L}_{\overline{\nu},\,\overline{2}}$ *is a space of summable functions* $f(x)= f(x_1,\dots,x_n)$ *on* $\mathbb{R}^n_+$*, such that*

$$\|f\|_{\overline{\nu},\overline{2}} = \left\{\int\limits_{\mathbb{R}^1_+} x_n^{\nu_n\cdot 2-1}\left[\dots\left(\int\limits_{\mathbb{R}^1_+} x_2^{\nu_2\cdot 2-1}\right.\right.\right.$$
$$\left.\left.\left.\times \int\limits_{\mathbb{R}^1_+} x_1^{\nu_1\cdot 2-1}|f(x_1,\dots,x_n)|^2 dx_1 dx_2\right)\dots\right]dx_n\right\}^{1/2} < \infty,$$

where ($\overline{2} = (2,\dots,2)$, $\overline{\nu} = (\nu_1,\dots,\nu_n) \in \mathbb{R}^n$, $\nu_1 = \nu_2 = \dots = \nu_n$).

Our studies are based on representations of (5.74) and (5.74) via the modified H-transform of the form (5.73). Mapping properties such as the boundedness, the range, the representation, and the inversion of the considered transforms are established.

5.3.2 The n-dimensional Mellin transform and its properties

We denote by $[X, Y]$ a set of bounded linear operators acting from a Banach space X into a Banach space Y.

The n-dimensional Mellin transform $(\mathfrak{M}f)(x)$ of a function $f(x) = f(x_1, x_2, ..., x_n)$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n_+$, is defined by

$$(\mathfrak{M}f)(s) = \int\limits_{\mathbb{R}^n_+} f(t)t^{s-1}dt, \qquad \mathrm{Re}(s) = \overline{\nu}, \tag{5.76}$$

where $s = (s_1, s_2, ..., s_n) \in \mathbb{C}^n$; while the inverse Mellin transform is given for $x \in \mathbb{R}^n_+$ by the formula

$$(\mathfrak{M}^{-1}g)(x) = \mathfrak{M}^{-1}[g(\mathbf{p})](x) = \frac{1}{(2\pi i)^n} \int\limits_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \cdots \int\limits_{\gamma_n - i\infty}^{\gamma_n + i\infty} x^{-s} g(s) ds,$$

with $\gamma_j = Re(s_j)$ $(j = 1, \cdots, n)$. The theory for these multi-dimensional Mellin transforms appears in the book by Brychkov and others [40] (see also [241], Chapter 1).

Let M_ζ, R be elementary operators (see [241], Chapter 1):

$$(M_\zeta f)(x) = x^\zeta f(x) \ (\zeta = (\zeta_1, \zeta_2, ..., \zeta_n) \in \mathrm{C}^n), \ \ (Rf)(x) = \frac{1}{x} f\left(\frac{1}{x}\right).$$

The following assertion holds, which follows from [241], formulas (1.4.44), (1.4.45), and (1.4.46) (see [239], Lemma 3.2).

Lemma 17. *Let $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$ $(\nu_1 = \nu_2 = ... = \nu_n)$ and $1 \le \overline{r} < \infty$.*

(a) M_ζ is an isometric isomorphism of $\mathfrak{L}_{\overline{\nu},\overline{r}}$ onto $\mathfrak{L}_{\overline{\nu} - \mathrm{Re}(\zeta),\overline{r}}$ and if $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \le \overline{r} \le 2)$, then

$$(\mathfrak{M}M_\zeta f)(s) = (\mathfrak{M}f)(s + \zeta) \ \ (\mathrm{Re}(s) = \overline{\nu} - \mathrm{Re}(\zeta)).$$

(b) R is an isometric isomorphism of $\mathfrak{L}_{\overline{\nu},\overline{r}}$ onto $\mathfrak{L}_{1 - \overline{\nu},\overline{r}}$ and if $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \le \overline{r} \le 2)$, then

$$(\mathfrak{M}Rf)(s) = (\mathfrak{M}f)(1 - s) \ \ (\mathrm{Re}(s) = \overline{\nu}).$$

Let $\mathrm{I}^\alpha_{0+;\,\sigma,\eta}$ and $\mathrm{I}^\alpha_{-;\,\sigma,\eta}$ be the Erdélyi–Kober operators of fractional integration, defined for $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathrm{C}^n$ $(\mathrm{Re}(\alpha) > 0)$, $\sigma > 0$, $\eta \in C^n$, by

$$\left(\mathrm{I}^\alpha_{0+;\,\sigma,\,\eta} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int\limits_0^x \left(x^\sigma - t^\sigma\right)^{\alpha-1} t^{\sigma\eta+\sigma-1} f(t) dt \ \ (x > 0),$$

$$\left(\mathrm{I}^\alpha_{-;\,\sigma,\,\eta} f\right)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int\limits_x^\infty \left(t^\sigma - x^\sigma\right)^{\alpha-1} t^{\sigma(1-\alpha-\eta)-1} f(t) dt \ \ (x > 0).$$

5.3.3 $\mathfrak{L}_{\overline{\nu},2}$-theory and the inversion formulas for the modified $\mathscr{H}$-transform

This subsection is devoted to the study of transforms $\mathrm{P}^{\gamma}_{\delta,k}f$ $(k=1,2)$ in $\mathfrak{L}_{\overline{\nu},\overline{2}}$. To formulate presented results from $\mathfrak{L}_{\overline{\nu},\overline{2}}$-theory and inversion formulas for the modified $\mathscr{H}$-transform (5.73) we need the following constants and the generalization of the H-function (5.71) (see [239], formulas (3.4.1), (3.4.2), (1.1.7), (1.1.8), and (1.1.10)):

$$\alpha_1=\begin{cases}-\min\limits_{1\le j_1\le m_1}\left[\frac{\mathrm{Re}(b_{j_1})}{\beta_{j_1}}\right] \text{ for } m_1>0,\\ 0 \text{ for } m_1=0,\end{cases}\quad \beta_1=\begin{cases}\min\limits_{1\le i_1\le \overline{n}_1}\left[\frac{1-\mathrm{Re}(a_{i_1})}{\alpha_{i_1}}\right] \text{ for } \overline{n}_1>0,\\ 0 \text{ for } \overline{n}_1=0,\end{cases}$$

$$\alpha_2=\begin{cases}-\min\limits_{1\le j_2\le m_2}\left[\frac{\mathrm{Re}(b_{j_2})}{\beta_{j_2}}\right] \text{ for } m_2>0,\\ 0 \text{ for } m_2=0,\end{cases}\quad \beta_2=\begin{cases}\min\limits_{1\le i_2\le \overline{n}_2}\left[\frac{1-\mathrm{Re}(a_{i_2})}{\alpha_{i_2}}\right] \text{ for } \overline{n}_2>0,\\ 0 \text{ for } \overline{n}_2=0,\end{cases}$$

and so on,

$$\alpha_n=\begin{cases}-\min\limits_{1\le j_n\le m_n}\left[\frac{\mathrm{Re}(b_{j_n})}{\beta_{j_n}}\right] \text{ for } m_n>0,\\ 0 \text{ for } m_2=0,\end{cases}\quad \beta_n=\begin{cases}\min\limits_{1\le i_n\le \overline{n}_n}\left[\frac{1-\mathrm{Re}(a_{i_n})}{\alpha_{i_n}}\right] \text{ for } \overline{n}_n>0,\\ 0 \text{ for } \overline{n}_n=0,\end{cases}$$

$$a_1^*=\sum_{i=1}^{\overline{n}_1}\alpha_{i_1}-\sum_{i=\overline{n}_1+1}^{p_1}\alpha_{i_1}+\sum_{j=1}^{m_1}\beta_{j_1}-\sum_{j=m_1+1}^{q_1}\beta_{j_1},\ \ \Delta_1=\sum_{j=1}^{q_1}\beta_{j_1}-\sum_{i=1}^{p_1}\alpha_{i_1},$$

$$a_2^*=\sum_{i=1}^{\overline{n}_2}\alpha_{i_2}-\sum_{i=\overline{n}_2+1}^{p_2}\alpha_{i_2}+\sum_{j=1}^{m_2}\beta_{j_2}-\sum_{j=m_1+1}^{q_2}\beta_{j_2},\ \ \Delta_2=\sum_{j=1}^{q_2}\beta_{j_2}-\sum_{i=1}^{p_2}\alpha_{i_2},$$

and so on,

$$a_n^*=\sum_{i=1}^{\overline{n}_n}\alpha_{i_n}-\sum_{i=\overline{n}_n+1}^{p_n}\alpha_{i_n}+\sum_{j=1}^{m_n}\beta_{j_n}-\sum_{j=m_n+1}^{q_n}\beta_{j_n},\ \ \Delta_n=\sum_{j=1}^{q_n}\beta_{j_n}-\sum_{i=1}^{p_n}\alpha_{i_n},$$

$$\mu_1=\sum_{j=1}^{q_1}b_{j_1}-\sum_{i=1}^{p_1}a_{i_1}+\frac{p_1-q_1}{2},\ \mu_2=\sum_{j=1}^{q_2}b_{j_2}-\sum_{i=1}^{p_2}a_{i_2}+\frac{p_2-q_2}{2},\ldots,$$

$$\mu_n=\sum_{j=1}^{q_n}b_{j_n}-\sum_{i=1}^{p_n}a_{i_n}+\frac{p_n-q_n}{2},$$

$$\alpha_0^1=\begin{cases}1+\max\limits_{m_1+1\le j_1\le q_1}\left[\frac{\mathrm{Re}(b_{j_1})-1}{\beta_{j_1}}\right] \text{ for } q_1>m_1,\\ \infty \text{ for } q_1=m_1,\end{cases}$$

$$\beta_0^1 = \begin{cases} 1+\min\limits_{\overline{n}_1+1\le i_1\le p_1}\left[\frac{\mathrm{Re}(a_{i_1})}{\alpha_{i_1}}\right] \text{ for } p_1>\overline{n}_1, \\ \infty \text{ for } p_1=\overline{n}_1, \end{cases}$$

$$\alpha_0^2 = \begin{cases} 1+\max\limits_{m_2+1\le j_2\le q_2}\left[\frac{\mathrm{Re}(b_{j_2})-1}{\beta_{j_2}}\right] \text{ for } q_2>m_2, \\ \infty \text{ for } q_2=m_2, \end{cases}$$

$$\beta_0^2 = \begin{cases} 1+\min\limits_{\overline{n}_2+1\le i_2\le p_2}\left[\frac{\mathrm{Re}(a_{i_2})}{\alpha_{i_2}}\right] \text{ for } p_2>\overline{n}_2, \\ \infty \text{ for } p_2=\overline{n}_2, \end{cases}$$

....

$$\alpha_0^n = \begin{cases} 1+\max\limits_{m_n+1\le j_n\le q_n}\left[\frac{\mathrm{Re}(b_{j_n})-1}{\beta_{j_n}}\right] \text{ for } q_n>m_n, \\ \infty \text{ for } q_n=m_n, \end{cases}$$

$$\beta_0^n = \begin{cases} 1+\min\limits_{\overline{n}_n+1\le i_n\le p_2}\left[\frac{\mathrm{Re}(a_{i_n})}{\alpha_{i_n}}\right] \text{ for } p_n>\overline{n}_n, \\ \infty \text{ for } p_n=\overline{n}_n. \end{cases}$$

The exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of a function $\overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},q}(s)$,

$$\overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}(s) \equiv \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}\left[\begin{array}{c} (a_i,\alpha_i)_{1,\mathbf{p}} \\ (b_j,\beta_j)_{1,\mathbf{q}} \end{array}\middle| s\right] = \prod_{k=1}^{n} \mathcal{H}^{m_k,\overline{n}_k}_{p_k,\,q_k}\left[\begin{array}{c} (a_{i_k},\alpha_{i_k})_{1,p_k} \\ (b_{j_k},\beta_{j_k})_{1,q_k} \end{array}\middle| s\right],$$

is called a set of vectors $\overline{\nu} = (\nu_1, \nu_2, ..., \nu_n) \in \mathbb{R}^n$ $(\nu_1 = \nu_2 = ... = \nu_n)$, such that $\alpha_1 < 1-\nu_1 < \beta_1, \alpha_2 < 1-\nu_2 < \beta_2, ..., \alpha_n < 1-\nu_n < \beta_n$, and functions $\mathcal{H}^{m_1,\overline{n}_1}_{p_1,\,q_1}(s_1)$, $\mathcal{H}^{m_2,\overline{n}_2}_{p_2,\,q_2}(s_2)$,...,$\mathcal{H}^{m_n,\overline{n}_n}_{p_n,\,q_n}(s_n)$ have zeros on lines $\mathrm{Re}(s_1) < 1-\nu_1$, $\mathrm{Re}(s_2) < 1-\nu_2$, ..., $\mathrm{Re}(s_n) < 1-\nu_n$, respectively.

Applying the multi-dimensional Mellin transform (5.76) to (5.73), taking into account the results for the one-dimensional case ([239], formula (5.1.14)), we obtain

$$(\mathfrak{M}\mathscr{H}^1_{\sigma,\kappa} f)(s) = \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},q}\left[\begin{array}{c} (a_i,\alpha_i)_{1,\mathbf{p}} \\ (b_j,\beta_j)_{1,\mathbf{q}} \end{array}\middle| s+\sigma\right](\mathfrak{M}f)(s+\sigma+\kappa). \tag{5.77}$$

The following assertion presents the $\mathfrak{L}_{\overline{\nu},\overline{2}}$-theory of the modified $\mathscr{H}$-transform (5.73). For the one-dimensional case, see [239], Theorem 5.37.

Here we will use spaces $\mathfrak{L}_{\overline{\nu},\overline{2}}$ and $\mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ (see Definition 41) with different $\overline{\nu}$ and $\widetilde{\overline{\nu}}$. If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ then we will use the notation $\mathscr{H}^1_{\sigma,\kappa} f$ and if $f \in \mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ then we will use the notation $\widetilde{\mathscr{H}}^1_{\sigma,\kappa} f$ where both these transforms are defined by formula (5.77).

Theorem 65. *Let*

$$\alpha_1 < \nu_1 - \mathrm{Re}(\kappa_1) < \beta_1, \tag{5.78}$$

$$\alpha_2 < \nu_2 - \mathrm{Re}(\kappa_2) < \beta_1, \dots, \alpha_n < \nu_n - \mathrm{Re}(\kappa_1) < \beta_n, \tag{5.79}$$

$$\nu_1 = \nu_2 = \ldots = \nu_n, \tag{5.80}$$
$$a_1^* = 0, a_2^* = 0, \ldots, a_n^* = 0, \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) \le 0, \tag{5.81}$$
$$\Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) \le 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) \le 0. \tag{5.82}$$

The following assertions hold:

(a) There exists a one-to-one map $\mathscr{H}^1_{\sigma,\kappa} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{\overline{\nu}-\mathrm{Re}(\kappa+\sigma),\overline{2}}]$ *such that relation* (5.77) *holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(s) = \overline{\nu} - \mathrm{Re}(\kappa+\sigma)$.

If $a_1^* = 0,\ a_2^* = 0, \ldots, a_n^* = 0,\ \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) = 0,\ \Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) = 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) = 0$, *and* $1 - \overline{\nu} + \mathrm{Re}(\kappa) \notin \mathcal{E}_{\overline{\mathcal{H}}}$, *then* $\mathscr{H}^1_{\sigma,\kappa}$ *maps* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *onto* $\mathfrak{L}_{\overline{\nu}-\mathrm{Re}(\kappa+\sigma),\overline{2}}$.

(b) The transform $\mathscr{H}^1_{\sigma,\kappa}$ *does not depend on* $\overline{\nu}$ *in the sense that if* $\overline{\nu}$ *and* $\widetilde{\overline{\nu}}$ *satisfy equations* (5.78)–(5.82) *and if the transforms* $\mathscr{H}^1_{\sigma,\kappa}$ *and* $\widetilde{\mathscr{H}}^1_{\sigma,\kappa}$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ *by formula* (5.77)*, then* $\mathscr{H}^1_{\sigma,\kappa} f = \widetilde{\mathscr{H}}^1_{\sigma,\kappa} f$ *for* $f \in \mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}} \bigcap \mathfrak{L}_{\overline{\nu},\overline{2}}$.

(c) If $a_1^* = 0,\ a_2^* = 0, \ldots, a_n^* = 0,\ \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) < 0$, $\Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) < 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) < 0$, *then for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *transform* $\mathscr{H}^1_{\sigma,\kappa} f$ *is given by* (5.73).

(d) Let $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathrm{C}^n$, $\overline{h} = (h_1, \ldots, h_n) > 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $\mathrm{Re}(\overline{\lambda}) > (\overline{\nu} - \mathrm{Re}(\kappa))\overline{h} - 1$, *then* $\mathscr{H}^1_{\sigma,\kappa} f$ *is represented in the form*

$$\left(\mathscr{H}^1_{\sigma,\kappa} f\right)(x) = \overline{h} x^{\sigma+1-(\overline{\lambda}+1)/\overline{h}}$$
$$\times \frac{d}{dx} x^{(\overline{\lambda}+1)/\overline{h}} \int\limits_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{x}{t}\,\middle|\, \begin{matrix} (-\overline{\lambda},\overline{h}),(a_i,\alpha_i)_{1,\mathbf{p}} \\ (b_j,\beta_j)_{1,q},\,(-\overline{\lambda}-1,\overline{h}) \end{matrix}\right] t^{\kappa-1} f(t)dt,$$

while for $\mathrm{Re}(\overline{\lambda}) < (\overline{\nu} - \mathrm{Re}(k))\overline{h} - 1$ *it is given by*

$$\left(\mathscr{H}^1_{\sigma,\kappa} f\right)(x) = -\overline{h} x^{\sigma+1-(\overline{\lambda}+1)/\overline{h}}$$
$$\times \frac{d}{dx} x^{(\overline{\lambda}+1)/\overline{h}} \int\limits_0^\infty \mathrm{H}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{x}{t}\,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p},(-\overline{\lambda},\overline{h}) \\ (-\overline{\lambda}-1,\overline{h}),(b_j,\beta_j)_{1,q} \end{matrix}\right] t^{\kappa-1} f(t)dx.$$

(e) If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(\kappa+\sigma),\overline{2}}$, *then there the following relation holds:*

$$\int\limits_0^\infty f(x)\left(\mathscr{H}^1_{\sigma,\kappa} g\right)(x)dx = \int\limits_0^\infty \left(\mathscr{H}^2_{\sigma,\kappa} f\right)(x)g(x)dx, \tag{5.83}$$

where

$$\left(\mathscr{H}^2_{\sigma,\kappa} f\right)(x) = x^\sigma \int\limits_0^\infty \mathrm{H}^{\mathbf{m},n}_{\mathbf{p},\mathbf{q}}\left[\frac{t}{x}\,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,\mathbf{p}} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] t^\kappa f(t)\frac{dt}{x}. \tag{5.84}$$

5.3.4 Inversion of $\mathscr{H}^1_{\sigma,\kappa}$

Inversion formulas for the transform $\mathscr{H}^1_{\sigma,\kappa}$ are given by the following equalities (for the one-dimensional case, see [5, formulas (5.5.23) and (5.5.24)]):

$$f(x) = -\overline{h}x^{(\overline{\lambda}+1)/\overline{h}-\kappa}\frac{d}{dx}x^{-(\overline{\lambda}+1)/\overline{h}} \times \int\limits_0^\infty \mathrm{H}^{\mathbf{q-m},\mathbf{p-n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{t}{x}\left|\begin{matrix}(-\overline{\lambda},\overline{h}),(1-a_i-\alpha_i,\alpha_i)_{\mathbf{m}+1,\mathbf{p}},(1-a_i-\alpha_i,\alpha_i)_{1,\mathbf{m}}\\(1-b_j-\beta_j,\beta_j)_{\mathbf{m}+1,\mathbf{q}},(1-b_j-\beta_j,\beta_j)_{1,\mathbf{m}}\ (-\overline{\lambda}-1,\overline{h})\end{matrix}\right.\right] \times t^{-\sigma}(\mathscr{H}^1_{\sigma,\kappa}f)(t)dt \tag{5.85}$$

or

$$f(x) = \overline{h}x^{(\overline{\lambda}+1)/\overline{h}-1}\frac{d}{dx}x^{-(\overline{\lambda}+1)/\overline{h}} \times \int\limits_0^\infty \mathrm{H}^{\mathbf{q-m}+1,\mathbf{p-n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{t}{x}\left|\begin{matrix}(1-a_i-\alpha_i,\alpha_i)_{\mathbf{m}+1,p},(1-a_i-\alpha_i,\alpha_i)_{1,n},\ (-\overline{\lambda},\overline{h})\\(-\overline{\lambda}-1,\overline{h}),(1-b_j-\beta_j,\beta_j)_{\mathbf{m}+1,\mathbf{q}},(1-b_j-\beta_j,\beta_j)_{1,\mathbf{m}}\end{matrix}\right.\right] \times t^{-\sigma}(\mathscr{H}^1_{\sigma,\kappa}f)(t)dt. \tag{5.86}$$

Conditions for the validity of these formulas are given by the following assertion (for the one-dimensional case, see [239], Theorem 5.47)).

Theorem 66. *Let* $a_1^* = 0, a_2^* = 0, ..., a_n^* = 0,\ \alpha_1 < \nu_1 - \mathrm{Re}(\kappa_1) < \beta_1,\ \alpha_2 < \nu_2 - \mathrm{Re}(\kappa_2) < \beta_2, ..., \alpha_n < \nu_n - \mathrm{Re}(\kappa_n) < \beta_n,\ \alpha_0^1 < 1 - \nu_1 + \mathrm{Re}(\kappa_1) < \beta_0^1,\ \alpha_0^2 < 1 - \nu_2 + \mathrm{Re}(\kappa_2) < \beta_0^2, ..., \alpha_0^n < 1 - \nu_n + \mathrm{Re}(\kappa_n) < \beta_0^n$, *and let* $\overline{\lambda} \in \mathbb{C}^n,\ \overline{h} > 0$.

If $\Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) = 0,\ \Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) = 0, ...,\ \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) = 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}(\nu_1, \nu_2, ..., \nu_n)$, *then the inversion formulas* (5.85) *and* (5.86) *are valid for* $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu}+\mathrm{Re}(\kappa))\overline{h}-1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu}+\mathrm{Re}(\kappa))\overline{h}-1$, *respectively.*

5.4 Representations in the form of modified $\mathscr{H}$-transform

5.4.1 Mellin transform of auxiliary functions $\mathrm{K}_1(x)$ and $\mathrm{K}_2(x)$

We introduce the so-called one-sided functions

$$\mathrm{K}_1(x) = (x^2-1)_+^{-\gamma/2}\,\mathrm{P}_\gamma^\delta(x) = \begin{cases}(x^2-1)^{-\gamma/2}\mathrm{P}_\gamma^\delta(x) & \text{for } x > 1,\\ 0 & \text{for } 0 < x < 1,\end{cases} \tag{5.87}$$

$$\mathrm{K}_2(x) = (1-x^2)_+^{-\gamma/2}\,\mathrm{P}_\gamma^\delta(x) = \begin{cases} (1-x^2)^{-\gamma/2}\mathrm{P}_\gamma^\delta(x) & \text{for } 0 < x < 1, \\ 0 & \text{for } x > 1. \end{cases} \tag{5.88}$$

Using (5.87) and (5.88) we can present transforms (5.74) and (5.75) in respective forms

$$\left(\mathrm{P}_{\delta,1}^{\gamma} f\right)(x) = \int\limits_0^\infty \mathrm{K}_1\left(\frac{x}{t}\right)\left(\mathrm{M}_{-\gamma} f\right)(t)dt,$$

$$\left(\mathrm{P}_{\delta,2}^{\gamma} f\right)(x) = x^{1-\gamma}\int\limits_0^\infty \left(\mathrm{R}\mathrm{K}_2\right)\left(\frac{x}{t}\right)\left(\mathrm{M}_{-1} f\right)(t)dt.$$

The following assertion yields the multi-dimensional Mellin transform formulas of $\mathrm{K}_1(x)$ and $\mathrm{K}_2(x)$.

Lemma 18. *Let* $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$, $\delta = (\delta_1, \delta_2, ..., \delta_n)$, $s = (s_1,\, s_2, ..., s_n) \in \mathrm{C}^n$.

(a) If $\mathrm{Re}(\gamma) < 1$, $\mathrm{Re}(s) < 1 + \mathrm{Re}(\gamma+\delta)$, $\mathrm{Re}(s) < \mathrm{Re}(\gamma-\delta)$*, then*

$$\left(\mathfrak{M}\mathrm{K}_1\right)(s) = 2^{\gamma-1}\frac{\Gamma\left(\frac{1+\gamma+\delta-s}{2}\right)\Gamma\left(\frac{\gamma-\delta-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}. \tag{5.89}$$

(b) If $\mathrm{Re}(\gamma) < 1$, $\mathrm{Re}(s) > 0$*, then*

$$\left(\mathfrak{M}\mathrm{K}_2\right)(s) = 2^{\gamma-1}\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{1-\gamma-\delta+s}{2}\right)\Gamma\left(1+\frac{\delta-\gamma+s}{2}\right)}. \tag{5.90}$$

Proof. By [457], formula (2.172.9), under the conditions in (a), the following formula holds:

$$\begin{aligned}\left(\mathfrak{M}\mathrm{K}_1\right)(s) &= \frac{2^{\gamma_1-s_1-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+\gamma_1+\delta_1-s_1}{2}\right)\Gamma\left(\frac{\gamma_1-\delta_1-s_1}{2}\right)}{\Gamma(1-s_1)} \\ &\times \frac{2^{\gamma_2-s_2-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+\gamma_2+\delta_2-s_2}{2}\right)\Gamma\left(\frac{\gamma_2-\delta_2-s_2}{2}\right)}{\Gamma(1-s_2)}\cdots \\ &\cdots\frac{2^{\gamma_1-s_n-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+\gamma_n+\delta_1-s_n}{2}\right)\Gamma\left(\frac{\gamma_n-\delta_n-s_n}{2}\right)}{\Gamma(1-s_n)} \\ &= \frac{2^{\gamma-s-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+\gamma+\delta-s}{2}\right)\Gamma\left(\frac{\gamma-\delta-s}{2}\right)}{\Gamma(1-s)}.\end{aligned} \tag{5.91}$$

Using the duplication formula (1.7) for the gamma function with $z = \frac{1-s}{2}$, from Eq. (5.91) we deduce equality (5.89).

If conditions in (b) are satisfied, then according to [457], formula (2.172.),

$$\left(\mathfrak{M}\mathrm{K}_2\right)(s) = 2^{\gamma-s}\sqrt{\pi}\frac{\Gamma(s)}{\Gamma\left(\frac{1-\gamma-\delta+s}{2}\right)\Gamma\left(1+\frac{\delta-\gamma+s}{2}\right)}. \tag{5.92}$$

Applying equalities (5.92) and (1.7) with $z=\frac{s}{2}$ we get equality (5.90). The lemma is proved. □

Applying the convolution Mellin formula ([241], (1.4.56)),

$$\left(\mathfrak{M}\int_0^\infty \mathrm{K}\left(\frac{x}{t}\right)y(t)\frac{dt}{t}\right)(s)=\left(\mathfrak{M}\mathrm{K}\right)(s)\left(\mathfrak{M}f\right)(s),\qquad (x\in\mathbb{R}_+^n),\tag{5.93}$$

being valid for suitable $\mathrm{K}\left(\frac{x}{t}\right)=\mathrm{K}\left(\frac{x_1}{t_1},\frac{x_2}{t_2},...,\frac{x_n}{t_n}\right)$ and $y(x)$.

5.4.2 Mellin transform of $\mathrm{P}^\gamma_{\delta,1}(x)$ and $\mathrm{P}^\gamma_{\delta,2}(x)$

Applying (5.89), for $\left(\mathrm{P}^\gamma_{\delta,1}f\right)(x)$ we have

$$\begin{aligned}\left(\mathfrak{M}\mathrm{P}^\gamma_{\delta,1}f\right)(s)&=\left(\mathfrak{M}\int_0^\infty \mathrm{K}_1\left(\frac{x}{t}\right)\left(\mathrm{M}_{1-\gamma}f\right)(t)\frac{dt}{t}\right)(s)\\
&=\left(\mathfrak{M}\mathrm{K}_1\right)(s)\left(\mathfrak{M}\mathrm{M}_{1-\gamma}f\right)(s)\\
&=2^{\gamma-1}\frac{\Gamma\Big((1+\gamma+\delta-s)/2\Big)\Gamma\Big((\gamma-\delta-s)/2\Big)}{\Gamma\Big(1-s/2\Big)\Gamma\Big((1-s)/2\Big)}\left(\mathfrak{M}f\right)(1-\gamma+s).\end{aligned}$$

In accordance with (5.72) we obtain

$$\begin{aligned}\left(\mathfrak{M}\mathrm{P}^\gamma_{\delta,1}f\right)(s)&=2^{\gamma-1}\frac{\Gamma\Big((1+\gamma+\delta-s)/2\Big)\Gamma\Big((\gamma-\delta-s)/2\Big)}{\Gamma\Big(1-s/2\Big)\Gamma\Big((1-s)/2\Big)}\left(\mathfrak{M}f\right)\\
&\quad\times(1-\gamma+s)\\
&=2^{\gamma-1}\overline{\mathcal{H}}^{0,2}_{2,2}\left[\left.\begin{matrix}\left(\frac{1-\gamma-\delta}{2},\frac{1}{2}\right), & \left(1+\frac{\delta-\gamma}{2},\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right), & \left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right|s\right]\left(\mathfrak{M}f\right)(s+1-\gamma).\end{aligned}$$

Therefore, the initial integral transform (5.74) is the modified $\mathscr{H}$-transform (5.73) with $\sigma=0$, $\kappa=1-\gamma$:

$$\left(\mathrm{P}^\gamma_{\delta,1}f\right)(s)=2^{\gamma-1}\int_0^\infty \mathrm{H}^{0,2}_{2,2}\left[\frac{x}{t}\left|\begin{matrix}\left(\frac{1-\gamma-\delta}{2},\frac{1}{2}\right) & \left(1+\frac{\delta-\gamma}{2},\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right) & \left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right.\right]t^{-\gamma}f(t)dt.$$

Similarly to the above, using Eq. (5.90) for $\left(\mathrm{P}^\gamma_{\delta,2}f\right)(x)$ we have

$$\left(\mathfrak{M}\mathrm{P}^\gamma_{\delta,2}f\right)(s)=\left(\mathfrak{M}\left(x^{1-\gamma}\int_0^\infty(\mathrm{R}\mathrm{K}_2)\left(\frac{x}{t}\right)(\mathrm{M}_{-1}f)(t)dt\right)\right)(s)$$

$$
\begin{aligned}
&= \Big(\mathfrak{M}\int_0^\infty (\mathrm{RK}_2)\Big(\frac{x}{t}\Big) f(t)\frac{dt}{t}\Big)(s+1-\gamma)\\
&= \big(\mathfrak{M}(\mathrm{RK}_2)\big)(s+1-\gamma)\big(\mathfrak{M}f\big)(s+1-\gamma)\\
&= \big(\mathfrak{M}\mathrm{K}_2\big)(\gamma-s)\big(\mathfrak{M}f\big)(s+1-\gamma)\\
&= 2^{\gamma-1}\frac{\Gamma\big((\gamma-s)/2\big)\Gamma\big((\gamma-s+1)/2\big)}{\Gamma\big((1-\delta-s)/2\big)\Gamma\big(1+(\delta-s)/2\big)}\big(\mathfrak{M}f\big)(1-\gamma+s).
\end{aligned}
$$

According to equality (5.72) we obtain

$$
\begin{aligned}
\big(\mathfrak{M}\mathrm{P}^{\gamma}_{\delta,2}f\big)(s) &= 2^{\gamma-1}\frac{\Gamma\big((\gamma-s)/2\big)\Gamma\big((\gamma-s+1)/2\big)}{\Gamma\big((1-\delta-s)/2\big)\Gamma\big(1+(\delta-s)/2\big)}\big(\mathfrak{M}f\big)(1-\gamma+s)\\
&= 2^{\gamma-1}\overline{\mathcal{H}}^{0,2}_{2,2}\left[\begin{matrix}\left(\frac{1-\gamma}{2},\frac{1}{2}\right), & \left(1-\frac{\gamma}{2},\frac{1}{2}\right)\\ \left(\frac{1+\delta}{2},\frac{1}{2}\right), & \left(-\frac{\delta}{2},\frac{1}{2}\right)\end{matrix}\,\middle|\, s\right]\Big(\mathfrak{M}f\Big)(s+1-\gamma),
\end{aligned}
$$

and hence, the initial transform $\big(\mathrm{P}^{\gamma}_{\delta,2}f\big)(x)$ is also a modified $\mathscr{H}$-transform (5.73), with $\sigma=0$, $\kappa=1-\gamma$:

$$
\big(\mathrm{P}^{\gamma}_{\delta,2}f\big)(s) = 2^{\gamma-1}\int_0^\infty \mathrm{H}^{0,2}_{2,2}\left[\frac{x}{t}\,\middle|\,\begin{matrix}\left(1-\frac{\gamma}{2},\frac{1}{2}\right), & \left(\frac{1-\gamma}{2},\frac{1}{2}\right)\\ \left(\frac{1+\delta}{2},\frac{1}{2}\right), & \left(-\frac{\delta}{2},\frac{1}{2}\right)\end{matrix}\right]t^{-\gamma}f(t)dt.
$$

5.4.3 $\mathfrak{L}_{\overline{\nu},\,\overline{2}}$-theory of the transforms $\mathrm{P}^{\gamma}_{\delta,k}f$ $(k=1,2)$

With respect to $\mathfrak{L}_{\nu,\,\overline{2}}$-theory of the transforms (5.74) and (5.75), using Theorem 65, for the $\mathscr{H}^{1}_{\sigma,\kappa}$-transform we have $a_1^*=a_2^*=\ldots=a_n^*=0$, $\Delta_1=\Delta_2=\ldots=\Delta_n=0$, $p=(p_1,p_2,\ldots,p_n)=(2,2,\ldots,2)$, $q=(q_1,q_2,\ldots,q_n)=(2,2,\ldots,2)$, $\alpha_i=(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_n})=(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$, $\beta_j=(\beta_{j1},\beta_{j2},\ldots,\beta_{jn})=(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$ $(i=1,\ldots,p,\ j=1,\ldots,q)$, $\mu=\gamma-1$.

As for $\mathbf{m}$, $\mathbf{n}$, and other parameters we obtain

$$
\begin{aligned}
&\mathbf{m}=0,\ n=2,\ \alpha=-\infty,\ \beta=\min[\mathrm{Re}(1+\gamma+\delta),\ \mathrm{Re}(\gamma-\delta)],\\
&\mathbf{n}=0,\ n=2,\ \alpha=-\infty,\ \beta=\mathrm{Re}(\gamma),
\end{aligned}
$$

respectively, for the operators (5.74) and (5.75).

The number $1-\overline{\nu}$ does not belong to the exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of the $\overline{\mathcal{H}}^{0,2}_{2,2}$-function if

$$
s\neq 2m+1,\ s\neq 2l+2\,(l=(l_1,l_2,\ldots,l_n)),m=(\overline{m}_1,\overline{m}_2,\ldots,\overline{m}_n)\in\mathbb{N}_0^n),
$$

for $\mathrm{Re}(s)=1-\overline{\nu}$.

From the other side, $1-\overline{\nu}$ does not belong to the exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of the $\overline{\mathcal{H}}^{2,0}_{2,2}$-function if

$$s \neq -\delta+2m+1,\; s\neq \delta+2l+2\,(l=(l_1,l_2,...,l_n),$$
$$m=(\overline{m}_1,\overline{m}_2,...,\overline{m}_n)\in\mathbb{N}_0^n),$$

for $\mathrm{R}e(s)=1-\overline{\nu}$.

From Theorem 65 we deduce the $\mathfrak{L}_{\overline{\nu},\,\overline{2}}$-theory of the transforms $\mathrm{P}^{\gamma}_{\delta,k}f$ $(k=1,2)$.

Theorem 67. *Let*

$$\begin{aligned}
&-\infty<\nu_1-\mathrm{Re}(1-\gamma_1)<\min[\mathrm{Re}(1+\gamma_1+\delta_1),\;\mathrm{Re}(\gamma_1-\delta_1)],\\
&\quad\mathrm{Re}(\gamma_1-1)\leq 0,\\
&-\infty<\nu_2-\mathrm{Re}(1-\gamma_2)<\min[\mathrm{Re}(1+\gamma_2+\delta_2),\;\mathrm{Re}(\gamma_2-\delta_2)],\\
&\quad\mathrm{Re}(\gamma_2-1)\leq 0,...,\\
&-\infty<\nu_n-\mathrm{Re}(1-\gamma_n)<\min[\mathrm{Re}(1+\gamma_n+\delta_n),\;\mathrm{Re}(\gamma_n-\delta_n)],\\
&\quad\mathrm{Re}(\gamma_n-1)\leq 0.
\end{aligned}$$

The following assertions hold:

(a) There exists a one-to-one map $\mathrm{P}^{\gamma}_{\delta,1}\in[\mathfrak{L}_{\overline{\nu},\overline{2}},\;\mathfrak{L}_{\overline{\nu}-\mathrm{Re}(1-\gamma),\overline{2}}]$ *such that the relation* (5.74) *holds for* $f\in\mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(s)=\overline{\nu}-\mathrm{Re}(1-\gamma)$. *If* $\mathrm{Re}(\gamma-1)=0$*, then* $\mathrm{P}^{\gamma}_{\delta,1}$ *is one-to-one on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$.

(b) The transform $\mathrm{P}^{\gamma}_{\delta,1}f$ *does not depend on* $\overline{\nu}$ *in the sense that if* $\overline{\nu}_1$ *and if the transforms* $\mathrm{P}^{\gamma}_{\delta,1}f$ *and* $\widetilde{\mathrm{P}}^{\gamma}_{\delta,1}f$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu}_1,\overline{2}}$ *and* $\mathfrak{L}_{\overline{\nu}_2,\overline{2}}$ *by relation* (5.74)*, then* $\mathrm{P}^{\gamma}_{\delta,1}f=\widetilde{\mathrm{P}}^{\gamma}_{\delta,1}f$ *for* $f\in\mathfrak{L}_{\overline{\nu}_1,\overline{2}}\bigcap\mathfrak{L}_{\overline{\nu}_2,\overline{2}}$.

(c) If $\mathrm{Re}(\gamma-1)<0$*, then for* $f\in\mathfrak{L}_{\overline{\nu},\overline{2}}$, $\mathrm{P}^{\gamma}_{\delta,1}f$ *is given by* (5.74).

(d) Let $\overline{\lambda}=(\lambda_1,\lambda_2,...,\lambda_n)\in\mathbb{C}^n$, $\overline{h}=(h_1,...,h_n)>0$*, and* $f\in\mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $\mathrm{Re}(\overline{\lambda})>(\overline{\nu}-\mathrm{Re}(1-\gamma))\overline{h}-1$*, then* $\mathrm{P}^{\gamma}_{\delta,1}f$ *is represented in the form*

$$\left(\mathrm{P}^{\gamma}_{\delta,1}f\right)(x)=2^{\gamma-1}\overline{h}x^{1-(\overline{\lambda}+1)/\overline{h}}\frac{d}{dx}x^{(\overline{\lambda}+1)/\overline{h}}$$
$$\times\int\limits_0^{\infty}\mathrm{H}^{0,3}_{3,3}\left[\frac{x}{t}\left|\begin{matrix}(-\overline{\lambda},h),\left(\frac{1-\gamma-\delta}{2},\frac{1}{2}\right),\;\left(1+\frac{\delta-\gamma}{2},\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right),\;\left(\frac{1}{2},\frac{1}{2}\right),(-\overline{\lambda}-1,\overline{h})\end{matrix}\right.\right]t^{-\gamma}f(t)dt,$$

while for $\mathrm{Re}(\overline{\lambda})<(\overline{\nu}-\mathrm{Re}(1-\gamma))\overline{h}-1$ *it is given by*

$$\left(\mathrm{P}^{\gamma}_{\delta,1}f\right)(x)=-2^{\gamma-1}\overline{h}x^{1-(\overline{\lambda}+1)/\overline{h}}\frac{d}{dx}x^{(\overline{\lambda}+1)/\overline{h}}$$
$$\times\int\limits_0^{\infty}\mathrm{H}^{1,2}_{3,3}\left[\frac{x}{t}\left|\begin{matrix}\left(\frac{1-\gamma-\delta}{2},\frac{1}{2}\right),\;\left(1+\frac{\delta-\gamma}{2},\frac{1}{2}\right),(-\overline{\lambda},\overline{h})\\ (-\overline{\lambda}-1,\overline{h}),\left(0,\frac{1}{2}\right),\;\left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right.\right]t^{-\gamma}f(t)dt.$$

(e) If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(1-\gamma),\overline{2}}$*, then the following relation holds:*

$$\int\limits_0^\infty f(x)\big(\mathrm{P}^\gamma_{\delta,1}g\big)(x)dx = \int\limits_0^\infty 2^{\gamma-1}\big(\mathrm{P}^{*\gamma}_{\delta,2}f\big)(x)g(x)dx,$$

where $\big(\mathrm{P}^{*\gamma}_{\delta,2}f\big)(x)$ *is the transform*

$$\big(\mathrm{P}^{*\gamma}_{\delta,2}f\big)(x) = \int\limits_x^\infty \big(t^2-x^2\big)^{-\gamma/2}\mathrm{P}^\gamma_\delta\left(\frac{t}{x}\right)f(t)dt = g(x) \ \ (x>0).$$

Theorem 68. *Let*

$$\begin{aligned}
&-\infty < \nu_1 - \mathrm{Re}(1-\gamma_1) < \mathrm{Re}(\gamma_1),\ \mathrm{Re}(\gamma_1-1) \le 0,\\
&-\infty < \nu_2 - \mathrm{Re}(1-\gamma_2) < \mathrm{Re}(\gamma_2),\ \mathrm{Re}(\gamma_2-1) \le 0, ...,\\
&-\infty < \nu_n - \mathrm{Re}(1-\gamma_n) < \mathrm{Re}(\gamma_n),\ \mathrm{Re}(\gamma_n-1) \le 0.
\end{aligned}$$

The following assertions hold:

(a) There exists a one-to-one map $\mathrm{P}^\gamma_{\delta,2} \in [\mathfrak{L}_{\overline{\nu},\overline{2}},\ \mathfrak{L}_{\overline{\nu}-\mathrm{Re}(1-\gamma),2}]$ *such that the relation* (5.75) *holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(s) = \overline{\nu} - \mathrm{Re}(1-\gamma)$*. If* $\mathrm{Re}(\gamma-1) = 0$*, then* $\mathrm{P}^\gamma_{\delta,2}$ *is one-to-one on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$*.*

(b) The transform $\mathrm{P}^\gamma_{\delta,2}f$ *does not depend on* $\overline{\nu}$ *in the sense that if* $\overline{\nu}_1$ *and if the transforms* $\mathrm{P}^\gamma_{\delta,2}f$ *and* $\widetilde{\mathrm{P}}^\gamma_{\delta,2}f$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu}_1,\overline{2}}$*, then* $\mathrm{P}^\gamma_{\delta,2}f = \widetilde{\mathrm{P}}^\gamma_{\delta,2}f$ *for* $f \in \mathfrak{L}_{\overline{\nu}_1,\overline{2}} \bigcap \mathfrak{L}_{\overline{\nu}_2,\overline{2}}$*.*

(c) If $\mathrm{Re}(\gamma-1) < 0$*, then for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$*,* $\mathrm{P}^\gamma_{\delta,2}f$ *is given by* (5.75)*.*

(d) Let $\overline{\lambda} \in \mathrm{C}^n$*,* $\overline{h} > 0$*, and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$*. If* $\mathrm{Re}(\overline{\lambda}) > (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$*, then* $\mathrm{P}^\gamma_{\delta,2}f$ *is represented in the form*

$$\begin{aligned}
\big(\mathrm{P}^\gamma_{\delta,2}f\big)(x) &= 2^{\gamma-1}\overline{h}x^{1-(\overline{\lambda}+1)/\overline{h}}\frac{d}{dx}x^{(\overline{\lambda}+1)/\overline{h}}\\
&\times \int\limits_0^\infty \mathrm{H}^{0,3}_{3,3}\left[\frac{x}{t}\,\middle|\, \begin{matrix} (-\overline{\lambda},\overline{h}),\left(1-\frac{\gamma}{2},\frac12\right),\ \left(\frac{1-\gamma}{2},\frac12\right)\\ \left(\frac{1+\delta}{2},\frac12\right),\ \left(-\frac{\delta}{2},\frac12\right),(-\overline{\lambda}-1,\overline{h})\end{matrix}\right]t^{-\gamma}f(t)dt,
\end{aligned}$$

while for $\mathrm{Re}(\overline{\lambda}) < (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$ *it is given by*

$$\begin{aligned}
\big(\mathrm{P}^\gamma_{\delta,2}f\big)(x) &= -2^{\gamma-1}\overline{h}xalign^{1-(\overline{\lambda}+1)/\overline{h}}\frac{x}{dx}x^{(\overline{\lambda}+1)/\overline{h}}\\
&\times \int\limits_0^\infty \mathrm{H}^{1,2}_{3,3}\left[\frac{x}{t}\,\middle|\, \begin{matrix} \left(1-\frac{\gamma}{2},\frac12\right),\ \left(\frac{1-\gamma}{2},\frac12\right),(-\overline{\lambda},\overline{h})\\ (-\overline{\lambda}-1,\overline{h}),\left(\frac{1+\delta}{2},\frac12\right),\ \left(-\frac{\delta}{2},\frac12\right)\end{matrix}\right]t^{-\gamma}f(t)dt.
\end{aligned}$$

(e) If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ and $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(1-\gamma),\overline{2}}$, then the following relation holds:

$$\int_0^\infty f(x)\big(\mathrm{P}^{\gamma}_{\delta,2}g\big)(x)dx = \int_0^\infty 2^{\gamma-1}\big(\mathrm{P}^{*\gamma}_{\delta,2}f\big)(x)g(x)dx,$$

where $\big(\mathrm{P}^{*\gamma}_{\delta,2}f\big)$ is given by

$$\big(\mathrm{P}^{*\gamma}_{\delta,2}f\big)(x) = \int_x^\infty \big(t^2-x^2\big)^{-\gamma/2}\mathrm{P}^{\gamma}_{\delta}\left(\frac{x}{t}\right)f(t)dt = g(x) \ \ (x>0).$$

5.4.4 Inversion formulas for transforms $\mathrm{P}^{\gamma}_{\delta,k}f \ \ (k=1,2)$

Let

$$\alpha_0 = 0, \ \beta_0 = \infty,$$
$$\alpha_0 = 1+\max[\mathrm{Re}(\delta-1), \ \mathrm{Re}(-\delta-2)], \ \beta_0 = \infty,$$

respectively, for the operators (5.74) and (5.75).

Inversion formulas for $\mathrm{P}^{\gamma}_{\delta,1}f$ take the following forms:

$$f(x) = -2^{1-\gamma}\overline{h}x^{(\overline{\lambda}+1)/\overline{h}-1+\gamma}\frac{d}{dx}x^{-(\overline{\lambda}+1)/\overline{h}} \times \int_0^\infty \mathrm{H}^{2,1}_{3,3}\left[\frac{t}{x}\middle| \begin{matrix} -(\overline{\lambda},\overline{h}), \left(\frac{\gamma+\delta}{2},\frac{1}{2}\right), \ \left(\frac{\gamma-\delta-1}{2},\frac{1}{2}\right) \\ \left(\frac{1}{2},\frac{1}{2}\right), \ \left(0,\frac{1}{2}\right), (-\overline{\lambda}-1,\overline{h}) \end{matrix}\right]\big(\mathrm{P}^{\gamma}_{\delta,1}f\big)(t)dt \tag{5.94}$$

or

$$f(x) = 2^{1-\gamma}\overline{h}x^{(\overline{\lambda}+1)/\overline{h}-1}\frac{d}{dx}x^{-(\overline{\lambda}+1)/\overline{h}} \times \int_0^\infty \mathrm{H}^{3,0}_{3,3}\left[\frac{t}{x}\middle| \begin{matrix} \left(\frac{\gamma+\delta}{2},\frac{1}{2}\right), \ \left(\frac{\gamma-\delta-1}{2},\frac{1}{2}\right), (-\overline{\lambda},\overline{h}) \\ (-\overline{\lambda}-1,\overline{h}), \left(\frac{1}{2},\frac{1}{2}\right), \ \left(0,\frac{1}{2}\right) \end{matrix}\right]\big(\mathrm{P}^{\gamma}_{\delta,1}f\big)(t)dt. \tag{5.95}$$

Inversion formulas for $\mathrm{P}^{\gamma}_{\delta,4}f$ take the following forms:

$$f(x) = -2^{1-\gamma}\overline{h}x^{(\overline{\lambda}+1)/\overline{h}-1+\gamma}\frac{d}{dx}x^{-(\overline{\lambda}+1)/\overline{h}} \times \int_0^\infty \mathrm{H}^{2,1}_{3,3}\left[\frac{t}{x}\middle| \begin{matrix} (-\overline{\lambda},\overline{h}), \left(\frac{\gamma-1}{2},\frac{1}{2}\right), \ \left(\frac{\gamma}{2},\frac{1}{2}\right) \\ \left(-\frac{\delta}{2},\frac{1}{2}\right), \ \left(\frac{\delta+1}{2},\frac{1}{2}\right), (-\overline{\lambda}-1,\overline{h}) \end{matrix}\right]\big(\mathrm{P}^{\gamma}_{\delta,4}f\big)(t)dt \tag{5.96}$$

or

$$f(x) = 2^{1-\gamma}\overline{h}x^{(\overline{\lambda}+1)/\overline{h}-1}\frac{d}{dx}x^{-(\overline{\lambda}+1)/\overline{h}}$$
$$\times\int_0^\infty \mathrm{H}^{3,0}_{3,3}\left[\frac{t}{x}\,\middle|\,\begin{matrix}\left(\frac{\gamma-1}{2},\frac{1}{2}\right), & \left(\frac{\gamma}{2},\frac{1}{2}\right), (-\overline{\lambda},\overline{h}) & \\ (-\overline{\lambda}-1,\overline{h}), \left(-\frac{\delta}{2},\frac{1}{2}\right), & \left(\frac{\delta+1}{2},\frac{1}{2}\right)\end{matrix}\right]\left(\mathrm{P}^{\gamma}_{\delta,4}f\right)(t)dt. \tag{5.97}$$

Theorem 69. *Let* $\mathrm{Re}(\gamma) = 1$, $-\infty < \overline{\nu} < \min[1, \mathrm{Re}(2+\delta), \mathrm{Re}(1-\delta)]$, *and let* $\overline{\lambda} \in \mathrm{C}^n$, $\overline{h} > 0$.

If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, *then the inversion formulas* (5.94) *and* (5.95) *are valid for* $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu})\overline{h} - 1$, *respectively.*

Theorem 70. *Let* $\mathrm{Re}(\gamma) = 1$, $-\infty < \overline{\nu} < \min[1, \mathrm{Re}(1-\delta), \mathrm{Re}(2+\delta)]$, *and let* $\overline{\lambda} \in \mathrm{C}^n$, $\overline{h} > 0$.

If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, *then the inversion formulas* (5.96) *and* (5.97) *are valid for* $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu})\overline{h} - 1$, *respectively.*

Integral transforms composition method for transmutations

6

In this chapter we study applications of the **integral transforms composition method** (ITCM) for obtaining transmutations via integral transforms. It is possible to derive a wide range of transmutation operators by this method. Classical integral transforms are involved in the ITCM as basic blocks; among them are Fourier, sine- and cosine-Fourier, Hankel, Mellin, Laplace, and some generalized transforms. The ITCM and transmutations obtained by it are applied to derive connection formulas for solutions of singular differential equations and more simple nonsingular ones. We consider well-known classes of singular differential equations with Bessel operators, such as the classical and generalized Euler–Poisson–Darboux equations and the generalized radiation problem of A. Weinstein. Methods of this chapter are applied to more general linear partial differential equations with Bessel operators, such as multivariate Bessel type equations, generalized axially symmetric potential theory (GASPT) equations of A. Weinstein, Bessel type generalized wave equations with variable coefficients, B-ultrahyperbolic equations, and others. So with many results and examples the main conclusion of this chapter is illustrated: ***the ITCM of constructing transmutations is a very important and effective tool also for obtaining connection formulas and explicit representations of solutions to a wide class of singular differential equations, including ones with Bessel operators.***

6.1 Basic ideas and definitions of the integral transforms composition method for the study of transmutations

To construct transmutation operators, the ITCM, introduced and thoroughly developed in [146,229,230,234,524,533,535], can be used. The essence of this method is to construct the necessary transmutation operator and corresponding connection formulas among solutions of perturbed and nonperturbed equations as a composition of classical integral transforms with properly chosen weighted functions.

6.1.1 Background of ITCM

We note that other possible generalizations of considered equations are equations with fractional powers of the Bessel operator considered in [89,252,367,515,516,527,531, 555]. In fractional differential equations theory the so-called "principle of subordination" was proposed (cf. [12,13,118,458]). In the cited literature the principle of subordination is reduced to formulas relating the solutions to equations of various fractional orders. Special cases of the subordination principle are formulas connecting

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solutions of fractional differential equations to solutions of integer order equations. Such formulas are also in fact parameter shift formulas, in which the parameter is the order of the fractional differential equation. So the popular "principle of subordination" may be considered as an example of parameter shift formulas, and consequently is in close connection with transmutation theory and the ITCM developed here.

Note that we specially restrict ourselves to linear problems, but of course nonlinear problems are also very important (cf. [469,470] for further references).

In transmutation theory explicit operators have been derived based on different ideas and methods, often not connecting altogether. So there is an urgent need in transmutation theory to develop a general method for obtaining known and new classes of transmutations.

We give such general method for constructing transmutation operators. It is the ***integral transforms composition method*** (ITCM). The method is based on the representation of transmutation operators as compositions of basic integral transforms. The ITCM gives the algorithm not only for constructing new transmutation operators, but also for all now explicitly known classes of transmutations, including Poisson, Sonine, Vekua–Erdélyi–Lowndes, Buschman–Erdélyi, Sonin–Katrakhov, and Poisson–Katrakhov ones (cf. [51–53,56,229,230,524,533,535] as well as the classes of elliptic, hyperbolic, and parabolic transmutation operators introduced by R. Carroll [51–53]).

6.1.2 What is ITCM and how to use it?

The formal algorithm of ITCM is the following. Let us take as input a pair of arbitrary operators A, B, and also connecting with them generalized Fourier transforms F_A, F_B, which are invertible and act by the formulas

$$F_A A = g(t) F_A, \quad F_B B = g(t) F_B, \tag{6.1}$$

where t is a dual variable and g is an arbitrary function with suitable properties. It is often convenient to choose $g(t) = -t^2$ or $g(t) = -t^\alpha$, $\alpha \in \mathbb{R}$.

Then the essence of the ITCM is to obtain formally a pair of transmutation operators P ***and*** S ***as the method output by the following formulas:***

$$S = F_B^{-1} \frac{1}{w(t)} F_A, \qquad P = F_A^{-1} w(t) F_B, \tag{6.2}$$

with arbitrary function $w(t)$***. When*** P ***and*** S ***are transmutation operators intertwining*** A ***and*** B,

$$SA = BS, \qquad PB = AP. \tag{6.3}$$

A formal checking of (6.3) can be obtained by direct substitution. The main difficulty is the calculation of compositions (6.2) in an explicit integral form, as well as the choice of domains of operators P and S. Also, we should note that the formulas in (6.2) are formal and the situation is possible when one operator, for example P, exists and is generated by the formula $P = F_A^{-1} w(t) F_B$, but its inverse operator S cannot

be constructed by the formula $F_B^{-1}\frac{1}{w(t)}F_A$ since this integral, for example, diverges. In this case if it is needed to construct an inverse operator for P it is necessary to use some regularization methods.

Let us list the main advantages of the ITCM.

- Simplicity – many classes of transmutations are obtained by explicit formulas from elementary basic blocks, which are classical integral transforms.
- The ITCM gives by a unified approach all previously explicitly known classes of transmutations.
- The ITCM gives by a unified approach many new classes of transmutations for different operators.
- The ITCM gives a unified approach to obtain both direct and inverse transmutations in the same composition form.
- The ITCM directly leads to estimates of norms of direct and inverse transmutations using known norm estimates for classical integral transforms on different functional spaces.
- The ITCM directly leads to connection formulas for solutions to perturbed and unperturbed differential equations.

Some obstacle to apply ITCM is the following one. We know classical integral transforms usually act on standard spaces like L_2, L_p, C^k, variable exponent Lebesgue spaces [465], and so on. But for the application of transmutations to differential equations we usually need some more conditions to hold, say, at zero or at infinity. For these problems we may first construct a transmutation by the ITCM and then expand it to the needed functional classes.

Let us stress that formulas of the type (6.2) of course are not new for integral transforms and their applications to differential equations. ***But the ITCM is new when applied to transmutation theory!*** In other fields of integral transforms and connected differential equations theory compositions (6.2) for the choice of the classical Fourier transform leads to famous pseudodifferential operators with symbol function $w(t)$. For the choice of the classical Fourier transform and the function $w(t)=(\pm it)^{-s}$ we get fractional integrals on the whole real axis, for $w(t)=|x|^{-s}$ we get the Riesz potential, for $w(t)=(1+t^2)^{-s}$ in (6.2) we get the Bessel potential, and for $w(t)=(1\pm it)^{-s}$ we obtain modified Bessel potentials [494].

The choice for the ITCM algorithm

$$A=B=B_\nu,\, F_A=F_B=H_\nu,\, g(t)=-t^2,\, w(t)=j_\nu(st) \tag{6.4}$$

leads to generalized translation operators of Delsart [315,319,321]. For this case we have to choose in the ITCM algorithm defined by (6.1)–(6.2) the above values (6.4) in which B_ν is the Bessel operator (1.87), F_ν is the Hankel transform (1.56), and j_ν is the normalized (or "small") Bessel function (1.19). In the same manner other families of operators commuting with a given one may be obtained by the ITCM for the choice $A=B$, $F_A=F_B$ with arbitrary functions $g(t), w(t)$ (generalized translation commutes with the Bessel operator). In the case of the choice of differential operator A as quantum oscillator and the connected integral transform F_A as fractional or quadratic Fourier transform [437], we may obtain by the ITCM transmutations also

for this case [230]. It is possible to apply the ITCM instead of classical approaches for obtaining fractional powers of Bessel operators [230,515,516,527,531].

Direct applications of the ITCM to multi-dimensional differential operators are obvious; in this case t is a vector and $g(t), w(t)$ are vector functions in (6.1)–(6.2). Unfortunately for this case we know and may derive some new explicit transmutations just for simple special cases. But among them are well-known and interesting classes of potentials. In the case of using the ITCM by (6.1)–(6.2) with Fourier transform when $w(t)$ is a positive definite quadratic form, we come to elliptic Riesz potentials [475,494]; when $w(t)$ is an indefinite quadratic form we come to hyperbolic Riesz potentials [426,475,494]; when $w(x,t) = (|x|^2 - it)^{-\alpha/2}$ we come to parabolic potentials [494]. In the case of using the ITCM by (6.1)–(6.2) with Hankel transform and when $w(t)$ is a quadratic form we come to elliptic Riesz B-potentials [206,344] or hyperbolic Riesz B-potentials [503]. For all abovementioned potentials we need to use distribution theory and consider for the ITCM convolutions of distributions; for inversion of such potentials we need some cutting and approximation procedures (cf. [426,503]). For this class of problems it is appropriate to use Schwartz and/or Lizorkin spaces for probe functions and dual spaces for distributions.

So we may conclude that the ITCM we consider in this chapter for obtaining transmutations is effective, it is connected to many known methods and problems, it gives all known classes of explicit transmutations, and it works as a tool to construct new classes of transmutations. Application of the ITCM requires the following three steps.

- Step 1. For a given pair of operators A, B and connected integral transforms F_A, F_B, define and calculate a pair of transmutations P, S by basic formulas (6.1)–(6.2).
- Step 2. Derive exact conditions and find classes of functions for which transmutations obtained by step 1 satisfy proper intertwining properties.
- Step 3. Apply now correctly defined transmutations by steps 1 and 2 on proper classes of functions to derive connection formulas for solutions of differential equations.

Based on this plan the next part of the chapter is organized as follows. First we illustrate step 1 of the above plan and apply the ITCM for obtaining some new and known transmutations. For step 2 we prove a general theorem for the case of Bessel operators; it is enough to complete strict definitions of necessary transmutations and start to solve problems using them. After that we give an example to illustrate step 3 of applying transmutations obtained by ITCM to derive formulas for solutions of a model differential equation.

6.2 Application of the ITCM to derive transmutations connected with the Bessel operator

The topic of this section is the application of the ITCM to obtain different classes of transmutations connected with the Bessel operator.

6.2.1 Index shift for the Bessel operator

Here we derive connection formulas of Bessel operators with the indices μ and ν. Such relations are called **parameter shift formulas** and such operator is called **index shift transmutation**. Aforesaid formulas arise when the classical wave equation is solved by the mean values method. The descent parameter in this case is the space dimension. Essentially, such parameter shift formulas define transmutation operators which are responsible for connection formulas among solutions of perturbed and nonperturbed equations. In this subsection we apply the ITCM to obtain integral representations for index shift transmutations. This corresponds to step 1 of the above plan for the ITCM algorithm.

Let us look at the operator T transmuting the operator B_ν defined by (1.87) into the same operator but with another parameter B_μ. To find such a transmutation we use the ITCM with Hankel transform. Applying the ITCM we obtain an interesting and important family of transmutations, including index shift transmutations, "descent" operators, classical Sonine and Poisson type transmutations, explicit integral representations for fractional powers of the Bessel operator, generalized translations of Delsart, and others.

So we are looking for an operator $T_{\nu,\mu}^{(\varphi)}$ such that

$$T_{\nu,\mu}^{(\varphi)} B_\nu = B_\mu T_{\nu,\mu}^{(\varphi)}, \tag{6.5}$$

or in the form factorized by the ITCM,

$$T_{\nu,\mu}^{(\varphi)} = H_\mu^{-1}\Big(\varphi(t) H_\nu\Big), \tag{6.6}$$

where H_ν is a Hankel transform (1.59). Assuming $\varphi(t) = Ct^\alpha$, $C \in \mathbb{R}$ does not depend on t, and $T_{\nu,\mu}^{(\varphi)} = T_{\nu,\mu}^{(\alpha)}$, we derive the following theorem.

Theorem 71. *Let $f \in L^2(0,\infty)$,*

$$\operatorname{Re}(\alpha+\mu+1) > 0, \qquad \operatorname{Re}\left(\alpha + \frac{\mu-\nu}{2}\right) < 0.$$

Then for the transmutation operator $T_{\nu,\mu}^{(\alpha)}$ obtained by the ITCM and such that

$$T_{\nu,\mu}^{(\alpha)} B_\nu = B_\mu T_{\nu,\mu}^{(\alpha)},$$

the following integral representation is true:

$$\left(T_{\nu,\mu}^{(\alpha)} f\right)(x) = C \cdot \frac{2^{\alpha+3}\Gamma\left(\frac{\alpha+\mu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)} \times$$

$$\left[\frac{x^{-1-\mu-\alpha}}{\Gamma\left(-\frac{\alpha}{2}\right)} \int\limits_0^x f(y)\, {}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\right) y^\nu dy + \right.$$

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu-\alpha}{2}\right)}\int\limits_x^\infty f(y)\times$$

$${}_2F_1\left(\frac{\alpha+\mu+1}{2},\frac{\alpha+\mu-\nu}{2}+1;\frac{\mu+1}{2};\frac{x^2}{y^2}\right)y^{\nu-\mu-\alpha-1}dy\Bigg], \tag{6.7}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. We have

$$\left(T^{(\alpha)}_{\nu,\,\mu}f\right)(x)=C\cdot H_\mu^{-1}\left[t^\alpha H_\nu[f](t)\right](x)=$$

$$C\cdot\frac{2^{1-\mu}}{\Gamma^2\left(\frac{\mu+1}{2}\right)}\int\limits_0^\infty j_{\frac{\mu-1}{2}}(xt)\,t^{\mu+\alpha}\,dt\int\limits_0^\infty j_{\frac{\nu-1}{2}}(ty)f(y)y^\nu dy=$$

$$C\cdot\frac{2^{\frac{\nu-\mu}{2}+2}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}\int\limits_0^\infty(xt)^{\frac{1-\mu}{2}}J_{\frac{\mu-1}{2}}(xt)t^{\mu+\alpha}\,dt\int\limits_0^\infty(ty)^{\frac{1-\nu}{2}}J_{\frac{\nu-1}{2}}(ty)f(y)y^\nu dy=$$

$$C\cdot\frac{2^{\frac{\nu-\mu}{2}+2}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}x^{\frac{1-\mu}{2}}\int\limits_0^\infty y^{\frac{\nu+1}{2}}f(y)dy\int\limits_0^\infty t^{\alpha+1+\frac{\mu-\nu}{2}}J_{\frac{\mu-1}{2}}(xt)J_{\frac{\nu-1}{2}}(ty)dt=$$

$$C\cdot\frac{2^{\frac{\nu-\mu}{2}+2}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}x^{\frac{1-\mu}{2}}\int\limits_0^x y^{\frac{\nu+1}{2}}f(y)dy\int\limits_0^\infty t^{\alpha+1+\frac{\mu-\nu}{2}}J_{\frac{\mu-1}{2}}(xt)J_{\frac{\nu-1}{2}}(ty)dt+$$

$$C\cdot\frac{2^{\frac{\nu-\mu}{2}+2}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)}x^{\frac{1-\mu}{2}}\int\limits_x^\infty y^{\frac{\nu+1}{2}}f(y)dy\int\limits_0^\infty t^{\alpha+1+\frac{\mu-\nu}{2}}J_{\frac{\mu-1}{2}}(xt)J_{\frac{\nu-1}{2}}(ty)dt.$$

Using formula (2.12.31.1) from [456], p. 209, of the form

$$\int\limits_0^\infty t^{\beta-1}J_\rho(xt)J_\gamma(yt)\,dt=$$

$$\begin{cases}2^{\beta-1}x^{-\gamma-\beta}y^\gamma\dfrac{\Gamma\left(\frac{\gamma+\rho+\beta}{2}\right)}{\Gamma(\gamma+1)\Gamma\left(\frac{\rho-\gamma-\beta}{2}+1\right)}{}_2F_1\left(\frac{\gamma+\rho+\beta}{2},\frac{\gamma-\rho+\beta}{2};\gamma+1;\frac{y^2}{x^2}\right) & 0<y<x,\\ 2^{\beta-1}x^\rho y^{-\rho-\beta}\dfrac{\Gamma\left(\frac{\gamma+\rho+\beta}{2}\right)}{\Gamma(\rho+1)\Gamma\left(\frac{\gamma-\rho-\beta}{2}+1\right)}{}_2F_1\left(\frac{\gamma+\rho+\beta}{2},\frac{\beta+\rho-\gamma}{2};\rho+1;\frac{x^2}{y^2}\right) & 0<x<y,\end{cases}$$

$$x,y,\operatorname{Re}(\beta+\rho+\gamma)>0,\ \operatorname{Re}\beta<2$$

and putting $\beta = \alpha + \frac{\mu-\nu}{2} + 2$, $\rho = \frac{\mu-1}{2}$, $\gamma = \frac{\nu-1}{2}$ we obtain the formula (6.7). We have

$$\int_0^\infty t^{\alpha+1+\frac{\mu-\nu}{2}} J_{\frac{\mu-1}{2}}(xt) J_{\frac{\nu-1}{2}}(ty)dt =$$

$$\begin{cases} \dfrac{2^{\alpha+1+\frac{\mu-\nu}{2}} y^{\frac{\nu-1}{2}}}{x^{\alpha+2-\frac{1-\mu}{2}}} \dfrac{\Gamma\left(\frac{\alpha+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)} {}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\right) & 0 < y < x, \\ \dfrac{2^{\alpha+1+\frac{\mu-\nu}{2}} x^{\frac{\mu-1}{2}}}{y^{\mu+\alpha-\frac{\nu-3}{2}}} \dfrac{\Gamma\left(\frac{\alpha+\mu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu-\alpha}{2}\right)} {}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha+\mu-\nu}{2}+1; \frac{\mu+1}{2}; \frac{x^2}{y^2}\right) & 0 < x < y, \end{cases}$$

$$\operatorname{Re}(\alpha+\mu+1) > 0, \qquad \operatorname{Re}\left(\alpha + \frac{\mu-\nu}{2}\right) < 0$$

and

$$\left(T_{\nu,\mu}^{(\alpha)} f\right)(x) =$$

$$C \cdot \frac{2^{\alpha+3}\Gamma\left(\frac{\alpha+\mu+1}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)} x^{-1-\mu-\alpha} \int_0^x f(y) {}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\right) y^\nu dy +$$

$$C \cdot \frac{2^{\alpha+3}\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\alpha+\mu+1}{2}\right)}{\Gamma^2\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu-\alpha}{2}\right)} \int_x^\infty f(y) {}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha+\mu-\nu}{2}+1; \frac{\mu+1}{2}; \frac{x^2}{y^2}\right) \times$$

$$y^{\nu-\mu-\alpha-1} dy.$$

This completes the proof. □

Constant C in (6.7) should be chosen based on convenience. Very often it is reasonable to choose this constant so that $T_{\nu,\mu}^{(\alpha)} 1 = 1$.

Using formula

$$ {}_2F_1(a, b; b; z) = (1-z)^{-a} \tag{6.8}$$

we give several useful transmutation operators that are special cases of operator (6.7). In Section 6.3 we will use these operators to find the solutions to the perturbed wave equations.

6.2.2 Poisson and "descent" operators, negative fractional power of the Bessel operator

Here we give important consequences of Theorem 71.

Statement 14. *Let $f \in L^2(0,\infty)$, $\alpha = -\mu$, $\nu = 0$. In this case for $\mu > 0$ we obtain the operator*

$$\left(T_{0,\,\mu}^{(-\mu)} f\right)(x) = \frac{2\Gamma\left(\frac{\mu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\mu}{2}\right)} x^{1-\mu} \int\limits_0^x f(y)(x^2 - y^2)^{\frac{\mu}{2}-1} dy, \tag{6.9}$$

such that

$$T_{0,\,\mu}^{(-\mu)} D^2 = B_\mu T_{0,\,\mu}^{(-\mu)} \tag{6.10}$$

and $T_{0,\,\mu}^{(-\mu)}\, 1 = 1$*,*

Proof. From Theorem 71 we obtain

$$\left(T_{0,\,\mu}^{(-\mu)} f\right)(x) = C \cdot \frac{2^{3-\mu}\sqrt{\pi}}{x\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)} \int\limits_0^x f(y)\,{}_2F_1\left(\frac{1}{2}, 1 - \frac{\mu}{2}; \frac{1}{2}; \frac{y^2}{x^2}\right) dy.$$

Using formula (6.8) we get

$${}_2F_1\left(\frac{1}{2}, 1 - \frac{\mu}{2}; \frac{1}{2}; \frac{y^2}{x^2}\right) = \left(1 - \frac{y^2}{x^2}\right)^{\frac{\mu}{2}-1} = x^{2-\mu}(x^2 - y^2)^{\frac{\mu}{2}-1}$$

and

$$\left(T_{0,\,\mu}^{(-\mu)} f\right)(x) = C \cdot \frac{x^{1-\mu} 2^{3-\mu}\sqrt{\pi}}{\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)} \int\limits_0^x f(y)(x^2 - y^2)^{\frac{\mu}{2}-1} dy.$$

It is easy to see that

$$x^{1-\mu} \int\limits_0^x (x^2 - y^2)^{\frac{\mu}{2}-1} dy = \{y = xz\} = \int\limits_0^1 (1 - z^2)^{\frac{\mu}{2}-1} dz =$$

$$\{z^2 = t\} = \frac{1}{2}\int\limits_0^1 (1-t)^{\frac{\mu}{2}-1} t^{-\frac{1}{2}} dt = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\mu}{2}\right)}{2\Gamma\left(\frac{\mu+1}{2}\right)}$$

and taking $C = \frac{\Gamma^2\left(\frac{\mu+1}{2}\right)}{2^{2-\mu}\pi}$ we get $T_{0,\,\mu}^{(-\mu)} 1 = 1$. This completes the proof. □

The operator (6.9) is the well-known *Poisson operator* (3.120). We will use the conventional symbol $\mathcal{P}_x^\mu$ for it:

$$\mathcal{P}_x^\mu f(x) = C(\mu) x^{1-\mu} \int\limits_0^x f(y)(x^2 - y^2)^{\frac{\mu}{2}-1} dy, \tag{6.11}$$

$$\mathcal{P}_x^{\mu}1 = 1, \qquad C(\mu) = \frac{2\Gamma\left(\frac{\mu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\mu}{2}\right)}.$$

Remark 11. *It is easy to see that if* $u = u(x,t)$, $x, t \in \mathbb{R}$, *and*

$$u(x,0) = f(x), \qquad u_t(x,0) = 0,$$

then

$$\mathcal{P}_t^{\mu} u(x,t)|_{t=0} = f(x), \qquad \frac{\partial}{\partial t}\mathcal{P}_t^{\mu} u(x,t)\bigg|_{t=0} = 0. \tag{6.12}$$

Indeed, we have

$$\mathcal{P}_t^{\mu} u(x,t)|_{t=0} = C(\mu) t^{1-\mu} \int\limits_0^t u(x,y)(t^2-y^2)^{\frac{\mu}{2}-1} dy\bigg|_{t=0} =$$

$$C(\mu) \int\limits_0^1 u(x,ty)|_{t=0} (1-y^2)^{\frac{\mu}{2}-1} dy = f(x)$$

and

$$\frac{\partial}{\partial t}\mathcal{P}_t^{\mu} u(x,t)\bigg|_{t=0} = C(\mu) \int\limits_0^1 u_t(x,ty)|_{t=0} (1-y^2)^{\frac{\mu}{2}-1} dy = 0.$$

Statement 15. *Let* $f \in L^2(0,\infty)$, $\alpha = 0$, $\mu = 0$. *In this case for* $\mu > 0$ *we obtain the operator*

$$\left(T_{\nu,0}^{(0)} f\right)(x) = \frac{2^{1-\gamma}\sqrt{\pi}}{\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \int\limits_x^{\infty} f(y)(y^2-x^2)^{\frac{\nu}{2}-1} dy \tag{6.13}$$

with the intertwining property

$$T_{\nu,0}^{(0)} B_\nu = D^2 T_{\nu,0}^{(0)}.$$

Proof. Using formula (6.8) we obtain

$$\left(T_{\nu,0}^{(0)} f\right)(x) = C \cdot 2^3 \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \int\limits_x^{\infty} f(y) {}_2F_1\left(\frac{1}{2}, 1-\frac{\nu}{2}; \frac{1}{2}; \frac{x^2}{y^2}\right) y^{\nu-1} dy =$$

$$C \cdot 2^3 \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \int\limits_x^{\infty} f(y)\left(1-\frac{x^2}{y^2}\right)^{\frac{\nu}{2}-1} y^{\nu-1} dy =$$

$$C \cdot 2^3 \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu}{2}\right)} \int\limits_x^\infty f(y)(y^2-x^2)^{\frac{\nu}{2}-1} dy.$$

Putting $C = \frac{\pi}{2^{\gamma+2}\Gamma^2\left(\frac{\nu+1}{2}\right)}$, we get (6.13). □

Statement 16. *For $f \in L^2(0,\infty)$, $\alpha{=}\nu{-}\mu$, $-1{<}\operatorname{Re}\nu{<}\operatorname{Re}\mu$, we obtain the first "descent" operator*

$$\left(T_{\nu,\,\mu}^{(\nu-\mu)} f\right)(x) = \frac{2\,\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)} x^{1-\mu} \int\limits_0^x f(y)(x^2-y^2)^{\frac{\mu-\nu}{2}-1} y^\nu dy =$$

$$\frac{2\,\Gamma\left(\frac{\mu+1}{2}\right)}{\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)} \int\limits_0^1 f(xy)(1-y^2)^{\frac{\mu-\nu}{2}-1} y^\nu dy \tag{6.14}$$

such that

$$T_{\nu,\,\mu}^{(\nu-\mu)} B_\nu = B_\mu T_{\nu,\,\mu}^{(\nu-\mu)}$$

and

$$T_{\nu,\,\mu}^{(\nu-\mu)} 1 = 1.$$

Proof. Substituting the value $\alpha = \nu - \mu$ into (6.7) we can write

$$\left(T_{\nu,\,\mu}^{(\nu-\mu)} f\right)(x) = C \cdot \frac{2^{\nu-\mu+3}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)} x^{-1-\nu} \times$$

$$\int\limits_0^x f(y)\,{}_2F_1\left(\frac{\nu+1}{2}, \frac{\nu-\mu}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\right) y^\nu dy.$$

Taking into account the identity (6.8) for a hypergeometric function the last equality reduces to $x^{\nu-\mu+2}(x^2-y^2)^{\frac{\mu-\nu}{2}-1}$ and the operator $T_{\nu,\,\mu}^{(\nu-\mu)}$ is written in the form

$$\left(T_{\nu,\,\mu}^{(\nu-\mu)} f\right)(x) = C \cdot \frac{2^{\nu-\mu+3}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)} x^{1-\mu} \int\limits_0^x f(y)(x^2-y^2)^{\frac{\mu-\nu}{2}-1} y^\nu dy.$$

Clearly,

$$x^{1-\mu} \int\limits_0^x (x^2-y^2)^{\frac{\mu-\nu}{2}-1} y^\nu dy = \{y = xz\} = \int\limits_0^1 (1-z^2)^{\frac{\mu-\nu}{2}-1} z^\nu dz =$$

$$\{z^2=t\}=\frac{1}{2}\int_0^1 (1-t)^{\frac{\mu-\nu}{2}-1}t^{\frac{\nu-1}{2}}dt=\frac{\Gamma\left(\frac{\mu-\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{\mu+1}{2}\right)},$$

and taking $C=\frac{2^{\mu-\nu-2}\Gamma^2\left(\frac{\mu+1}{2}\right)}{\Gamma^2\left(\frac{\nu+1}{2}\right)}$ we get $T_{\nu,\,\mu}^{(\nu-\mu)}1=1$. This completes the proof. □

Statement 17. *Let $f\in L^2(0,\infty)$, $\alpha=0$, $-1<\operatorname{Re}\mu<\operatorname{Re}\nu$. In this case we obtain the second "descent" operator:*

$$\left(T_{\nu,\,\mu}^{(0)}f\right)(x)=\frac{2\Gamma\left(\nu-\mu\right)}{\Gamma^2\left(\frac{\nu-\mu}{2}\right)}\int_x^\infty f(y)(y^2-x^2)^{\frac{\nu-\mu}{2}-1}ydy. \tag{6.15}$$

Proof. We have

$$\left(T_{\nu,\,\mu}^{(0)}f\right)(x)=$$

$$C\cdot\frac{2^3\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}\right)}\int_x^\infty f(y){}_2F_1\left(\frac{\mu+1}{2},\frac{\mu-\nu}{2}+1;\frac{\mu+1}{2};\frac{x^2}{y^2}\right)y^{\nu-\mu-1}dy.$$

Using formula (6.8) we get

$${}_2F_1\left(\frac{\mu+1}{2},\frac{\mu-\nu}{2}+1;\frac{\mu+1}{2};\frac{x^2}{y^2}\right)=\left(1-\frac{x^2}{y^2}\right)^{\frac{\nu-\mu}{2}-1}=y^{2+\mu-\nu}(y^2-x^2)^{\frac{\nu-\mu}{2}-1}$$

and

$$\left(T_{\nu,\,\mu}^{(0)}f\right)(x)=C\cdot\frac{2^3\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}\right)}\int_x^\infty f(y)(y^2-x^2)^{\frac{\nu-\mu}{2}-1}ydy.$$

It is obvious that

$$\int_x^\infty (y^2-x^2)^{\frac{\nu-\mu}{2}-1}ydy=\left\{y=\frac{x}{z}\right\}=$$

$$x^{\nu-\mu}\int_0^1(1-z^2)^{\frac{\nu-\mu}{2}-1}z^{\mu-\nu-1}dz=\{z^2=t\}=$$

$$\frac{x^{\nu-\mu}}{2}\int_0^1(1-t)^{\frac{\nu-\mu}{2}-1}t^{\frac{\nu-\mu}{2}-1}dt=\frac{x^{\nu-\mu}\Gamma^2\left(\frac{\nu-\mu}{2}\right)}{2\Gamma(\nu-\mu)}.$$

Therefore, for $C = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma(\nu-\mu)}{4\Gamma\left(\frac{\nu+1}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}\right)}$, we get $T_{\nu,\mu}^{(\nu-\mu)}1 = x^{\nu-\mu}$. This completes the proof. □

In [535] formula (6.15) was obtained as a particular case of the Buschman–Erdélyi operator of the third kind but with different constant:

$$\left(T_{\nu,\mu}^{(0)} f\right)(x) = \frac{2^{1-\frac{\nu-\mu}{2}}}{\Gamma\left(\frac{\nu-\mu}{2}\right)} \int_x^\infty f(y) y \left(y^2 - x^2\right)^{\frac{\nu-\mu}{2}-1} dy. \tag{6.16}$$

As might be seen in the form (6.15) as well as (6.16), the operator $T_{\nu,\mu}^{(0)}$ does not depend on the values ν and μ but only on the difference between ν and μ.

Statement 18. *Let $f \in L^2(0,\infty)$, $\mathrm{Re}\,(\alpha+\nu+1) > 0$, $\mathrm{Re}\,\alpha < 0$. If we take $\mu = \nu$ in (6.7) we obtain the operator*

$$\left(T_{\nu,\nu}^{(\alpha)} f\right)(x) = \frac{2^{\alpha+3}\Gamma\left(\frac{\alpha+\nu+1}{2}\right)}{\Gamma\left(-\frac{\alpha}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)} \times$$

$$\left[x^{-1-\nu-\alpha} \int_0^x f(y) {}_2F_1\left(\frac{\alpha+\nu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\right) y^\nu dy + \right.$$

$$\left. \int_x^\infty f(y) {}_2F_1\left(\frac{\alpha+\nu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{x^2}{y^2}\right) y^{-\alpha-1} dy \right], \tag{6.17}$$

which is an explicit integral representation of the negative fractional power α of the Bessel operator B_ν^α.

So it is possible and easy to obtain fractional powers of the Bessel operator by the ITCM. For different approaches to fractional powers of the Bessel operator and its explicit integral representations, cf. [89,230,252,367,515,516,527,531,555].

6.2.3 ITCM for generalized translation and the weighted spherical mean

Now we show how the generalized translation (3.144) and the weighted spherical mean (3.183) can be constructed by the ITCM.

Theorem 72. *If we apply the ITCM with $\varphi(t) = j_{\frac{\nu-1}{2}}(zt)$ in (6.6) and with $\mu = \nu$, then the operator*

$$\left(T_{\nu,\nu}^{(\varphi)} f\right)(x) = {}^\nu T_x^z f(x) = H_\nu^{-1}\left[j_{\frac{\nu-1}{2}}(zt) H_\nu[f](t)\right](x) =$$

$$\frac{2^\nu \Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}(4xz)^{\nu-1}\Gamma\left(\frac{\nu}{2}\right)} \int\limits_{|x-z|}^{x+z} f(y) y [(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\nu}{2}-1} dy \tag{6.18}$$

coincides with the generalized translation operator (see [315,319,321,537]), for which the following properties are valid

$$ {}^\nu T_x^z (B_\nu)_x = (B_\nu)_z \, {}^\nu T_x^z, \tag{6.19}$$

$$ {}^\nu T_x^z f(x)|_{z=0} = f(x), \qquad \frac{\partial}{\partial z} {}^\nu T_x^z f(x)\bigg|_{z=0} = 0. \tag{6.20}$$

Proof. We have

$$\left(T^{(z)}_{\nu,\,\nu} f\right)(x) = H_\nu^{-1}\left[j_{\frac{\nu-1}{2}}(zt) H_\nu[f](t)\right](x) =$$

$$\frac{2^{1-\nu}}{\Gamma^2\left(\frac{\nu+1}{2}\right)} \int\limits_0^\infty j_{\frac{\nu-1}{2}}(xt)\, j_{\frac{\nu-1}{2}}(zt)\, t^\nu \, dt \int\limits_0^\infty j_{\frac{\nu-1}{2}}(ty) f(y) y^\nu dy =$$

$$\frac{2^{1-\nu}}{\Gamma^2\left(\frac{\nu+1}{2}\right)} \int\limits_0^\infty f(y) y^\nu dy \int\limits_0^\infty j_{\frac{\nu-1}{2}}(xt)\, j_{\frac{\nu-1}{2}}(ty)\, j_{\frac{\nu-1}{2}}(zt)\, t^\nu \, dt.$$

Using the formula (see (3.156))

$$\int\limits_0^\infty j_{\frac{\nu-1}{2}}(tx) j_{\frac{\nu-1}{2}}(ty) j_{\frac{\nu-1}{2}}(tz) t^\nu dt =$$

$$\begin{cases} 0 & 0<y<|x-z| \text{ or } y>x+z, \\ \frac{2\Gamma^3\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)} \frac{[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\nu}{2}-1}}{(xyz)^{\nu-1}} & |x-z|<y<x+z, \end{cases}$$

we obtain

$$\left(T^{(z)}_{\nu,\,\nu} f\right)(x) =$$

$$\frac{2^{1-\nu}}{\Gamma^2\left(\frac{\nu+1}{2}\right)} \frac{2\Gamma^3\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}(xz)^{\nu-1}\Gamma\left(\frac{\nu}{2}\right)} \int\limits_{|x-z|}^{x+z} f(y) y [(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\nu}{2}-1} dy =$$

$$\frac{2^\nu \Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}(4xz)^{\nu-1}\Gamma\left(\frac{\nu}{2}\right)} \int\limits_{|x-z|}^{x+z} f(y) y [(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\nu}{2}-1} dy = {}^\nu T_x^z f(x).$$

From the derived representation it is clear that ${}^\nu T_x^z f(x) = {}^\nu T_z^x f(z)$ and from (6.5) it follows that ${}^\nu T_x^z (B_\nu)_x = (B_\nu)_x \, {}^\nu T_x^z$, and consequently ${}^\nu T_x^z (B_\nu)_x = (B_\nu)_z \, {}^\nu T_x^z$.

Properties (6.20) follow easily from the representation (6.18). This completes the proof. □

A more frequently used representation of the generalized translation operator ${}^{\nu}T_z^x$ is (3.144). In Statement 8 it was proved that (6.18) and (3.144) are the same operator.

So it is possible and easy to obtain generalized translation operators by the ITCM, and its basic properties follow immediately from the ITCM integral representation.

Now we apply the ITCM for the construction of the weighted spherical mean using the second formula from (6.1) when $A=(\Delta_\gamma)_x$, $x\in\mathbb{R}^n_+$ $B=(B_{n+|\gamma|-1})_r$, $r\in\mathbb{R}^1_+$, $F_A=F_B=\mathbf{F}_\gamma$ (see Definition 12) and $w(x)=j_{\frac{n+|\gamma|}{2}-1}(r|x|)$.

Theorem 73. *For $f\in S_{ev}$ the representation of the weighted spherical mean*

$$M_r^\gamma[f(x)]=\mathbf{F}_\gamma^{-1}\left[j_{\frac{n+|\gamma|}{2}-1}(r|\xi|)\mathbf{F}_\gamma[f](\xi)\right](x) \tag{6.21}$$

is valid.

Proof. Let us consider

$$\mathbf{F}_\gamma[M_r^\gamma[f(x)]](\xi)=\int\limits_{\mathbb{R}^n_+}\mathbf{j}_\gamma(x,\xi)M_r^\gamma[f(x)]x^\gamma\,dx=$$

$$\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{\mathbb{R}^n_+}\mathbf{j}_\gamma(x,\xi)x^\gamma\,dx\int\limits_{S_1^+(n)}{}^{\gamma}\mathbf{T}_x^{r\theta}f(x)\theta^\gamma dS=$$

$$\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{\mathbb{R}^n_+}\mathbf{j}_\gamma(x,\xi)\mathbf{T}_x^{r\theta}\left(\int\limits_{S_1^+(n)}{}^{\gamma}f(x)\theta^\gamma dS\right)x^\gamma\,dx.$$

Formulas (3.172) and (3.170) give

$$\mathbf{F}_\gamma[M_r^\gamma[f(x)]](\xi)=\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{\mathbb{R}^n_+}f(x)x^\gamma\,dx\int\limits_{S_1^+(n)}{}^{\gamma}\mathbf{T}_x^{r\theta}\mathbf{j}_\gamma(x,\xi)\theta^\gamma dS=$$

$$\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{\mathbb{R}^n_+}f(x)\mathbf{j}_\gamma(x,\xi)x^\gamma\,dx\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(r\theta,\xi)\theta^\gamma dS.$$

Since (3.140) taking into account (1.107), we obtain

$$\mathbf{F}_\gamma[M_r^\gamma[f(x)]](\xi)=\frac{1}{|S_1^+(n)|_\gamma}\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\times$$

$$j_{\frac{n+|\gamma|}{2}-1}(r|\xi|)\int\limits_{\mathbb{R}^n_+}f(x)\mathbf{j}_\gamma(x,\xi)x^\gamma\,dx=j_{\frac{n+|\gamma|}{2}-1}(r|\xi|)\mathbf{F}_\gamma[f](\xi).$$

Application of the inverse Hankel transform gives formula (6.21). □

6.2.4 Integral representations of transmutations for perturbed differential Bessel operators

Now let us prove a general result on transmutations for perturbed Bessel operator with potentials. These results are of technical form; they were proved many times for some special cases. It is convenient to prove the general result accurately here. It is the necessary step 2 from the ITCM algorithm; it turns operators obtained by the ITCM with formal transmutation property into transmutations with exact conditions on input parameters and classes of functions.

We will construct a transmutation operator $S_{\nu,\mu}$ intertwining Bessel operators $B_\nu + q$ and $B_\mu + r$. In this case it is reasonable to use the Hankel transforms of orders ν and μ, respectively. So for a pair of perturbed Bessel differential operators

$$A = B_\nu + q(x)I, \qquad B = B_\mu + r(x)I, \qquad I \text{ is a unique operator},$$

we seek a transmutation operator $S_{\nu,\mu}$ such that

$$S_{\nu,\mu}(B_\nu + q(x))u = (B_\mu + r(x))S_{\nu,\mu}u. \tag{6.22}$$

Let us apply the ITCM and obtain it in the form

$$S_{\nu,\mu} = F_\mu^{-1} \frac{1}{w(t)} F_\nu$$

with arbitrary $w(t)$, $w(t) \neq 0$. So we have formally

$$S_{\nu,\mu} f(x) = \frac{2^{1-\mu}}{\Gamma^2\left(\frac{\mu+1}{2}\right)} \int\limits_0^\infty j_{\frac{\mu-1}{2}}(xt) \frac{t^\mu}{w(t)} \, dt \int\limits_0^\infty j_{\frac{\nu-1}{2}}(ty) f(y) y^\nu dy =$$
$$\frac{2^{1-\mu}}{\Gamma^2\left(\frac{\mu+1}{2}\right)} \int\limits_0^\infty f(y) y^\nu dy \int\limits_0^\infty j_{\frac{\nu-1}{2}}(ty) j_{\frac{\mu-1}{2}}(xt) \frac{t^\mu}{w(t)} \, dt.$$

For all known cases we may represent transmutations $S_{\nu,\mu}$ in the following general form (see [558,586]):

$$S_{\nu,\mu} f(x) = a(x) f(x) + \int\limits_0^x K(x,y) f(y) y^\nu dy + \int\limits_x^\infty L(x,y) f(y) y^\nu dy.$$

Necessary conditions on kernels K and L as well as on functions $a(x)$, $f(x)$ to satisfy (6.22) are given in the following theorem.

Theorem 74. *Let $u \in L_2(0,\infty)$ be twice continuously differentiable on $[0,\infty)$ such that $u'(0) = 0$ and let q and r be functions such that*

$$\int\limits_0^\infty t^\delta |q(t)| dt < \infty, \qquad \int\limits_0^\infty t^\varepsilon |r(t)| dt < \infty$$

for some $\delta < \frac{1}{2}$ *and* $\varepsilon < \frac{1}{2}$. *There exists a transmutation operator of the form*

$$S_{\nu,\mu}u(x) = a(x)u(x) + \int_0^x K(x,t)u(t)t^\nu dt + \int_x^\infty L(x,t)u(t)t^\nu dt, \tag{6.23}$$

such that

$$S_{\nu,\mu}\Big[B_\nu + q(x)\Big]u(x) = \Big[B_\mu + r(x)\Big]S_{\nu,\mu}u(x) \tag{6.24}$$

with twice continuously differentiable kernels $K(x,t)$ *and* $L(x,t)$ *on* $[0,\infty)$ *such that*

$$\lim_{t\to 0} t^\nu K(x,t)u'(t) = 0, \qquad \lim_{t\to 0} t^\nu K_t(x,t)u(t) = 0$$

and

$$\lim_{t\to\infty} t^\nu L(x,t)u'(t) = 0, \qquad \lim_{t\to\infty} t^\nu L_t(x,t)u(t) = 0,$$

satisfying the following relations:

$$\Big[(B_\nu)_t + q(t)\Big]K(x,t) = \Big[(B_\mu)_x + r(x)\Big]K(x,t),$$
$$\Big[(B_\nu)_t + q(t)\Big]L(x,t) = \Big[(B_\mu)_x + r(x)\Big]L(x,t),$$

and

$$a(x)\Big[B_\nu + q(x)\Big]u(x) - \Big[B_\mu + r(x)\Big]a(x)u(x) =$$
$$(\mu+\nu)x^{\nu-1}u(x)\Big[K(x,x) - L(x,x)\Big] + 2x^\nu u(x)\Big[K'(x,x) - L'(x,x)\Big].$$

Proof. First we have

$$S_{\nu,\mu}(B_\nu u(x) + q(x)u(x)) = a(x)[B_\nu u(x) + q(x)u(x)] +$$
$$\int_0^x K(x,t)(B_\nu u(t) + q(t)u(t))t^\nu dt + \int_x^\infty L(x,t)(B_\nu u(t) + q(t)u(t))t^\nu dt.$$

Substituting the Bessel operator in the form $B_\nu = \frac{1}{t^\nu}\frac{d}{dt}t^\nu\frac{d}{dt}$ and integrating by parts we obtain

$$\int_0^x K(x,t)(B_\nu u(t))\,t^\nu dt = \int_0^x K(x,t)\frac{d}{dt}t^\nu\frac{d}{dt}u(t)dt =$$

$$K(x,t)t^{\nu}u'(t)\Big|_{t=0}^{x}-\int\limits_0^x t^{\nu}K_t(x,t)\frac{d}{dt}u(t)dt=$$

$$K(x,t)t^{\nu}u'(t)\Big|_{t=0}^{x}-t^{\nu}K_t(x,t)u(t)\Big|_{t=0}^{x}+\int\limits_0^x((B_\nu)_tK_t(x,t))\,u(t)t^{\nu}dt.$$

Since

$$\lim_{t\to 0}t^{\nu}K(x,t)u'(t)=0,\qquad \lim_{t\to 0}t^{\nu}K_t(x,t)u(t)=0,$$

we obtain

$$\int\limits_0^x K(x,t)(B_\nu u(t))\,t^{\nu}dt=$$

$$K(x,x)x^{\nu}u'(x)-x^{\nu}u(x)K_t(x,t)\Big|_{t=x}+\int\limits_0^x((B_\nu)_tK(x,t))\,u(t)t^{\nu}dt.$$

Similarly,

$$\int\limits_x^\infty L(x,t)(B_\nu u(t))\,t^{\nu}dt=\int\limits_x^\infty L(x,t)\frac{d}{dt}t^{\nu}\frac{d}{dt}u(t)dt=$$

$$L(x,t)t^{\nu}u'(t)\Big|_{t=x}^{\infty}-\int\limits_x^\infty t^{\nu}L_t(x,t)\frac{d}{dt}u(t)dt=$$

$$L(x,t)t^{\nu}u'(t)\Big|_{t=x}^{\infty}-t^{\nu}L_t(x,t)u(t)\Big|_{t=x}^{\infty}+\int\limits_x^\infty((B_\nu)_tL(x,t))\,u(t)t^{\nu}dt.$$

Since

$$\lim_{t\to\infty}t^{\nu}L(x,t)u'(t)=0,\qquad \lim_{t\to\infty}t^{\nu}L_t(x,t)u(t)=0,$$

we obtain

$$\int\limits_x^\infty L(x,t)(B_\nu u(t))\,t^{\nu}dt=$$

$$-L(x,x)x^{\nu}u'(x)+x^{\nu}u(x)L_t(x,t)\Big|_{t=x}+\int\limits_x^\infty((B_\nu)_tL(x,t))\,u(t)t^{\nu}dt.$$

Therefore,

$$S_{\nu,\mu}(B_\nu u(x)+q(x)u(x))=a(x)\Big[B_\nu+q(x)\Big]u(x)+$$
$$x^\nu K(x,x)u'(x)-x^\nu u(x)K_t(x,t)\Big|_{t=x}-x^\nu L(x,x)u'(x)+x^\nu u(x)L_t(x,t)\Big|_{t=x}+$$
$$\int\limits_0^x((B_\nu)_t K(x,t)+q(t)K(x,t))\,u(t)t^\nu dt+$$
$$\int\limits_x^\infty((B_\nu)_t L(x,t)+q(t)L(x,t))\,u(t)t^\nu dt.$$

Further we have

$$(B_\mu+r(x))S_{\nu,\mu}u(x)=$$
$$(B_\mu+r(x))\left(a(x)u(x)+\int\limits_0^x K(x,t)u(t)t^\nu dt+\int\limits_x^\infty L(x,t)u(t)t^\nu dt\right)=$$
$$B_\mu\,[a(x)u(x)]+a(x)r(x)u(x)+B_\mu\int\limits_0^x K(x,t)u(t)t^\nu dt+B_\mu\int\limits_x^\infty L(x,t)u(t)t^\nu dt+$$
$$r(x)\int\limits_0^x K(x,t)u(t)t^\nu dt+r(x)\int\limits_x^\infty L(x,t)u(t)t^\nu dt.$$

Using the formula of differentiation of integrals depending on the parameter we get

$$(B_\mu)_x\int\limits_0^x K(x,t)u(t)t^\nu dt=\frac{1}{x^\mu}\frac{d}{dx}x^\mu\frac{d}{dx}\int\limits_0^x K(x,t)u(t)t^\nu dt=$$
$$\frac{1}{x^\mu}\frac{d}{dx}\left(x^{\mu+\nu}K(x,x)u(x)+x^\mu\int\limits_0^x K_x(x,t)u(t)t^\nu dt\right)=$$
$$\frac{1}{x^\mu}\Big((\mu+\nu)x^{\mu+\nu-1}K(x,x)u(x)+x^{\mu+\nu}K'(x,x)u(x)+x^{\mu+\nu}K(x,x)u'(x)+$$
$$\mu x^{\mu-1}\int\limits_0^x K_x(x,t)u(t)t^\nu dt+x^\mu\frac{d}{dx}\int\limits_0^x K_x(x,t)u(t)t^\nu dt\Big)=$$
$$\frac{1}{x^\mu}\Big((\mu+\nu)x^{\mu+\nu-1}K(x,x)u(x)+x^{\mu+\nu}K'(x,x)u(x)+x^{\mu+\nu}K(x,x)u'(x)+$$

$$x^{\mu+\nu}u(x)K_x(x,t)\Big|_{t=x}+\mu x^{\mu-1}\int\limits_0^x K_x(x,t)u(t)t^\nu dt+x^\mu\int\limits_0^x K_{xx}(x,t)u(t)t^\nu dt\Bigg)=$$

$$(\mu+\nu)x^{\nu-1}K(x,x)u(x)+x^\nu K'(x,x)u(x)+x^\nu K(x,x)u'(x)+$$

$$x^\nu u(x)K_x(x,t)\Big|_{t=x}+\int\limits_0^x (B_\mu)_x K(x,t)u(t)t^\nu dt$$

and also

$$(B_\mu)_x\int\limits_x^\infty L(x,t)u(t)t^\nu dt=\frac{1}{x^\mu}\frac{d}{dx}x^\mu\frac{d}{dx}\int\limits_x^\infty L(x,t)u(t)t^\nu dt=$$

$$\frac{1}{x^\mu}\frac{d}{dx}\left(-x^{\mu+\nu}L(x,x)u(x)+x^\mu\int\limits_x^\infty L_x(x,t)u(t)t^\nu dt\right)=$$

$$\frac{1}{x^\mu}\Bigg(-(\mu+\nu)x^{\mu+\nu-1}L(x,x)u(x)-x^{\mu+\nu}L'(x,x)u(x)-x^{\mu+\nu}L(x,x)u'(x)+$$

$$\mu x^{\mu-1}\int\limits_x^\infty L_x(x,t)u(t)t^\nu dt+x^\mu\frac{d}{dx}\int\limits_x^\infty L_x(x,t)u(t)t^\nu dt\Bigg)=$$

$$\frac{1}{x^\mu}\Bigg(-(\mu+\nu)x^{\mu+\nu-1}L(x,x)u(x)-x^{\mu+\nu}L'(x,x)u(x)-x^{\mu+\nu}L(x,x)u'(x)-$$

$$-x^{\mu+\nu}u(x)L_x(x,t)\Big|_{t=x}+\mu x^{\mu-1}\int\limits_x^\infty L_x(x,t)u(t)t^\nu dt+x^\mu\int\limits_x^\infty L_{xx}(x,t)u(t)t^\nu dt\Bigg)=$$

$$-(\mu+\nu)x^{\nu-1}L(x,x)u(x)-x^\nu L'(x,x)u(x)-x^\nu L(x,x)u'(x)-$$

$$x^\nu u(x)L_x(x,t)\Big|_{t=x}+\int\limits_x^\infty (B_\mu)_x L(x,t)u(t)t^\nu dt.$$

So

$$(B_\mu+r(x))S_{\nu,\mu}u(x)=$$

$$\Big[B_\mu+r(x)\Big]a(x)u(x)+(\mu+\nu)x^{\nu-1}K(x,x)u(x)+x^\nu K'(x,x)u(x)+$$

$$x^\nu K(x,x)u'(x)+x^\nu u(x)K_x(x,t)\Big|_{t=x}-$$

$$(\mu+\nu)x^{\nu-1}L(x,x)u(x)-x^\nu L'(x,x)u(x)-x^\nu L(x,x)u'(x)-$$

$$x^\nu u(x)L_x(x,t)\Big|_{t=x}+$$

$$\int_0^x \left[(B_\mu)_x K(x,t) + r(x) K(x,t) \right] u(t) t^\nu dt +$$

$$\int_x^\infty \left[(B_\mu)_x L(x,t) + r(x) L(x,t) \right] u(t) t^\nu dt.$$

Since we should have an equality

$$S_{\nu,\mu}(B_\nu + q(x)) = (B_\mu + r(x)) S_{\nu,\mu},$$

equating the corresponding terms in both parts we obtain

$$\int_0^x \left[(B_\nu)_t K(x,t) + q(t) K(x,t) \right] u(t) t^\nu dt =$$

$$\int_0^x \left[(B_\mu)_x K(x,t) + r(x) K(x,t) \right] u(t) t^\nu dt$$

and

$$\int_x^\infty \left[(B_\nu)_t L(x,t) + q(t) L(x,t) \right] u(t) t^\nu dt =$$

$$\int_x^\infty \left[(B_\mu)_x L(x,t) + r(x) L(x,t) \right] u(t) t^\nu dt.$$

From that we derive two equations

$$\left[(B_\nu)_t + q(t) \right] K(x,t) = \left[(B_\mu)_x + r(x) \right] K(x,t)$$

and

$$\left[(B_\nu)_t + q(t) \right] L(x,t) = \left[(B_\mu)_x + r(x) \right] L(x,t).$$

Because

$$K'(x,x) = K_x(x,t)\Big|_{t=x} + K_t(x,t)\Big|_{t=x}$$

and

$$L'(x,x) = L_x(x,t)\Big|_{t=x} + L_t(x,t)\Big|_{t=x},$$

we obtain

$$
\begin{aligned}
& a(x)\Big[B_\nu + q(x)\Big]u(x)+ \\
& x^\nu K(x,x)u'(x) - x^\nu u(x)K_t(x,t)\Big|_{t=x} - x^\nu L(x,x)u'(x) + x^\nu u(x)L_t(x,t)\Big|_{t=x} = \\
& \Big[B_\mu + r(x)\Big]a(x)u(x) + (\mu+\nu)x^{\nu-1}K(x,x)u(x) + x^\nu K'(x,x)u(x)+ \\
& x^\nu K(x,x)u'(x) + x^\nu u(x)K_x(x,t)\Big|_{t=x} - \\
& (\mu+\nu)x^{\nu-1}L(x,x)u(x) - x^\nu L'(x,x)u(x) - x^\nu L(x,x)u'(x)- \\
& x^\nu u(x)L_x(x,t)\Big|_{t=x},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
& a(x)\Big[B_\nu + q(x)\Big]u(x) - \Big[B_\mu + r(x)\Big]a(x)u(x) = \\
& (\mu+\nu)x^{\nu-1}u(x)\Big[K(x,x) - L(x,x)\Big] + 2x^\nu u(x)\Big[K'(x,x) - L'(x,x)\Big].
\end{aligned}
$$

This completes the proof of the theorem. □

We consider here some special cases of the transmutation operator $S_{\nu,\mu}$ for functions q and r from Theorem 74. Let functions u, q, r satisfy the conditions of Theorem 74.

1. For the transmutation in (6.23) of the form

$$
S_{\nu,\mu}u(x) = a(x)u(x) + \int\limits_0^x K(x,t)u(t)t^\nu dt
$$

with intertwining property

$$
S_{\nu,\mu}\Big[B_\nu + q(x)\Big]u(x) = \Big[B_\mu + r(x)\Big]S_{\nu,\mu}u(x),
$$

a kernel $K(x,t)$ and function $a(x)$ should satisfy the relations

$$
\Big[(B_\nu)_t + q(t)\Big]K(x,t) = \Big[(B_\mu)_x + r(x)\Big]K(x,t)
$$

and

$$a(x)\Big[B_\nu + q(x)\Big]u(x) - \Big[B_\mu + r(x)\Big]a(x)u(x) =$$
$$(\mu+\nu)x^{\nu-1}u(x)K(x,x) + 2x^\nu u(x)K'(x,x).$$

In the particular case when $\nu = \mu$, $r(x) = 0$, $a(x) = 1$ transmutations with such representations were obtained in [558,586].

2. For the transmutation in (6.23) of the form

$$S_{\nu,\mu}u(x) = a(x)u(x) + \int\limits_x^\infty L(x,t)u(t)t^\nu dt,$$

such that

$$S_{\nu,\mu}(B_\nu + q(x))u = (B_\mu + r(x))S_{\nu,\mu}u,$$

a kernel $L(x,t)$ and function $a(x)$ should satisfy the relations

$$\Big[(B_\nu)_t + q(t)\Big]L(x,t) = \Big[(B_\mu)_x + r(x)\Big]L(x,t)$$

and

$$a(x)\Big[B_\nu + q(x)\Big]u(x) - \Big[B_\mu + r(x)\Big]a(x)u(x) =$$
$$-(\mu+\nu)x^{\nu-1}u(x)L(x,x) - 2x^\nu u(x)L'(x,x).$$

3. When one potential in (6.24) is equal to zero we get for a transmutation operator

$$S_{\nu,\mu}u(x) = a(x)u(x) + \int\limits_0^x K(x,t)u(t)t^\nu dt + \int\limits_x^\infty L(x,t)u(t)t^\nu dt,$$

such that

$$S_{\nu,\mu}\Big[B_\nu + q(x)\Big]u(x) = B_\mu S_{\nu,\mu}u(x),$$

and for kernels $K(x,t)$, $L(x,t)$ and function $a(x)$ we have

$$\Big[(B_\nu)_t + q(t)\Big]K(x,t) = (B_\mu)_x K(x,t),$$
$$\Big[(B_\nu)_t + q(t)\Big]L(x,t) = (B_\mu)_x L(x,t),$$

and

$$a(x)\Big[B_\nu + q(x)\Big]u(x) - B_\mu[a(x)u(x)] =$$

$$(\mu+\nu)x^{\nu-1}u(x)\Big[K(x,x)-L(x,x)\Big]+2x^{\nu}u(x)\Big[K'(x,x)-L'(x,x)\Big].$$

4. When $\mu=0$ and $r(x)\equiv 0$ in (6.24) for a transmutation operator

$$S_\nu u(x)=a(x)u(x)+\int\limits_0^x K(x,t)u(t)t^\nu dt+\int\limits_x^\infty L(x,t)u(t)t^\nu dt,$$

such that

$$S_\nu(B_\nu+q(x))u=D^2S_\nu u,$$

kernels $K(x,t)$, $L(x,t)$ and function $a(x)$ should satisfy the relations

$$\Big[(B_\nu)_t+q(t)\Big]K(x,t)=D_x^2K(x,t),$$
$$\Big[(B_\nu)_t+q(t)\Big]L(x,t)=D_x^2L(x,t),$$

and

$$a(x)\Big[B_\nu+q(x)\Big]u(x)-D_x^2a(x)u(x)=$$
$$\nu x^{\nu-1}u(x)\Big[K(x,x)-L(x,x)\Big]+2x^{\nu}u(x)\Big[K'(x,x)-L'(x,x)\Big].$$

5. When both potentials in (6.24) are equal to zero for a transmutation operator

$$S_{\nu,\mu}u(x)=a(x)u(x)+\int\limits_0^x K(x,t)u(t)t^\nu dt+\int\limits_x^\infty L(x,t)u(t)t^\nu dt,$$

such that

$$S_{\nu,\mu}B_\nu u=B_\mu S_{\nu,\mu}u,$$

kernels $K(x,t)$, $L(x,t)$ and function $a(x)$ should satisfy the relations

$$(B_\nu)_tK(x,t)=(B_\mu)_xK(x,t),$$
$$(B_\nu)_tL(x,t)=(B_\mu)_xL(x,t),$$

and

$$a(x)B_\nu u(x)-B_\mu[a(x)u(x)]=$$
$$(\mu+\nu)x^{\nu-1}u(x)\Big[K(x,x)-L(x,x)\Big]+2x^{\nu}u(x)\Big[K'(x,x)-L'(x,x)\Big].$$

6. When both potentials in (6.24) are equal to zero and $\mu = \nu$ for a transmutation operator

$$S_{\nu,\nu} u(x) = a(x)u(x) + \int_0^x K(x,t)u(t)t^\nu dt + \int_x^\infty L(x,t)u(t)t^\nu dt,$$

such that

$$S_{\nu,\nu} B_\nu u = B_\nu S_{\nu,\nu} u,$$

kernels $K(x,t)$, $L(x,t)$ and function $a(x)$ should satisfy the relations

$$(B_\nu)_t K(x,t) = (B_\nu)_x K(x,t),$$
$$(B_\nu)_t L(x,t) = (B_\nu)_x L(x,t),$$

and

$$a(x)B_\nu u(x) - B_\nu[a(x)u(x)] =$$
$$2\nu x^{\nu-1} u(x)\Big[K(x,x) - L(x,x)\Big] + 2x^\nu u(x)\Big[K'(x,x) - L'(x,x)\Big].$$

7. When both potentials in (6.24) are equal to zero and $\mu = 0$ for a transmutation operator

$$S_{\nu,0} u(x) = a(x)u(x) + \int_0^x K(x,t)u(t)t^\nu dt + \int_x^\infty L(x,t)u(t)t^\nu dt,$$

such that

$$S_{\nu,0} B_\nu u = D^2 S_{\nu,0} u,$$

kernels $K(x,t)$, $L(x,t)$ and function $a(x)$ should satisfy the relations

$$(B_\nu)_t K(x,t) = D_x^2 K(x,t),$$
$$(B_\nu)_t L(x,t) = D_x^2 L(x,t),$$

and

$$a(x)B_\nu u(x) - D_x^2[a(x)u(x)] =$$
$$\nu x^{\nu-1} u(x)\Big[K(x,x) - L(x,x)\Big] + 2x^\nu u(x)\Big[K'(x,x) - L'(x,x)\Big].$$

6.3 Connection formulas for solutions to singular differential equations via the ITCM

Suppose we solved the problem of obtaining transmutations by the ITCM (step 1) and justified an integral representation and proper function classes for it (step 2). Now we consider applications of these transmutations to integral representations of solutions to hyperbolic equations with Bessel operators (step 3). For simplicity we consider model equations, as for them integral representations of solutions are mostly known. More complex problems need more detailed and spacious calculations. But even for the model problems considered below, application of the transmutation method based on the ITCM is new; it allows a more unified and simplified approach to hyperbolic equations with Bessel operators of Euler–Poisson–Darboux and general Euler–Poisson–Darboux types.

6.3.1 Application of transmutations for finding general solutions to Euler–Poisson–Darboux type equations

A standard approach to solving a differential equation is to find its general solution first, and then substitute given functions to find particular solutions. Here we will show how to obtain general solutions of Euler–Poisson–Darboux type equations using transmutation operators.

Statement 19. *The general solution of the equation*

$$\frac{\partial^2 u}{\partial x^2} = (B_\mu)_t u, \qquad u = u(x, t; \mu) \tag{6.25}$$

for $0 < \mu < 1$ *is represented in the form*

$$u = \int_0^1 \frac{\Phi(x + t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}}\, dp + t^{1-\mu} \int_0^1 \frac{\Psi(x + t(2p-1))}{(p(1-p))^{\frac{\mu}{2}}}\, dp, \tag{6.26}$$

with a pair of arbitrary functions Φ, Ψ.

Proof. First, we consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{6.27}$$

The general solution to this equation has the form

$$F(x + t) + G(x - t), \tag{6.28}$$

where F and G are arbitrary functions. Applying operator (6.11) by variable t we obtain that one solution to Eq. (6.25) is

$$u_1 = 2C(\mu)\frac{1}{t^{\mu-1}}\int\limits_0^t [F(x+z)+G(x-z)](t^2-z^2)^{\frac{\mu}{2}-1}\,dz.$$

Let us transform the resulting general solution as follows:

$$u_1 = \frac{C(\mu)}{t^{\mu-1}}\int\limits_{-t}^t \frac{F(x+z)+F(x-z)+G(x+z)+G(x-z)}{(t^2-z^2)^{1-\frac{\mu}{2}}}\,dz.$$

Introducing a new variable p by the formula $z = t(2p-1)$ we get

$$u_1 = \int\limits_0^1 \frac{\Phi(x+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}}\,dp,$$

where

$$\Phi(x+z)=[F(x+z)+F(x-z)+G(x+z)+G(x-z)]$$

is an arbitrary function.

It is easy to see that if $u(x,t;\mu)$ is a solution of (6.25), then a function $t^{1-\mu}u(x,t;2-\mu)$ is also a solution of (6.25). Therefore the second solution to (6.25) is

$$u_2 = t^{1-\mu}\int\limits_0^1 \frac{\Psi(x+t(2p-1))}{(p(1-p))^{\frac{\mu}{2}}}\,dp,$$

where Ψ is an arbitrary function, not coinciding with Φ. Summing u_1 and u_2 we obtain the general solution to (6.25) of the form (6.26). From (6.26) we can see that for summable functions Φ and Ψ such a solution exists for $0 < \mu < 1$. □

6.3.2 Application of transmutations for finding solutions to general Euler–Poisson–Darboux type equations

Now we derive a general solution to general Euler–Poisson–Darboux type equations by the transmutation method.

Statement 20. *The general solution to the equation*

$$(B_\nu)_x u = (B_\mu)_t u, \qquad u = u(x,t;\nu,\mu) \tag{6.29}$$

for $0<\mu<1$, $0<\nu<1$ *is*

$$u=\frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(x^{1-\nu}\int\limits_0^x(x^2-y^2)^{\frac{\nu}{2}-1}dy\int\limits_0^1\frac{\Phi(y+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}}\,dp+\right.$$

$$\left.t^{1-\mu}x^{1-\nu}\int\limits_0^x(x^2-y^2)^{\frac{\nu}{2}-1}dy\int\limits_0^1\frac{\Psi(y+t(2p-1))}{(p(1-p))^{\frac{\mu}{2}}}\,dp\right). \tag{6.30}$$

Proof. Applying the Poisson operator (6.11) (again obtained by the ITCM in Subection 6.2.2) with index ν by variable x to (6.26) we derive the general solution (6.30) to Eq. (6.29). □

Now let us apply transmutations for finding general solutions to general Euler–Poisson–Darboux type equations with spectral parameter.

Proposition 3. The general solution to the equation

$$(B_\nu)_x u=(B_\mu)_t u+b^2u,\qquad u=u(x,t;\nu,\mu) \tag{6.31}$$

for $0<\mu<1$, $0<\nu<1$ is

$$u=\frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\times$$

$$\left(x^{1-\nu}\int\limits_0^x(x^2-y^2)^{\frac{\nu}{2}-1}dy\int\limits_0^1\frac{\Phi(y+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}}j_{\frac{\mu}{2}-1}(2bt\sqrt{p(1-p)})\,dp+\right.$$

$$\left.t^{1-\mu}x^{1-\nu}\int\limits_0^x(x^2-y^2)^{\frac{\nu}{2}-1}dy\int\limits_0^1\frac{\Psi(y+t(2p-1))}{(p(1-p))^{\frac{\mu}{2}}}j_{-\frac{\mu}{2}}(2bt\sqrt{p(1-p)})\,dp\right). \tag{6.32}$$

Proof. The general solution to the equation

$$\frac{\partial^2u}{\partial x^2}=(B_\mu)_t u+b^2u,\qquad u=u(x,t;\mu),\qquad 0<\mu<1,$$

is (see [453], p. 328)

$$u=\int\limits_0^1\frac{\Phi(x+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}}j_{\frac{\mu}{2}-1}(2bt\sqrt{p(1-p)})\,dp+$$

$$t^{1-\mu}\int\limits_0^1\frac{\Psi(x+t(2p-1))}{(p(1-p))^{\frac{\mu}{2}}}j_{-\frac{\mu}{2}}(2bt\sqrt{p(1-p)})\,dp.$$

Applying the Poisson operator (6.11) with index ν by variable x to (6.26), we derive the general solution (6.30) to Eq. (6.29). □

6.3.3 Application of transmutations for finding general solutions to singular Cauchy problems

Using (6.26) for $0 < \mu < 1$ we find the solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} = (B_\mu)_t u, \qquad u = u(x,t;\mu), \qquad 0 < \mu < 1, \tag{6.33}$$

$$u(x,0;\mu) = f(x), \qquad \left.\left(t^\mu \frac{\partial u}{\partial t}\right)\right|_{t=0} = g(x), \tag{6.34}$$

and this solution is

$$u = \frac{\Gamma(\mu)}{\Gamma^2\left(\frac{\mu}{2}\right)} \int\limits_0^1 \frac{f(x+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}} dp$$
$$+ t^{1-\mu} \frac{\Gamma(\mu+2)}{(1-\mu)\Gamma^2\left(\frac{\mu}{2}+1\right)} \int\limits_0^1 \frac{g(x+t(2p-1))}{(p(1-p))^{\frac{\mu}{2}}} dp. \tag{6.35}$$

The solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} = (B_\mu)_t u, \qquad u = u(x,t;\mu), \tag{6.36}$$

$$u(x,0;\mu) = f(x), \qquad \left.\left(\frac{\partial u}{\partial t}\right)\right|_{t=0} = 0 \tag{6.37}$$

exists for any $\mu > 0$ and has the form

$$u(x,t;\mu) = \frac{\Gamma(\mu)}{\Gamma^2\left(\frac{\mu}{2}\right)} \int\limits_0^1 \frac{f(x+t(2p-1))}{(p(1-p))^{1-\frac{\mu}{2}}} dp. \tag{6.38}$$

Taking into account (6.12), we can see that it is possible to obtain a solution of (6.36)–(6.37) applying the Poisson operator to the solution of the Cauchy problem (6.27)–(6.37) directly.

The solution of the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} = (B_\mu)_t u, \qquad u = u(x,t;\mu), \tag{6.39}$$

$$u(x,0;\mu) = 0, \qquad \left.\left(t^\mu \frac{\partial u}{\partial t}\right)\right|_{t=0} = g(x) \tag{6.40}$$

exists for any $\mu < 1$ and has the form

$$u(x,t;\mu) = t^{1-\mu}\,\frac{\Gamma(\mu+2)}{(1-\mu)\Gamma^2\left(\frac{\mu}{2}+1\right)}\int\limits_0^1 \frac{g(x+t(2p-1))}{(p(1-p))^{\frac{\mu}{2}}}dp.$$

The Cauchy problem (6.33)–(6.34) can be considered for $\mu \notin (0,1)$. In this case, to obtain the solution the transmutation operator (6.7) should be used. The case $\mu = -1, -3, -5, \ldots$ is exceptional and has to be studied separately.

It is easy to see that if we know that generalized translation has properties (6.19)–(6.20) we can in a straightforward way obtain that the solution to the equation

$$(B_\mu)_x u = (B_\mu)_t u, \qquad u = u(x,t;\mu)$$

with initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = 0$$

is $u = {}^{\mu}T_x^t f(x)$. Now the first and second descent operators (6.14) and (6.15) allow to represent the solution to the Cauchy problem

$$(B_\mu)_x u = (B_\nu)_t u, \qquad u = u(x,t;\mu,\nu), \tag{6.41}$$

$$u(x,0;\mu,\nu) = f(x), \qquad \left.\left(\frac{\partial u}{\partial t}\right)\right|_{t=0} = 0. \tag{6.42}$$

For $0 < \mu < \nu$ a solution of (6.41)–(6.42) is derived by using (6.14) and has the form

$$u(x,t;\mu,\nu)=\frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)}\,t^{1-\nu}\int\limits_0^t (t^2-y^2)^{\frac{\nu-\mu}{2}-1}\,{}^{\mu}T_x^y f(x)\,y^\mu dy. \tag{6.43}$$

In the case $0<\nu<\mu$ by using (6.15) we get a solution to (6.41)–(6.42) in the form

$$u(x,t;\mu,\nu)==\frac{2\Gamma(\mu-\nu)}{\Gamma^2\left(\frac{\mu-\nu}{2}\right)}\int\limits_t^\infty (y^2-t^2)^{\frac{\mu-\nu}{2}-1}\,{}^{\mu}T_x^y f(x)\,ydy. \tag{6.44}$$

Let us consider a Cauchy problem

$$(B_\mu)_x u = (B_\nu)_t u + b^2 u, \qquad u = u(x,t;\nu,\mu), \tag{6.45}$$

$$u(x,0;\nu,\mu) = f(x), \qquad u_t(x,0;\nu,\mu) = 0 \tag{6.46}$$

for $0 < \mu < 1$, $0 < \nu < 1$. Applying (6.18) and descent operators (6.14) and (6.15) we obtain that the solution of (6.45)–(6.46) in the case $0 < \mu < \nu$ is

$$u(x,t;\mu,\nu) = \frac{2\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}\right)\Gamma\left(\frac{\mu+1}{2}\right)}\times$$

$$t^{1-\nu}\int\limits_0^t (t^2-y^2)^{\frac{\nu-\mu}{2}-1} j_{\frac{\nu-\mu}{2}-1}\left(b\sqrt{t^2-y^2}\right)\, {}^\mu T_x^y f(x) y^\mu dy \tag{6.47}$$

and in the case $0<\nu<\mu$ it is

$$u(x,t;\mu,\nu)=$$

$$\frac{2\Gamma(\mu-\nu)}{\Gamma^2\left(\frac{\mu-\nu}{2}\right)}\int\limits_t^\infty (y^2-t^2)^{\frac{\mu-\nu}{2}-1} j_{\frac{\mu-\nu}{2}-1}\left(b\sqrt{t^2-y^2}\right)\, {}^\mu T_x^y f(x)\, y dy. \tag{6.48}$$

Differential equations with Bessel operator

7

7.1 General Euler–Poisson–Darboux equation

In this section we find solution representations in the compact integral form to the Cauchy problem for a general form of the Euler–Poisson–Darboux equation with Bessel operators via generalized translation and spherical mean operators for all values of the parameter k, including also exceptional odd negative values, which have not been studied before. We use a Hankel transform method to prove results in a unified way. Under additional conditions we prove that a distributional solution is a classical one too. A transmutation property for the connected generalized spherical mean is proved and the importance of applying transmutation methods for differential equations with Bessel operators is emphasized. The section also contains a short historical introduction on differential equations with Bessel operators and a rather detailed reference list of monographs and papers on mathematical theory and applications of this class of differential equations.

7.1.1 The first Cauchy problem for the general Euler–Poisson–Darboux equation

The classical Euler–Poisson–Darboux equation is defined by

$$\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2},\quad u=u(x,t;k),\quad x\in\mathbb{R}^n,\quad t>0,\quad k\in\mathbb{R}. \tag{7.1}$$

The operator acting by variable t in (7.1) is the **Bessel operator** $(B_k)_t$ (see (1.87)). When $n=1$, Eq. (7.1) appears in Leonard Euler's work (see [128], p. 227) and later was studied by Siméon Denis Poisson in [447], by Gaston Darboux in [77], and by Bernhard Riemann in [471].

For the Cauchy problem initial conditions to the solution of Eq. (7.1) are added:

$$u(x,0;k)=f(x),\qquad \left.\frac{\partial u(x,t;k)}{\partial t}\right|_{t=0}=0. \tag{7.2}$$

Interest in the multi-dimensional equation (7.1) has increased significantly after Alexander Weinstein's papers [593,595,596,598,599]. In [593,595] the Cauchy problem for (7.1) is considered with $k\in\mathbb{R}$, the first initial condition being nonzero and the second initial condition equaling zero. A solution of the Cauchy problem (7.1)–(7.2) in the classical sense was obtained in [595,596,599,602] and in the distributional sense

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in [38,56]. S. A. Tersenov in [564] solved the Cauchy problem for (7.1) in the general form where the first and second conditions are nonzeros. Singular and degenerate hyperbolic equations of one-dimentional EPD-type were considered in [539–541]. Different problems for Eq. (7.1) with many applications to gas dynamics, hydrodynamics, mechanics, elasticity, plasticity, and so on were also studied in [7,32,38,39, 56,61,74,88,96,97,127,140,148–150,159,203,306,383,461,462,539,550,552,553,559, 564,581,602,616]. Of course, this list of references is incomplete.

In this subsection we consider the singular with respect to all variables hyperbolic differential equation, which is a generalization of the multi-dimensional Euler–Poisson–Darboux equation (7.1):

$$\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}=(\Delta_\gamma)_x\, u, \qquad u=u(x,t;k), \quad k\in\mathbb{R}, \quad t>0, \tag{7.3}$$

with the singular elliptic operator defined by (1.88) together with initial conditions

$$u(x,0;k)=f(x), \qquad \lim_{t\to+0} t^k u_t(x,t;k)=g(x).$$

We will call Eq. (7.3) the **Euler–Poisson–Darboux equation in the general form**.

Here we will use an important approach based on the application of transmutation theory. This method is essential in the study of singular problems with the use of special classes of transmutations such as Sonine, Poisson, and Buschman–Erdélyi ones and different forms of fractional integro-differential operators (cf. [51,52,56,228, 230–232,277,524,525,533,535]). Abstract differential equations with Bessel operators were studied in and in fact were mostly initiated by the famous monograph [56] (cf. also papers [182,185,189–193]).

Considering the Cauchy problem (7.2)–(7.3) in more detail, David Fox in [147] (cf. also [56], p. 243, and [559]) proved solution uniqueness for $k\geq 0$ and found a solution representation in the explicit form for all k except odd negative values. The explicit solution was found via Lauricella functions in fact as n-times series, which is not convenient for applications and numerical solving. In all the above references the case $k\neq -1,-3,-5,\ldots$ was expelled and not studied. So in [16,350,506,512] different approaches from those used in [147] to the solution of this Cauchy problem were considered.

We start by finding a solution to the above first Cauchy problem,

$$(B_k)_t u=(\Delta_\gamma)_x u, \quad u=u(x,t;k), \qquad x\in\mathbb{R}^n_+, \quad t>0, \quad k\in\mathbb{R}, \tag{7.4}$$

$$u(x,0;k)=f(x), \qquad u_t(x,0;k)=0 \tag{7.5}$$

in the compact integral form via generalized translation and spherical mean operators for all values of the parameter k, including also exceptional odd negative values, which have not been studied before.

Theorem 75. *Let $f=f(x)\in C^2_{ev}$, $x\in\mathbb{R}^n_+$. Then for the case $k>n+|\gamma|-1$ the unique solution to (7.4)–(7.5) is*

$$u(x,t;k)=\frac{2^n\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\int\limits_{B_1^+(n)}[{}^\gamma\mathbf{T}_x^{ty}f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^\gamma dy$$
$$=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^1(1-r^2)^{\frac{k-n-|\gamma|-1}{2}}M^\gamma_{tr}[f(x)]r^{n+|\gamma|-1}dr. \tag{7.6}$$

The unique solution of the problem (7.4)–(7.5) for $k=n+|\gamma|-1$ is the weighted spherical mean $M_t^\gamma[f(x)]$ (see (3.183)).

Proof. Using Theorem 36 and the property (3.189) we obtain that the weighted spherical mean $M_t^\gamma[f(x)]$ satisfies the general Euler–Poisson–Darboux equation

$$(B_k)_tM_t^\gamma[f(x)]=(\Delta_\gamma)_xM_t^\gamma[f(x)],\quad k=n+|\gamma|-1$$

and initial conditions

$$M_0^\gamma[f(x)]=f(x),\quad M_t^\gamma[f(x)]\Big|_{t=0}=0.$$

It means the weighted spherical mean $M_t^\gamma[f(x)]$ is the solution of the problem (7.4)–(7.5) for $k=n+|\gamma|-1$.

In order to obtain the solution of (7.4)–(7.5) for $k>n+|\gamma|-1$, we will use the method of descent. First, we will seek a solution of the Cauchy problem (7.4)–(7.5) for the case $k>n+|\gamma|$.

Let $\gamma'=(\gamma_1,...,\gamma_n,\gamma'_{n+1})$, $\gamma'_{n+1}>0$, $x'=(x_1,...,x_{n+1})\in\mathbb{R}^{n+1}_+$, and

$$(\Delta_{\gamma'})_{x'}=(B_{\gamma_1})_{x_1}+...+(B_{\gamma_n})_{x_n}+(B_{\gamma'_{n+1}})_{x_{n+1}}.$$

Consider the equation of type (7.4)

$$(B_k)_tu=(\Delta_{\gamma'})_{x'}u,\quad u=u(x',t;k),\quad x'\in\mathbb{R}^{n+1}_+,\quad t>0$$

with the initial conditions

$$u(x',0;k)=f_1(x'),\qquad u_t(x',0;k)=0.$$

When $k=n+|\gamma'|=n+|\gamma|+\gamma'_{n+1}$, the weighted spherical mean $M_t^\gamma[f_1(x')]$ is a solution of this Cauchy problem:

$$u(x',t;k)=$$
$$\frac{1}{|S_1^+(n+1)|_{\gamma'}}\int\limits_{S_1^+(n+1)}[{}^{\gamma_1}T_{x_1}^{ty_1}...{}^{\gamma_n}T_{x_n}^{ty_n}\,{}^{\gamma'_{n+1}}T_{x_{n+1}}^{ty_{n+1}}f_1(x)](y')^{\gamma'}dS_{y'}, \tag{7.7}$$

$$y' = (y_1, ..., y_n, y_{n+1}) \in \mathbb{R}^{n+1}_+,$$

$$|S_1^+(n+1)|_{\gamma'} = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\gamma'_{n+1}+1}{2}\right)}{2^n\Gamma\left(\frac{n+1+|\gamma|+\gamma'_{n+1}}{2}\right)} = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}{2^n\Gamma\left(\frac{k+1}{2}\right)}.$$

Let us put $f_1(x_1, ..., x_n, 0) = f(x_1, ..., x_n)$, where f is the function which appears in initial conditions (7.5). In this way, u defined by (7.7) becomes a function only of $x_1, ..., x_n$ which satisfies Eq. (7.4) and initial conditions (7.5). We have

$$u(x,t;k) = \frac{1}{|S_1^+(n+1)|_{\gamma'}} \int\limits_{S_1^+(n+1)} [{}^\gamma \mathbf{T}_x^{ty} f(x)](y')^{\gamma'} dS_{y'}, \qquad \gamma'_{n+1} = k-n-|\gamma|.$$

Now we rewrite the integral over the part of the sphere $S_1^+(n+1)$ as an integral over the part of the ball $B_1^+(n)=\{y\in\mathbb{R}^n_+ : \sum\limits_{i=1}^n y_i^2 \leq 1\}$. We write the surface integral as a multiple integral:

$$\int\limits_{S_1^+(n+1)} [{}^\gamma \mathbf{T}_x^{ty} f(x)](y')^{\gamma'} dS_{y'} = \int\limits_{B_1^+(n)} [{}^\gamma \mathbf{T}_x^{ty} f(x)](1-y_1^2-...-y_n^2)^{\frac{\gamma'_{n+1}-1}{2}} y^\gamma dy$$
$$= \int\limits_{B_1^+(n)} [{}^\gamma \mathbf{T}_x^{ty} f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy,$$

where $B_1^+(n)$ is a projection of $S_1^+(n+1)$ on the equatorial plane $x_{n+1} = 0$. We have

$$u(x,t;k)=\frac{2^n\Gamma\left(\frac{k+1}{2}\right)}{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)} \int\limits_{B_1^+(n)} [{}^\gamma \mathbf{T}_x^{ty} f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy. \tag{7.8}$$

Although (7.8) was obtained as the solution of the problem (7.4)–(7.5) for the case $k>n+|\gamma|$, the integral on its right side converges for $k > n+|\gamma|-1$. We can verify by direct substitution of (7.8) in (7.4)–(7.5) that (7.8) satisfies the differential equation (7.4) and the initial conditions (7.5) for all values of k which are greater than $(n+|\gamma|-1)$. Let us show this. Changing coordinates from y to y/t and using that $(B_{\gamma_i})_{x_i}\, {}^{\gamma_i}T_{x_i}^{y_i} = (B_{\gamma_i})_{y_i}\, {}^{\gamma_i}T_{x_i}^{y_i}$ (see (3.146)), we obtain

$$I = (\triangle_\gamma)_x \int\limits_{B_1^+(n)} [{}^\gamma \mathbf{T}_x^{ty} f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} y^\gamma dy =$$

$$\sum_{i=1}^{n}(B_{\gamma_i})_{x_i}\int\limits_{B_1^+(n)}[{}^\gamma\mathbf{T}_x^{ty}f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^\gamma dy=$$

$$t^{1-k}\sum_{i=1}^{n}\int\limits_{B_t^+(n)}[(B_{\gamma_i})_{x_i}{}^\gamma\mathbf{T}_x^{y}f(x)](t^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^\gamma dy=$$

$$t^{1-k}\sum_{i=1}^{n}\int\limits_{B_t^+(n)}[(B_{\gamma_i})_{y_i}{}^\gamma\mathbf{T}_x^{y}f(x)](t^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^\gamma dy, \tag{7.9}$$

where $B_t^+(n)=\{y\in\mathbb{R}_+^n: \sum\limits_{i=1}^{n}y_i^2\leq t\}$.

For the functions w, $v\in C_{ev}^2(\overline{\Omega}^+)$ integrable over $\overline{\Omega}^+$, we have formula (1.101). By applying formula (1.101) to the right side of relation (7.9), we get

$$I=t^{1-k}\sum_{i=1}^{n}\int\limits_{S_t^+(n)}\left[\frac{\partial}{\partial y_i}{}^\gamma\mathbf{T}_x^{y}f(x)\right](t^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}\cos(\vec{\nu},\vec{e}_i)\,y^\gamma\,dS,$$

where $\vec{e}_i$ is the direction of the axis Oy_i, $i=1,...,n$, and thus $\cos(\vec{\nu},\vec{e}_i)=\frac{y_i}{t}$. Now, by using the fact that the direction of the outward normal to the boundary of a ball with center the origin coincides with the direction of the position vector of the point on the ball, we obtain the relation

$$I=\frac{1}{t^k}\frac{\partial}{\partial t}t^k\frac{\partial}{\partial t}\int\limits_{B_1^+(n)}\left[{}^\gamma\mathbf{T}_x^{ty}f(x)\right](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^\gamma\,dy.$$

Given that $\frac{1}{t^k}\frac{\partial}{\partial t}t^k\frac{\partial}{\partial t}=(B_k)_t$ and (7.9) we have

$$(\triangle_\gamma)_x\int\limits_{B_1^+(n)}[{}^\gamma\mathbf{T}_x^{ty}f(x)](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^\gamma dy=$$

$$(B_k)_t\int\limits_{B_1^+(n)}\left[{}^\gamma\mathbf{T}_x^{ty}f(x)\right](1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}}y^\gamma\,dy.$$

It means that $u(x,t;k)$ defined by formula (7.8) indeed satisfies Eq. (7.4) for $k>n+|\gamma|-1$. In order to obtain the representation (7.6) it is necessary to use spherical coordinates $y=r\theta$. Validity of the first and second initial conditions follows from the property (3.189) for $M_{tr}^\gamma[f(x)]$ and from the formula

$$\int\limits_0^1(1-r^2)^{\frac{k-n-|\gamma|-1}{2}}r^{n+|\gamma|-1}dr=\frac{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}{2\Gamma\left(\frac{k+1}{2}\right)}. \qquad\square$$

Remark 12. *It is easy to see that the solution of* (7.4)–(7.5) *can be obtained using the first "descent" operator* (6.14) *when* $\mu = k$, $\nu = n+|\gamma|-1$, $\nu < \mu$ *applied by the variable* t*:*

$$u(x,t;k)=\left(T^{(n+|\gamma|-1-k)}_{n+|\gamma|-1,\,k} M^{\gamma}_{t}[f(x)]\right)(t).$$

Theorem 76. *Let* $f = f(x) \in C_{ev}^{\left[\frac{n+|\gamma|-k}{2}\right]+2}$. *Then the solution of* (7.4)–(7.5) *for* $k < n+|\gamma|-1$, $k \neq -1,-3,-5,\ldots$ *is*

$$u(x,t;k) = t^{1-k}\left(\frac{\partial}{t\partial t}\right)^{m}(t^{k+2m-1}u(x,t;k+2m)), \tag{7.10}$$

where m *is a minimum integer such that* $m \geq \frac{n+|\gamma|-k-1}{2}$ *and* $u(x,t;k+2m)$ *is the solution of the Cauchy problem*

$$(B_{k+2m})_t u = (\Delta_\gamma)_x u, \tag{7.11}$$

$$u(x,0;k+2m) = \frac{f(x)}{(k+1)(k+3)\ldots(k+2m-1)}, \qquad u_t(x,0;k+2m) = 0. \tag{7.12}$$

Proof. In order to proof that (7.10) is the solution of (7.4)–(7.5) when $k<n+|\gamma|-1$, $k\neq -1,-3,-5,\ldots$, we will use the recursion formulas (1.109) and (1.110). Let us choose minimum integer m such that $k+2m \geq n+|\gamma|-1$. Now we can write the solution of the Cauchy problem

$$(B_{k+2m})_t u = (\Delta_\gamma)_x u,$$
$$u(x,0;k+2m) = g(x), \qquad u_t(x,0;k+2m) = 0, \quad g \in C^2_{ev}$$

by (7.8). We have

$$u(x,t;k+2m) =$$
$$\frac{2^n\Gamma\left(\frac{k+2m+1}{2}\right)}{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{k+2m-n-|\gamma|+1}{2}\right)}\int\limits_{B_1^+(n)}[{}^\gamma T^{ty}g(x)](1-|y|^2)^{\frac{k+2m-n-|\gamma|-1}{2}}y^\gamma dy,$$

and using (1.109) we obtain

$$t^{k+2m-1}u(x,t;k+2m) = u(x,t;2-k-2m).$$

Applying (1.110) to the last formula m times we get

$$\left(\frac{\partial}{t\partial t}\right)^{m}(t^{k+2m-1}u(x,t;k+2m) = u(x,t;2-k).$$

Applying again (1.109) we can write

$$u(x,t;k) = t^{1-k}\left(\frac{\partial}{t\partial t}\right)^{m}(t^{k+2m-1}u(x,t;k+2m)), \tag{7.13}$$

which gives the solution of (7.41). Now we obtain the function g such that (7.12) is true. From (7.13) we have an asymptotic relation for $t\to 0$,

$$u(x,t;k)=(k+1)(k+3)...(k+2m-1)u(x,t;k+2m)+$$
$$C\,t\,u(x,t;k+2m)+O(t^2),$$

where C is a constant. Therefore, if

$$g(x)=\frac{f(x)}{(k+1)(k+3)...(k+2m-1)},$$

then $u(x,t;k)$ defined by (7.13) satisfies the initial conditions (7.5).

Let us recall that for $u(x,t;k+2m)$ to be a solution of (7.11)–(7.12) it is sufficient that $f\in C^2_{ev}$. In order to be able to carry out the construction (7.13), it is sufficient to require that $f\in C_{ev}^{\left[\frac{n+|\gamma|-k}{2}\right]+2}$. □

Theorem 77. *If f is B–polyharmonic of order $\frac{1-k}{2}$ and even with respect to each variable, then one of the solutions of the Cauchy problem (7.4)–(7.5) for k=$-1,-3,-5,...$ is given by*

$$u(x,t;k)=f(x),\qquad k=-1,\tag{7.14}$$

$$u(x,t;k)=f(x)+\sum_{h=1}^{-\frac{k+1}{2}}\frac{\Delta_\gamma^h f}{(k+1)...(k+2h-1)}\,\frac{t^{2h}}{2\cdot 4\cdot ....\cdot 2h},\quad k=-3,-5,....\tag{7.15}$$

Proof. Let us first take $k=-1$ and assume that $\lim\limits_{t\to 0}\frac{\partial^2 u(x,t;-1)}{\partial t^2}$ exists. Let $t\to 0$ in

$$(\Delta_\gamma)_x u^{-1}(x,t;k)=\frac{\partial^2 u(x,t;-1)}{\partial t^2}-\frac{1}{t}\frac{\partial u(x,t;-1)}{\partial t},$$

i.e.,

$$(\Delta_\gamma)_x u(x,0;-1)=\lim_{t\to 0}\frac{\partial^2 u(x,t;-1)}{\partial t^2}-\lim_{t\to 0}\frac{1}{t}\frac{\partial u(x,t;-1)}{\partial t}=0.$$

We find that $(\Delta_\gamma)_x u(x,0;-1)=0$, which shows that f must be B-harmonic. So the function f satisfies (7.11)–(7.12) for k=-1.

When $k=-3$ we have

$$\lim_{t\to 0}\frac{\partial^2 u(x,t;-3)}{\partial t^2}=\lim_{t\to 0}\frac{1}{t}\frac{\partial u(x,t;-3)}{\partial t}.$$

From the general form of the Euler–Poisson–Darboux equation for k=-3 we obtain

$$\lim_{t\to 0}(\Delta_\gamma)_x u(x,t;-3)=\lim_{t\to 0}\frac{\partial^2 u(x,t;-3)}{\partial t^2}-3\lim_{t\to 0}\frac{1}{t}\frac{\partial u(x,t;-3)}{\partial t}$$
$$=-2\lim_{t\to 0}\frac{1}{t}\frac{\partial u(x,t;-3)}{\partial t}.$$

It follows from (1.110) that

$$\frac{1}{t}\frac{\partial u(x,t;-3)}{\partial t} = u(x,t;-1),$$

and hence

$$\lim_{t\to 0}(\Delta_\gamma)_x u(x,t;-3) = -2u(x,0;-1). \tag{7.16}$$

If the limit $\lim\limits_{t\to 0}\frac{\partial^4 u(x,t;-3)}{\partial t^4}$ exists and all odd derivatives of $u(x,t;-3)$ tend to zero when $t\to 0$, then $\lim\limits_{t\to 0}\frac{\partial^2 u(x,t;-1)}{\partial t^2}$ also exists. Therefore,

$$\lim_{t\to 0}(\Delta_\gamma)_x u(x,t;-1) = 0$$

and by (7.16) we have $\lim\limits_{t\to 0}(\Delta_\gamma)_x^2 u(x,t;-3) = 0$. This remark can easily be generalized to include all the exceptional values. So, in this case a solution of the Cauchy problem for the general form of the Euler–Poisson–Darboux equation for the case k=-3, -5, ... is given by the formula

$$u(x,t;k) = f(x) + \sum_{h=1}^{-\frac{k+1}{2}} \frac{\Delta_\gamma^h f}{(k+1)...(k+2h-1)}\,\frac{t^{2h}}{2\cdot 4\cdot ....\cdot 2h}, \qquad k=-3,-5,...,$$

and as we proved earlier $u(x,t;-1) = f(x)$. □

7.1.2 The second Cauchy problem for the general Euler–Poisson–Darboux equation

Now let us consider the second Cauchy problem,

$$(B_k)_t u = (\Delta_\gamma)_x u, \quad u = u(x,t;k), \qquad x\in\mathbb{R}_+^n, \quad t>0, \quad k\in\mathbb{R}, \tag{7.17}$$

$$u(x,0;k) = 0, \qquad \lim_{t\to +0} t^k u_t(x,t;k) = g(x). \tag{7.18}$$

Theorem 78. *If* $g = g(x) \in C_{ev}^{\left[\frac{n+|\gamma|+k-1}{2}\right]}$*, then the solution* $u = u(x,t;k)$ *of* (7.17)–(7.18) *for* $k<1$ *is given by*

$$u(x,t;k) = \frac{\Gamma\left(\frac{3-k+2q}{2}\right)\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{3-k+2q-n-|\gamma|}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)} \sum_{s=0}^{q} \frac{C_q^s\, t^{1-k+2s}}{2^s\,\Gamma\left(\frac{3-k}{2}+s\right)}\times$$

$$\int_0^1 (1-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}} r^{n+|\gamma|-1}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_{tr}^\gamma g(x)dr \tag{7.19}$$

if $n+|\gamma|+k$ is not an odd integer and

$$u(x,t;k)=\sum_{s=0}^{q}\frac{C_q^s\,\Gamma\left(\frac{1-k}{2}\right)}{2^{s+1}\Gamma\left(\frac{3-k}{2}+s\right)}\,t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_t^\gamma g(x) \tag{7.20}$$

if $n+|\gamma|+k$ is an odd integer, where $q\geq 0$ is the smallest positive integer such that $2-k+2q\geq n+|\gamma|-1$.

Proof. Let $q\geq 0$ be the smallest positive integer number such that $2-k+2q\geq n+|\gamma|-1$, i.e., $q=\left[\frac{n+|\gamma|+k-1}{2}\right]$, and let $u(x,t;2-k+2q)$ be a solution of (7.17) when we take $2-k+2q$ instead of k such that

$$u(x,0;2-k+2q)=g(x),\qquad u_t(x,0;2-k+2q)=0. \tag{7.21}$$

By the recurrent formula (1.109) we obtain that

$$u(x,t;k-2q)=t^{1-k+2q}u(x,t;2-k+2q)$$

is a solution of the equation

$$(\triangle_\gamma)_x u=\frac{\partial^2 u}{\partial t^2}+\frac{k-2q}{t}\frac{\partial u}{\partial t}.$$

Further, applying q times the formula (1.110) we obtain that

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q u(x,t;k-2q)=\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q (t^{1-k+2q}u(x,t;2-k+2q))$$

is a solution of (7.17).

Let us consider

$$u(x,t;k)=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q (t^{1-k+2q}u(x,t;2-k+2q)). \tag{7.22}$$

We have shown that (7.22) satisfies Eq. (7.17).

Now we will prove that $u(x,t;k)$ given by (7.22) satisfies the conditions (7.21). For $u\in C_{ev}^q(\Omega_+)$ we have the formula (see [564], p. 9)

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q (t^{1-k+2q}u(x,t;2-k+2q))=$$
$$\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{1-k}{2}+s+1\right)}\,t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s u(x,t;2-k+2q). \tag{7.23}$$

Taking into account formula (7.23) we obtain $u(x,0;k)=0$ and

$$\lim_{t\to 0} t^k u_t(x,t;k) =$$
$$\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)} \lim_{t\to 0} t^k \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q (t^{1-k+2q}u(x,t;2-k+2q)) =$$
$$\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)} \lim_{t\to 0} t^k \frac{\partial}{\partial t}\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{1-k}{2}+s+1\right)}\times$$
$$t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s u(x,t;2-k+2q) =$$
$$\frac{1}{1-k}\lim_{t\to 0} t^k \frac{\partial}{\partial t}\left(t^{1-k}u(x,t;2-k+2q)\right) =$$
$$\frac{1}{1-k}\lim_{t\to 0} t^k \left((1-k)t^{-k}u(x,t;2-k+2q)+t^{1-k}u_t(x,t;2-k+2q)\right) =$$
$$\frac{1}{1-k}\lim_{t\to 0}((1-k)u(x,t;2-k+2q)+tu_t(x,t;2-k+2q)) = g(x).$$

Now let us obtain the representation of $u(x,t;k)$ through the integral. Using formula (7.6) we get

$$u(x,t;2-k+2q) =$$
$$\frac{2\Gamma\left(\frac{3-k+2q}{2}\right)}{\Gamma\left(\frac{3-k+2q-n-|\gamma|}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int_0^1 (1-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}} r^{n+|\gamma|-1} M_{tr}^{\gamma} g(x)dr.$$

If $2-k+2q > n+|\gamma|-1$, then applying (7.22) and (7.23) we obtain

$$u(x,t;k) = \frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\times$$
$$\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{3-k}{2}+s\right)} t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s u(x,t;2-k+2q) =$$
$$\frac{\Gamma\left(\frac{3-k}{2}\right)}{1-k}\sum_{s=0}^{q}\frac{C_q^s\, t^{1-k+2s}}{2^s\Gamma\left(\frac{3-k}{2}+s\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s u(x,t;2-k+2q) =$$
$$\frac{\Gamma\left(\frac{3-k+2q}{2}\right)\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{3-k+2q-n-|\gamma|}{2}\right)\Gamma\left(\frac{n+|\gamma|}{2}\right)}\sum_{s=0}^{q}\frac{C_q^s\, t^{1-k+2s}}{2^s\Gamma\left(\frac{3-k}{2}+s\right)}\times$$

$$\int_0^1 (1-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}} r^{n+|\gamma|-1} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_{tr}^\gamma g(x) dr.$$

If $2-k+2q = n+|\gamma|-1$, then $u(x,t;2-k+2q) = M_t^\gamma g(x)$ and

$$u(x,t;k) = \frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q \left(t^{n+|\gamma|-2} M_t^\gamma g(x)\right) =$$

$$\frac{2^{-1-q}\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{3-k+2q}{2}\right)} \sum_{s=0}^q \frac{2^{q-s} C_q^s \Gamma\left(\frac{3-k}{2}+q\right)}{\Gamma\left(\frac{3-k}{2}+s\right)} t^{1-k+2s} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_t^\gamma g(x) =$$

$$\sum_{s=0}^q \frac{C_q^s \Gamma\left(\frac{1-k}{2}\right)}{2^{s+1}\Gamma\left(\frac{3-k}{2}+s\right)} t^{1-k+2s} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^s M_t^\gamma g(x).$$

This completes the proof. □

Remark 13. *A solution of the problem close to* (7.4)–(7.5) *was obtained in [147] (see also [56], p. 243) when* $k \neq -1,-3,-5,...$ *in terms of the Lauricella function* (1.37). *More precisely, in [147] the solution of the problem*

$$\frac{\partial^2 v}{\partial t^2} + \frac{k}{t}\frac{\partial v}{\partial t} - \sum_{i=1}^n \left(\frac{\partial^2 v}{\partial x_i^2} + \frac{\lambda_i}{x_i^2} v\right) = 0, \qquad v = v(t,x), \tag{7.24}$$

$$v(0,x) = T(x), \qquad \left.\frac{\partial v}{\partial t}\right|_{t=0} = 0 \tag{7.25}$$

has the form

$$v(t,x) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{n}{2}}\Gamma\left(\frac{k-n+1}{2}\right)} \int\limits_{|x-\xi|=|t|} |t|^{1-k}(t^2-|x-\xi|^2)^{\frac{k-n-1}{2}} T(\xi) \times$$

$$\mathbf{F}_\gamma^{(n)}\left(a_1,...,a_n,b_1,...,b_n;\frac{k-n+1}{2};z_1,...,z_n\right) dS_\xi, \tag{7.26}$$

where

$$a_1 = \frac{1+\sqrt{1-4\lambda_1}}{2}, ..., a_n \frac{1+\sqrt{1-4\lambda_n}}{2},$$

$$b_1 = \frac{1-\sqrt{1-4\lambda_1}}{2}, ..., b_n = \frac{1-\sqrt{1-4\lambda_n}}{2},$$

$$z_1 = \frac{t^2-|x-\xi|^2}{2x_1\xi_1}, ..., z_n = \frac{t^2-|x-\xi|^2}{2x_n\xi_n}.$$

If $\lambda_k = \frac{\gamma_k}{2}\left(1-\frac{\gamma_k}{2}\right)$, $i=1,...,n$, *and* $u = x^{\frac{\gamma}{2}} v = x_1^{\frac{\gamma_1}{2}} ... x_n^{\frac{\gamma_n}{2}} v$, *then we get our problem (7.79)–(7.80). As we see, expression (7.81) gives a much more convenient formula for solving the problem (7.79)–(7.80). Also note that in [513] in two different ways including* $k=-1,-3,-5,....$

Now we concentrate on the case when x is one-dimensional. Then problems and solutions constructed above are simplified. For these problems we consider below some illustrative examples with explicit solution representations (see Section 7.1.4).

In this case we have the first Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\gamma}{x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t}, \tag{7.27}$$

$$u(x,0;k) = f(x), \qquad \left.\frac{\partial u(x,t;k)}{\partial t}\right|_{t=0} = 0, \qquad f(x) \in C^2_{ev}. \tag{7.28}$$

When $k > \gamma > 0$ the solution of (7.27)–(7.28) is given by the formula (see (7.8))

$$u(x,t;k) = \frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)} \int\limits_0^1 (1-y^2)^{\frac{k-\gamma-2}{2}}\, {}^\gamma T_x^{ty} f(x) y^\gamma dy. \tag{7.29}$$

When $k < \gamma$ the solution of (7.27)–(7.28) is found by formula (7.13), (7.14), or (7.15).

As for the second Cauchy problem for $k < 1$, $\gamma > 0$, and $2-k > \gamma$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\gamma}{x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t}, \tag{7.30}$$

$$\lim_{t\to 0} u(x,t;k) = 0, \qquad \lim_{t\to 0} t^k \frac{\partial u(x,t;k)}{\partial t} = g(x), \qquad g(x) \in C_{ev}^{\left[\frac{k+\gamma}{2}\right]}. \tag{7.31}$$

The condition $2-k > \gamma$ means that we can take $q=0$ in (7.19). The solution of (7.30)–(7.31) for $2-k > \gamma$ is defined by

$$u(x,t;k) = \frac{\Gamma\left(\frac{1-k}{2}\right) t^{1-k}}{\Gamma\left(\frac{2-k-\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)} \int\limits_0^1 (1-r^2)^{-\frac{k+\gamma}{2}}\, {}^\gamma T_x^{tr} f(x) r^\gamma dr.$$

7.1.3 The singular Cauchy problem for the generalized homogeneous Euler–Poisson–Darboux equation

In this subsection, we solve the singular Cauchy problem for a generalized form of a homogeneous Euler–Poisson–Darboux equation with constant potential, where the Bessel operator acts instead of each second derivative. In the classical formulation, the Cauchy problem for this equation is not correct. However, S. A. Tersenov [564] observed that, considering the form of a general solution of the classical Euler–Poisson–Darboux equation, the derivative in the second initial condition must be multiplied by

a power function whose degree is equal to the index of the Bessel operator acting on the time variable. The first initial condition remains in the usual formulation. With the chosen form of the initial conditions, the considering equation has a solution. The obtained solution is represented as the sum of two terms. The first term is an integral containing the normalized Bessel function and the weighted spherical mean. The second term is expressed in terms of the derivative of the square of the time variable from the integral, which is similar in structure to the first term.

We study the initial value problem

$$Lu = \left[\sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right)-\left(\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}\right)\right]u = c^2 u, \tag{7.32}$$

$$u(x,0;k)=\varphi(x), \qquad \lim_{t\to+0} t^k u_t(x,t;k)=\psi(x), \qquad u=u(x,t;k), \tag{7.33}$$

where $\gamma_i>0$, $x_i>0$, $i=1,...,n$, $k\in\mathbb{R}$, $t>0$. Eq. (7.32) is called the **generalized Euler–Poisson–Darboux equation**.

Using the terminology from the book [56], a problem for the equation of the type

$$A(t)\frac{\partial^2 u}{\partial t^2}+B(t)\frac{\partial u}{\partial t}+C(t)u = Gu, \qquad u=u(t,x), \qquad x=(x_1,...,x_n),$$

where G is a linear operator, acting only by variables $x_1,...,x_n$, is called **singular** if at least one of the operator coefficients tends to infinity in some sense as $t\to 0$.

In [56] five general techniques were given for the solution of the singular Cauchy problem

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}=\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t}, \qquad u=u(x,t;k), \tag{7.34}$$

$$u(x,0;k)=\varphi(x), \quad u_t(x,0;k)=0. \tag{7.35}$$

These methods are:

1. Fourier transform method in a distribution space,
2. spectral technique in a Hilbert space,
3. transmutation method,
4. studying related simpler differential equations, and
5. energy methods.

Some of these methods were successfully applied to the generalized Euler–Poisson–Darboux equation (7.32) and to $Lu=0$ in other papers. Namely, using the Hankel transform instead of Fourier solutions to $Lu=0$, (7.32) with conditions (7.35) was obtained in [514] and [508], accordingly. The third and the closely connected forth method were used to solve $Lu=0$ in [514] and [509]. In [532] a transmutation method was used for obtaining new integral initial conditions for the Euler–Poisson–Darboux equation (7.34). Abstract differential equations with Bessel operator of Euler–Poisson–Darboux type were studied in [56] (cf. also [184]). In [577] the problem (7.34)–(7.33) was solved using "descent" operators, which are special cases of

Buschman–Erdélyi transmutation operators (see [533,535]). Here as a main result we obtain a solution to the problem (7.32)–(7.33).

We will be concerned with the solutions of the following singular initial value hyperbolic problem:

$$Lu = \left[\sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i} \right) - \left(\frac{\partial^2}{\partial t^2} + \frac{k}{t}\frac{\partial}{\partial t} \right) \right] u = c^2 u, \tag{7.36}$$

$$u(x,0;k) = \varphi(x), \qquad \lim_{t\to+0} t^k u_t(x,t;k) = \psi(x), \qquad u = u(x,t;k).$$

We will call (7.36) the **generalized Euler–Poisson–Darboux equation**.

In [508] the following theorem was proved.

Theorem 79. *The solution $u \in C^2_{ev}(\mathbb{R}^{n+1}_+)$ to*

$$\left[(\triangle_\gamma)_x - (B_k)_t\right] u = c^2 u, \qquad c > 0, \qquad u = u(x,t;k), \tag{7.37}$$

$$u(x,0;k) = \varphi(x), \qquad u_t(x,0;k) = 0 \tag{7.38}$$

for $k > n + |\gamma| - 1$ is unique and defined by the formula

$$u(x,t;k) = A(n,\gamma,k)\, t^{1-k} \int\limits_0^t (t^2 - r^2)^{\frac{k-n-|\gamma|-1}{2}}\, j_{\frac{k-n-|\gamma|-1}{2}} \left(c\sqrt{t^2 - r^2} \right) \times$$

$$r^{n+|\gamma|-1} M_r^\gamma [\varphi(x)] dr, \tag{7.39}$$

where

$$A(n,\gamma,k) = \frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}.$$

It is proved in Chapter 1 that if $u(x,t;k)$ is a solution to (7.37), then the following two fundamental recursion formulas (1.109) and (1.110) hold:

$$u(x,t;k) = t^{1-k} u(x,t;2-k), \qquad u(x,t;k)_t = t u(x,t;2+k).$$

Theorem 80. *Let $\varphi = \varphi(x)$, $\varphi \in C_{ev}^{\left[\frac{n+|\gamma|-k}{2}\right]+2}$. Then the solution of (7.37)–(7.38) for $k \le n + |\gamma| - 1$, $k \ne -1, -3, -5, \ldots$, is*

$$u(x,t;k) = t^{1-k} \left(\frac{\partial}{t\,\partial t} \right)^m \left(t^{k+2m-1} u(x,t;k+2m) \right), \tag{7.40}$$

where m is a minimum integer such that $m \ge \frac{n+|\gamma|-k-1}{2}$ and $u(x,t;k+2m)$ is the solution of the Cauchy problem

$$\left[(\triangle_\gamma)_x - (B_{k+2m})_t\right] u = c^2 u, \qquad c > 0, \tag{7.41}$$

$$u(x,0;k+2m)=\frac{\varphi(x)}{(k+1)(k+3)...(k+2m-1)},\qquad u_t(x,0;k+2m)=0. \tag{7.42}$$

Proof. In order to proof that (7.40) is a solution of (7.37)–(7.38) when $k{\leq}n+|\gamma|{-}1$, $k{\neq}{-}1,-3,-5,...$, we will use the recursion formulas (1.109) and (1.110). Let us choose the minimum integer m such that $k+2m>n+|\gamma|-1$. Now we can write the solution of the Cauchy problem

$$\begin{aligned}
&\left[(\triangle_\gamma)_x-(B_{k+2m})_t\right]u=c^2u,\qquad c>0,\\
&u(x,0;k+2m)=g(x),\qquad u_t(x,0;k+2m)=0,\quad g\in C^2_{ev}
\end{aligned}$$

by (7.39). We have

$$\begin{aligned}
u(x,t;k+2m)=A(n,\gamma,k+2m)\,t^{1-k-2m}\int\limits_0^t(t^2-r^2)^{\frac{k+2m-n-|\gamma|-1}{2}}\\
\times\, j_{\frac{k+2m-n-|\gamma|-1}{2}}\left(c\sqrt{t^2-r^2}\right)r^{n+|\gamma|-1}M_r^\gamma[g(x)]dr,
\end{aligned}$$

where

$$A(n,\gamma,k+2m)=\frac{2\Gamma\left(\frac{k+2m+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k+2m-n-|\gamma|+1}{2}\right)}.$$

Considering (1.109), it is easy to see that

$$t^{k+2m-1}u(x,t;k+2m)=u(x,t;2-k-2m).$$

Applying (1.110) to the last formula m times we get

$$\left(\frac{\partial}{t\partial t}\right)^m(t^{k+2m-1}u(x,t;k+2m)=u(x,t;2-k).$$

Applying again (1.109) we can write

$$u(x,t;k)=t^{1-k}\left(\frac{\partial}{t\partial t}\right)^m\left(t^{k+2m-1}u(x,t;k+2m)\right), \tag{7.43}$$

which gives the solution of (7.41). Now we obtain the function g such that (7.42) is true. From (7.43) it follows that

$$\begin{aligned}
u(x,t;k)=(k+1)(k+3)...(k+2m-1)u(x,t;k+2m)\\
+C\,t\,u(x,t;k+2m)+O(t^2),
\end{aligned}$$

when $t\to 0$, where C is a constant. Evidently, if

$$g(x) = \frac{\varphi(x)}{(k+1)(k+3)...(k+2m-1)},$$

then $u(x,t;k)$ defined by (7.40) satisfies the initial conditions (7.38).

Let us recall that for $u(x,t;k+2m)$ to be a solution of (7.41)–(7.42) it is sufficient that $f \in C^2_{ev}$. In order to be able to carry out the construction (7.43), it is sufficient to require that $f \in C_{ev}^{\left[\frac{n+|\gamma|-k}{2}\right]+2}$. □

Theorem 81. *Let $\psi \in C_{ev}^{\left[\frac{n+|\gamma|+k-1}{2}\right]}$. The solution $u = u(x,t;k)$ to*

$$\left[(\triangle_\gamma)_x - (B_k)_t\right] u = c^2 u, \qquad c > 0, \tag{7.44}$$

$$u(x,0;k) = 0, \qquad \lim_{t\to+0} t^k u_t(x,t;k) = \psi(x) \tag{7.45}$$

for $k < 1$ is defined by the formula

$$u(t,x;k) = B(n,\gamma,k,q)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q \left(\int\limits_0^t (t^2-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}} \times \right.$$

$$\left. j_{\frac{1-k+2q-n-|\gamma|}{2}}\left(c\sqrt{t^2-r^2}\right) r^{n+|\gamma|-1} M_r^\gamma[\psi(x)]dr\right), \tag{7.46}$$

where

$$B(n,\gamma,k,q) = \frac{2^{-q}\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{2-k+2q-n-|\gamma|+1}{2}\right)}.$$

Proof. Let $q \geq 0$ be the smallest positive integer such that $2-k+2q > n+|\gamma|-1$, i.e., $q > \frac{n+|\gamma|+k-3}{2}$, and let $u(x,t;2-k+2q)$ be a solution to (7.44) when we take $2-k+2q$ instead of k such that

$$u(x,0;2-k+2q) = \psi(x), \qquad u_t(x,0;2-k+2q) = 0. \tag{7.47}$$

By property (1.109) we obtain that

$$u(t,x;k-2q) = t^{1-k+2q} u(t,x;2-k+2q)$$

is a solution to the equation

$$(\triangle_\gamma)_x u - \frac{\partial^2 v}{\partial t^2} - \frac{k-2q}{t}\frac{\partial v}{\partial t} = c^2 u.$$

Further, applying q times formula (1.110) we obtain that

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q u(t,x;k-2q) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q \left(t^{1-k+2q} u(t,x;2-k+2q)\right)$$

is a solution to (7.44). In order to get a solution to (7.44) satisfying the conditions (7.45) we use the multiplier $\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}$. Let

$$u(t,x;k)=\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{q}\left(t^{1-k+2q}u(t,x;2-k+2q)\right). \tag{7.48}$$

We have shown that (7.48) satisfies Eq. (7.44). Now we will prove that $u(t,x;k)$ satisfies the conditions (7.47). Using formula (1.13) from [564], p. 9, we obtain

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{q}\left(t^{1-k+2q}u(t,x;2-k+2q)\right)=$$
$$\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{1-k}{2}+s+1\right)}t^{1-k+2s}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{s}u(t,x;2-k+2q),$$

and $u(0,x;k)=0$ for $k<1$. For the second condition in (7.47) we get

$$\lim_{t\to 0}t^{k}u_t(t,x;k)=$$
$$\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\lim_{t\to 0}t^{k}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{q}\left(t^{1-k+2q}u(t,x;2-k+2q)\right)=$$
$$\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\lim_{t\to 0}t^{k}\frac{\partial}{\partial t}\sum_{s=0}^{q}\frac{2^{q-s}C_q^s\Gamma\left(\frac{1-k}{2}+q+1\right)}{\Gamma\left(\frac{1-k}{2}+s+1\right)}t^{1-k+2s}\times$$
$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{s}u(t,x;2-k+2q)=$$
$$\frac{1}{1-k}\lim_{t\to 0}t^{k}\frac{\partial}{\partial t}\left(t^{1-k}u(t,x;2-k+2q)\right)=\frac{1}{1-k}\lim_{t\to 0}t^{k}\times$$
$$\left((1-k)t^{-k}u(t,x;2-k+2q)+t^{1-k}u_t(t,x;2-k+2q)\right)=$$
$$\frac{1}{1-k}\lim_{t\to 0}\left((1-k)u(t,x;2-k+2q)+tu_t(t,x;2-k+2q)\right)=$$
$$\lim_{t\to 0}u(t,x;2-k+2q)=\psi(x).$$

Now we write the representation of $u(t,x;k)$ through the integral. Using formula (7.39) we get

$$u(x,t;2-k+2q)=A(n,\gamma,2-k+2q)\,t^{k-1-2q}\int\limits_0^t(t^2-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}}\times$$

$$j_{\frac{1-k+2q-n-|\gamma|}{2}}\left(c\sqrt{t^2-r^2}\right) r^{n+|\gamma|-1} M_r^{\gamma}[\psi(x)]dr.$$

Considering (7.48) we write

$$u(t,x;k)=A(n,\gamma,2-k+2q)\,\frac{2^{-q}\Gamma\left(\frac{3-k}{2}\right)}{(1-k)\Gamma\left(\frac{3-k+2q}{2}\right)}\times$$

$$\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^q\int\limits_0^t (t^2-r^2)^{\frac{1-k+2q-n-|\gamma|}{2}} j_{\frac{1-k+2q-n-|\gamma|}{2}}\left(c\sqrt{t^2-r^2}\right) r^{n+|\gamma|-1} M_r^{\gamma}[\psi(x)]dr.$$

Simplifying we get (7.46), and this completes the proof. □

The union of Theorems 80 and 81 gives the following statement.

Theorem 82. *Let $\varphi=\varphi(x)$, $\varphi\in C_{ev}^{\left[\frac{n+|\gamma|-k}{2}\right]+2}$, $\psi=\psi(x)$, $\psi\in C_{ev}^{\left[\frac{n+|\gamma|+k-1}{2}\right]}$. Then the solution of*

$$\left[(\triangle_\gamma)_x-(B_k)_t\right]u=c^2u,\qquad c>0, \tag{7.49}$$

$$u(x,0;k)=\varphi(x),\qquad \lim_{t\to+0} t^k u_t(x,t;k)=\psi(x), \tag{7.50}$$

for $k\le\min\{n+|\gamma|-1,1\}$, $k\ne-1,-3,-5,...$, is given by the formula

$$u(x,t;k)=u_1(x,t;k)+u_2(x,t;k),$$

where $u_1(x,t;k)$ is found by Theorem 80 and $u_1(x,t;k)$ is found by Theorem 81.

7.1.4 Examples

Example 1. *We are looking for the solution of*

$$\frac{\partial^2 u}{\partial x^2}+\frac{\gamma}{x}\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial t^2}+\frac{k}{t}\frac{\partial u}{\partial t},$$

$$u(x,0;k)=j_{\frac{\gamma-1}{2}}(x),\qquad u_t(x,0;k)=0.$$

1) For $k>\gamma>0$ using (7.29) we obtain

$$u(x,t;k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)}\int\limits_0^1 (1-y^2)^{\frac{k-\gamma-2}{2}}\,{}^{\gamma}T_x^{ty} j_{\frac{\gamma-1}{2}}(x)y^{\gamma}dy.$$

Using (3.152) and formula (2.12.4.6) from [456] of the form

$$\int\limits_0^a x^{\nu+1}(a^2-x^2)^{\beta-1}J_\nu(cx)dx=\frac{2^{\beta-1}a^{\beta+\nu}}{c^\beta}\Gamma(\beta)J_{\beta+\nu}(ac), \tag{7.51}$$

$$a > 0, \qquad \operatorname{Re}\beta > 0, \qquad \operatorname{Re}\nu > -1,$$

we obtain

$$u(x,t;k) = j_{\frac{\gamma-1}{2}}(x)\, t^{\frac{1-\gamma}{2}} \frac{2^{\frac{\gamma+1}{2}}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-\gamma}{2}\right)} \int\limits_0^1 (1-y^2)^{\frac{k-\gamma-2}{2}} J_{\frac{\gamma-1}{2}}(ty)\, y^{\frac{\gamma+1}{2}} dy,$$

$$u(x,t;k) = j_{\frac{\gamma-1}{2}}(x)\, j_{\frac{k-1}{2}}(t). \tag{7.52}$$

Example 2. *The solution for all $k \in \mathbb{R}$ such that $k \neq -1, -3, -5, \ldots$ is given by* (7.13) *and it is easy to check that*

$$u(x,t;k) = j_{\frac{\gamma-1}{2}}(x)\, j_{\frac{k-1}{2}}(t).$$

Example 3. *When $k = -1, -3, -5, \ldots$, we have*

$$u(x,t;-1) = j_{\frac{\gamma-1}{2}}(x)$$

and for $k = -3, -5, \ldots$

$$u(x,t;k) = j_{\frac{\gamma-1}{2}}(x) + \sum_{h=1}^{-\frac{k+1}{2}} \frac{B_\gamma^h j_{\frac{\gamma-1}{2}}(x)}{(k+1)\ldots(k+2h-1)} \frac{t^{2h}}{2\cdot 4\cdot \ldots \cdot 2h}$$

$$= j_{\frac{\gamma-1}{2}}(x) \left(1 + \sum_{h=1}^{-\frac{k+1}{2}} \frac{(-1)^h}{(k+1)\ldots(k+2h-1)} \frac{t^{2h}}{2\cdot 4\cdot \ldots \cdot 2h}\right).$$

Example 4. *Now let us find a solution to*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\gamma}{x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t},$$

$$\lim_{t\to 0} u(x,t;k) = 0, \qquad \lim_{t\to +0} t^k\, u_t(x,t;k) = j_{\frac{\gamma-1}{2}}(x).$$

For $k < 1$ we have

$$u(x,t;k) = \frac{2\Gamma\left(\frac{3-k}{2}\right) t^{1-k}\, j_{\frac{\gamma-1}{2}}(x)}{(1-k)\Gamma\left(\frac{2-k-\gamma}{2}\right)\Gamma\left(\frac{\gamma+1}{2}\right)} \int\limits_0^1 (1-\xi^2)^{-\frac{k+\gamma}{2}} j_{\frac{\gamma-1}{2}}(t\xi)\xi^\gamma d\xi$$

$$= \frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(x)\, j_{\frac{1-k}{2}}(t). \tag{7.53}$$

Example 5. *Let us consider an example*

$$\left[(B_\gamma)_x - (B_k)_t\right] u = c^2 u, \qquad c > 0, \qquad u = u(t,x;k), \qquad (t,x) \in \mathbb{R}^2_+, \tag{7.54}$$

$$u(x,0;k)=j_\gamma(ax), \quad u_t(x,0;k)=0, \tag{7.55}$$

where $n{=}1$, $\gamma-2\leq k\leq\gamma$, $k\neq-1$, $0<\gamma$, $a\in\mathbb{R}$. When $m{=}1$ and considering that (see (3.144))

$$^\gamma T_x^r\, j_\gamma(ax)=j_{\frac{\gamma-1}{2}}(ax)\,j_{\frac{\gamma-1}{2}}(ar), \tag{7.56}$$

we obtain

$$u(x,t;k+2)=\frac{A(1,\gamma,k+2)}{k+1}\,j_{\frac{\gamma-1}{2}}(ax)\,t^{-k-1}\int\limits_0^t (t^2-r^2)^{\frac{k-\gamma}{2}}\times$$
$$j_{\frac{k-\gamma}{2}}\left(c\sqrt{t^2-r^2}\right) r^\gamma\, j_{\frac{\gamma-1}{2}}(ar)dr,$$

where

$$A(1,\gamma,k+2)=\frac{2\Gamma\left(\frac{k+3}{2}\right)}{\Gamma\left(\frac{1+\gamma}{2}\right)\Gamma\left(\frac{k+2-\gamma}{2}\right)}.$$

Passing to the functions J_ν in the integral we get

$$u(x,t;k+2)=$$
$$\frac{2^{\frac{k-1}{2}}A(1,\gamma,k+2)\Gamma\left(\frac{1+\gamma}{2}\right)\Gamma\left(1+\frac{k-\gamma}{2}\right)}{(k+1)a^{\frac{\gamma-1}{2}}c^{\frac{k-\gamma}{2}}}\,j_{\frac{\gamma-1}{2}}(ax)\,t^{-k-1}\int\limits_0^t (t^2-r^2)^{\frac{k-\gamma}{4}}\times$$
$$J_{\frac{k-\gamma}{2}}\left(c\sqrt{t^2-r^2}\right) r^{\frac{\gamma-1}{2}+1} J_{\frac{\gamma-1}{2}}(ar)dr.$$

Applying formula (2.12.35.2) from [456] of the form

$$\int\limits_0^t (t^2-x^2)^{m+\frac{\mu}{2}}x^{\nu+1+2l}J_\mu(c\sqrt{t^2-x^2})J_\nu(hx)dx=t^{\mu+\nu-m-l+1}c^\mu h^\nu\times$$
$$\left(\frac{\partial}{c\partial c}\right)^m\left(\frac{\partial}{h\partial h}\right)^l[(c^2+h^2)^{-\frac{\mu+\nu+m+l+1}{2}}J_{\mu+\nu+m+l+1}(t\sqrt{c^2+h^2})], \tag{7.57}$$
$$t>0, \qquad \operatorname{Re}\nu>-l-1, \qquad \operatorname{Re}\mu>-m-1,$$

we obtain

$$\int\limits_0^t (t^2-r^2)^{\frac{k-\gamma}{4}}J_{\frac{k-\gamma}{2}}\left(c\sqrt{t^2-r^2}\right) r^{\frac{\gamma-1}{2}+1}J_{\frac{\gamma-1}{2}}(ar)dr=$$
$$t^{\frac{k+1}{2}}a^{\frac{\gamma-1}{2}}c^{\frac{k-\gamma}{2}}(\sqrt{a^2+c^2})^{-\frac{k+1}{2}}J_{\frac{k+1}{2}}\left(t\sqrt{a^2+c^2}\right)$$

and

$$u(x,t;k+2)=$$
$$\frac{2^{\frac{k+1}{2}}\Gamma\left(\frac{k+3}{2}\right)}{k+1}\, j_{\frac{\gamma-1}{2}}(ax)\, t^{-\frac{k+1}{2}}(\sqrt{a^2+c^2})^{-\frac{k+1}{2}} J_{\frac{k+1}{2}}\left(t\sqrt{a^2+c^2}\right)=$$
$$\frac{1}{k+1} j_{\frac{\gamma-1}{2}}(ax)\, j_{\frac{k+1}{2}}\left(t\sqrt{a^2+c^2}\right).$$

Then the solution of (7.54)–(7.55) *is*

$$u(x,t;k)=t^{1-k}\frac{\partial}{t\partial t}\left(t^{k+1}u(x,t;k+2)\right)=$$
$$\frac{t^{1-k}}{k+1} j_{\frac{\gamma-1}{2}}(ax)\frac{\partial}{t\partial t}\left(t^{k+1} j_{\frac{k+1}{2}}\left(t\sqrt{a^2+c^2}\right)\right)=$$
$$\frac{2^{\frac{k+1}{2}}t^{1-k}\Gamma\left(\frac{k+3}{2}\right)}{(k+1)(\sqrt{a^2+c^2})^{\frac{k+1}{2}}} j_{\frac{\gamma-1}{2}}(ax)\frac{\partial}{t\partial t}\left(t^{\frac{k+1}{2}} J_{\frac{k+1}{2}}\left(t\sqrt{a^2+c^2}\right)\right)=$$
$$\frac{2^{\frac{k-1}{2}}\Gamma\left(\frac{k+1}{2}\right)}{(\sqrt{a^2+c^2})^{\frac{k-1}{2}}} j_{\frac{\gamma-1}{2}}(ax) t^{\frac{1-k}{2}} J_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right)= j_{\frac{\gamma-1}{2}}(ax)\, j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right).$$

As might be seen from (1.23) *and* (1.24)*,*

$$j_{\frac{\gamma-1}{2}}(ax)\lim_{t\to 0} j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right)=j_{\frac{\gamma-1}{2}}(ax),$$
$$j_{\frac{\gamma-1}{2}}(ax)\lim_{t\to 0}\frac{\partial}{\partial t} j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right)=j_{\frac{\gamma-1}{2}}(ax)=0,$$
$$(B_\gamma)_x\, j_{\frac{\gamma-1}{2}}(ax)\, j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right)=-a^2 j_{\frac{\gamma-1}{2}}(ax)\, j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right),$$
$$(B_k)_t\, j_{\frac{\gamma-1}{2}}(ax)\, j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right)=-(a^2+c^2) j_{\frac{\gamma-1}{2}}(ax)\, j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right),$$

which shows that the function

$$u(t,x;k)=j_{\frac{\gamma-1}{2}}(ax)\, j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right) \tag{7.58}$$

satisfies (7.54)–(7.55).

Example 6. *Consider the problem*

$$\left[(B_\gamma)_x-(B_k)_t\right]u=c^2u,\qquad c>0,\qquad u=u(t,x;k),\qquad (t,x)\in\mathbb{R}^2_+, \tag{7.59}$$

$$u(x,0;k)=0,\qquad \lim_{t\to+0} t^k u_t(x,t;k)=j_\gamma(bx), \tag{7.60}$$

where $n=1$, $k<1$, $0<\gamma<3$, $b\in\mathbb{R}$. When $q=1$ and considering (7.56) we obtain

$$u(t,x;k)=B(1,\gamma,k,1)\,j_{\frac{\gamma-1}{2}}(bx)\,\frac{1}{t}\frac{\partial}{\partial t}\Bigg(\int\limits_0^t (t^2-r^2)^{1-\frac{k+\gamma}{2}}\times$$

$$j_{1-\frac{k+\gamma}{2}}\left(c\sqrt{t^2-r^2}\right)j_{\frac{\gamma-1}{2}}(br)r^{\gamma}\,dr\Bigg)=$$

$$\frac{B(1,\gamma,k,1)\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(2-\frac{k+\gamma}{2}\right)}{2^{\frac{k-1}{2}}b^{\frac{\gamma-1}{2}}c^{1-\frac{k+\gamma}{2}}}\,j_{\frac{\gamma-1}{2}}(bx)\,\frac{1}{t}\frac{\partial}{\partial t}\Bigg(\int\limits_0^t (t^2-r^2)^{\frac{2-k-\gamma}{4}}\times$$

$$J_{1-\frac{k+\gamma}{2}}\left(c\sqrt{t^2-r^2}\right)J_{\frac{\gamma-1}{2}}(br)r^{\frac{\gamma-1}{2}+1}dr\Bigg),$$

where

$$B(1,\gamma,k,1)=\frac{\Gamma\left(\frac{1-k}{2}\right)}{2\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(2-\frac{k+\gamma}{2}\right)}.$$

Applying formula (2.12.35.2) from [456] of the form

$$\int\limits_0^t (t^2-x^2)^{m+\frac{\mu}{2}}x^{\nu+1+2l}J_{\mu}(c\sqrt{t^2-x^2})J_{\nu}(hx)dx=$$

$$t^{\mu+\nu-m-l+1}c^{\mu}h^{\nu}\left(\frac{\partial}{c\partial c}\right)^m\left(\frac{\partial}{h\partial h}\right)^l\times$$

$$[(c^2+h^2)^{-\frac{\mu+\nu+m+l+1}{2}}J_{\mu+\nu+m+l+1}(t\sqrt{c^2+h^2})],$$

$$t>0,\qquad \operatorname{Re}\nu>-l-1,\qquad \operatorname{Re}\mu>-m-1,$$

we obtain

$$\int\limits_0^t (t^2-r^2)^{\frac{2-k-\gamma}{4}}J_{1-\frac{k+\gamma}{2}}\left(c\sqrt{t^2-r^2}\right)J_{\frac{\gamma-1}{2}}(br)r^{\frac{\gamma-1}{2}+1}dr=$$

$$b^{\frac{\gamma-1}{2}}c^{1-\frac{k+\gamma}{2}}(b^2+c^2)^{\frac{k-3}{4}}t^{\frac{3-k}{2}}J_{\frac{3-k}{2}}\left(t\sqrt{b^2+c^2}\right)\Bigg)$$

and

$$u(t,x;k)=2^{\frac{1-k}{2}}B(1,\gamma,k,1)\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(2-\frac{k+\gamma}{2}\right)\times$$

$$(b^2+c^2)^{\frac{k-3}{4}}\,j_{\frac{\gamma-1}{2}}(bx)\frac{1}{t}\frac{\partial}{\partial t}\left(t^{\frac{3-k}{2}}J_{\frac{3-k}{2}}\left(t\sqrt{b^2+c^2}\right)\right)=$$

$$2^{-\frac{k+1}{2}}\Gamma\left(\frac{1-k}{2}\right)(\sqrt{b^2+c^2})^{\frac{k-1}{2}}\, j_{\frac{\gamma-1}{2}}(bx)\, t^{\frac{1-k}{2}} J_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right)=$$
$$\frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(bx)\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right).$$

Taking into account (1.23) *and* (1.24) *it is easy to check that*

$$(B_\gamma)_x \frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(bx)\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right)=$$
$$-b^2\frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(bx)\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right),$$
$$(B_k)_t \frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(bx)\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right)=$$
$$-(b^2+c^2)\frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(bx)\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right),$$
$$\lim_{t\to 0}\frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(bx)\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right)=0,$$

and

$$j_{\frac{\gamma-1}{2}}(bx)\lim_{t\to 0} t^k\frac{\partial}{\partial t}\left(\frac{t^{1-k}}{1-k}\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right)\right)=j_{\frac{\gamma-1}{2}}(bx),$$

which confirms that the function

$$u(t,x;k)=\frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(bx)\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right) \tag{7.61}$$

satisfies (7.59)–(7.60).

Example 7. *From Examples 5 and 6 it is plain to see that the solution of*

$$\left[(B_\gamma)_x-(B_k)_t\right]u=c^2u,\qquad c>0,\qquad u=u(t,x;k),\qquad (t,x)\in\mathbb{R}^2_+,$$
$$u(x,0;k)=j_\gamma(ax),\qquad \lim_{t\to+0} t^k u_t(x,t;k)=j_\gamma(bx),$$

where $n=1$, $0<\gamma<1$, $\gamma-2\leq k\leq\gamma$, $k\neq -1$, $a,b\in\mathbb{R}$, *is*

$$u(t,x;k)=j_{\frac{\gamma-1}{2}}(ax)j_{\frac{k-1}{2}}\left(t\sqrt{a^2+c^2}\right)+\frac{t^{1-k}}{1-k}\, j_{\frac{\gamma-1}{2}}(bx)\, j_{\frac{1-k}{2}}\left(t\sqrt{b^2+c^2}\right).$$

7.2 Hyperbolic and ultrahyperbolic equations with Bessel operator in spaces of weighted distributions

7.2.1 The generalized Euler–Poisson–Darboux equation and the singular Klein–Gordon equation

In this subsection we apply the Hankel transform method to solve the initial value problem

$$\left[(\triangle_\gamma)_x - (B_k)_t\right] u = c^2 u, \tag{7.62}$$

$$u(x,0;k) = f(x), \quad u_t(x,0;k) = 0, \quad u = u(x,t;k), \tag{7.63}$$

where $\gamma_i > 0$, $x_i > 0$, $i = 1, ..., n$, $t > 0$. We will call (7.62) the **generalized Euler–Poisson–Darboux equation**. We obtain the distributional solution of (7.62)–(7.63) in convenient space. Besides, we give formulas for regular solution of (7.62)–(7.63) in the particular case of k and of Cauchy for the singular Klein–Gordon equation.

We are looking for the solution $u \in S'_{ev}(\mathbb{R}^n_+) \times C^2_{ev}(0,\infty)$ of (7.62)–(7.63). The notation $u \in S'_{ev}(\mathbb{R}^n_+) \times C^2_{ev}(0,\infty)$ means that $u(x,t;k)$ belongs to $S'_{ev}(\mathbb{R}^n_+)$ by variable x and belongs to $C^2_{ev}(0,\infty)$ by variable t. Here we use methods of weighted generalized function, see [501].

Theorem 83. *The solution $u \in S'_{ev}(\mathbb{R}^n_+) \times C^2_{ev}(0,\infty)$ of (7.62)–(7.63) for $k \neq -1, -3, -5, ...$ is unique and defined by the formula*

$$u(x,t;k) = C(n,\gamma,k)\left(t^{1-k}(t^2-|x|^2)_+^{\frac{k-n-|\gamma|-1}{2}}\, j_{\frac{k-n-|\gamma|-1}{2}}\left((t^2-|x|^2)_+^{\frac{1}{2}} \cdot c\right) * f(x)\right)_\gamma, \tag{7.64}$$

where

$$C(n,\gamma,k) = \frac{2^n \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

In the case when $k < 0$ of (7.62)–(7.63) is not unique, when $k < 0$ and $k \neq -1, -3, -5, ...$, the difference between two arbitrary solutions is always of the form

$$At^{1-k} u(t,x;2-k), \qquad A = const, \tag{7.65}$$

where $u(t,x;2-k)$ is a solution of the Cauchy problem

$$\left[(\triangle_\gamma)_x - (B_{2-k})_t\right] u = c^2 u,$$
$$u(x,0;2-k) = \psi(x), \qquad u_t(x,0;2-k) = 0$$

and $\psi(x)$ *is an arbitrary function or distribution belonging to* S'_{ev}. *When* $k = -1,-3,-5,...$, *a nonunique solution of the Cauchy problem (7.62)–(7.63) will contain a term (7.65) and*

$$\frac{e^{\pm\frac{1}{2}\pi n i}\,\Gamma\left(\frac{n+|\gamma|-k+1}{2}\right)}{2^n\Gamma\left(\frac{1-k}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\,t^{1-k}\left((t^2-|x|^2-c^2\pm i0)_\gamma^{\frac{k-n-|\gamma|-1}{2}} * f(x)\right)_\gamma .$$

Proof. Applying the multi-dimensional Hankel transform to (7.62) with respect to the variables $x_1,...,x_n$ only and using (1.95), we obtain

$$\left(|\xi|^2+c^2+\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}\right)\widehat{u}(\xi,t;k)=0, \tag{7.66}$$
$$\widehat{u}(\xi,0;k)=\widehat{f}(\xi), \qquad \widehat{u}_t(\xi,0;k)=0, \tag{7.67}$$

where $\xi=(\xi_1,\xi_2,...,\xi_n)\in\mathbb{R}^n_+$ corresponds to $x=(x_1,...,x_n)\in\mathbb{R}^n_+$, $|\xi|^2=\xi_1^2+\xi_2^2+...+\xi_n^2$,

$$\widehat{u}(\xi,t;k)=(F_\gamma)_x[u(x,t;k)](\xi)=\int\limits_{\mathbb{R}^n_+} u(x,t;k)\,\mathbf{j}_\gamma(x;\xi)x^\gamma\,dx,$$

and $\widehat{f}(\xi)=\mathbf{F}_\gamma[f](\xi)$.

The solution $\widehat{G}^k(\xi,t)$ of the Cauchy problem

$$\left(|\xi|^2+c^2+\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t}\right)\widehat{G}^k(\xi,t)=0,$$
$$\widehat{G}^k(\xi,0)=1, \qquad \widehat{G}^k_t(\xi,0)=0$$

was obtained in [38]. We have different solutions for nonnegative and negative values of k, specifically:

1. for $k\geq 0$,

$$\widehat{G}^k(\xi,t)=j_{\frac{k-1}{2}}(\sqrt{|\xi|^2+c^2}\;t), \tag{7.68}$$

2. for $k<0$, $k\neq -1,-3,-5,...$,

$$\widetilde{G}^k(\xi,t)=At^{1-k}j_{\frac{1-k}{2}}(\sqrt{|\xi|^2+c^2}\;t)+j_{\frac{k-1}{2}}(\sqrt{|\xi|^2+c^2}\;t), \tag{7.69}$$

where A is an arbitrary complex number which depends on ξ and c, and

3. for $k=-1,-3,-5,...$,

$$\widehat{G}^k(\xi,t)=Bt^{1-k}j_{\frac{1-k}{2}}(\sqrt{|\xi|^2+c^2}\;t)-$$
$$\frac{\pi 2^{\frac{k-1}{2}}}{\Gamma\left(\frac{1-k}{2}\right)}\left[\sqrt{|\xi|^2+c^2}\,t\right]^{\frac{1-k}{2}}Y_{\frac{1-k}{2}}(\sqrt{|\xi|^2+c^2}\;t), \tag{7.70}$$

where B denotes an arbitrary complex number which depends on ξ and c.

From (7.68)–(7.70) we conclude that the problem (7.62)–(7.63) has a unique solution for $k\geq 0$ only. Besides, we can see that the difference between two different solutions (7.69) is always of the form

$$At^{1-k}j_{\frac{1-k}{2}}(\sqrt{|\xi|^2+c^2}\;t). \tag{7.71}$$

Now let us find $G^k(x,t)=\left((\mathbf{F}_\gamma^{-1})_\xi\widehat{G}^k(\xi,t)\right)(x)$. We call $G^k(x,t)$ the *fundamental solution* of problem (7.62)–(7.63). The inverse transform $\left((\mathbf{F}_\gamma^{-1})_\xi\widehat{G}^k(\xi,t)\right)(x)$ is most easily found by considering c as an additional independent variable. Setting $\xi'=(\xi_1,...,\xi_n,c)$, we can write $\widehat{G}^k(\xi,t)=j_{\frac{k-1}{2}}(|\xi'|t)$ for $k\geq 0$ and find an inverse Hankel transform of $j_{\frac{k-1}{2}}(|\xi'|t)$ by variable ξ' using (4.94). We obtain

$$\left((\mathbf{F}_{\gamma'}^{-1})_{\xi'}j_{\frac{k-1}{2}}(|\xi'|t)\right)(x')=\frac{2^{n+1}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma'|}{2}\right)\prod\limits_{i=1}^{n+1}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\,t^{1-k}(t^2-|x'|^2)_{+,\gamma'}^{\frac{k-n-|\gamma'|-2}{2}},$$

where $\gamma'=(\gamma_1,..,\gamma_n,\gamma_{n+1})$, γ_{n+1} is an arbitrary positive number, $x'=(x,\sigma)$, and $\sigma\in R_+$ is dual to the variable c. Now in order to find $G^k(x,t)$ we need to apply a direct Hankel transform only on a one-dimensional variable σ. We have

$$G^k(x,t)=\frac{2^{n+1}\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma'|}{2}\right)\prod\limits_{i=1}^{n+1}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\,t^{1-k}\times$$
$$\left((F_{\gamma_{n+1}})_\sigma(t^2-x^2-\sigma^2)_{+,\gamma'}^{\frac{k-n-|\gamma'|-2}{2}}\right)(c)=\frac{2^n\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\times$$

$$t^{1-k}(t^2-x^2)_+^{\frac{k-n-|\gamma|-1}{2}} \, j_{\frac{k-n-|\gamma|-1}{2}}\left((t^2-|x|^2)_+^{\frac{1}{2}}\cdot c\right).$$

Therefore, the solution of (7.62)–(7.63) for $k \geq 0$ is given by

$$u(x,t;k) = (G^k(x,t) * f(x))_\gamma. \tag{7.72}$$

It is easy to see that (7.72) still gives one of the possible solutions of (7.62)–(7.63) for $k < 0$, $k \neq -1,-3,-5,\ldots$. Moreover, in consideration of (7.71) the difference between two solutions for $k < 0$ has the form

$$(\mathbf{F}_\gamma^{-1})_\xi\left[At^{1-k}j_{\frac{1-k}{2}}(\sqrt{|\xi|^2+c^2}\;t)\right](x). \tag{7.73}$$

Consequently, the difference between two arbitrary solutions for $k < 0$ is always of the form

$$A(t^{1-k}G^{2-k}(t,x) * \psi(x))_\gamma = At^{1-k}u(t,x;2-k), \tag{7.74}$$

where $\psi(x)$ is an arbitrary function or distribution belonging to S'_{ev}, $u(t,x;2-k)$ is the solution of the Cauchy problem

$$\begin{aligned}
&\left[(\triangle_\gamma)_x - (B_{2-k})_t\right]u = c^2u,\\
&u(x,0;2-k) = \psi(x), \qquad u_t(x,0;2-k) = 0,
\end{aligned}$$

and $G^{2-k}(t,x)$ is the corresponding fundamental solution.

Finally we consider the case $k = -1,-3,-5,\ldots$. In this case the solution will be of a different character than the solutions for other values of k and will always contain a term

$$G^k(t,x) = \frac{\pi 2^{\frac{k-1}{2}}}{\Gamma\left(\frac{1-k}{2}\right)}\mathbf{F}_\gamma^{-1}\left[\left(\sqrt{|\xi|^2+c^2}\,t\right)^{\frac{1-k}{2}} Y_{\frac{1-k}{2}}(\sqrt{|\xi|^2+c^2}\;t)\right].$$

It is clear that

$$G^k(t,x) = \frac{i\pi 2^{\frac{k-1}{2}}}{\Gamma\left(\frac{1-k}{2}\right)}\mathbf{F}_\gamma^{-1}\left[\left(\sqrt{|\xi|^2+c^2}\,t\right)^{\frac{1-k}{2}} H^{(1)}_{\frac{1-k}{2}}(\sqrt{|\xi|^2+c^2}\;t)\right]$$

and

$$G^k(t,x) = -\frac{i\pi 2^{\frac{k-1}{2}}}{\Gamma\left(\frac{1-k}{2}\right)}\mathbf{F}_\gamma^{-1}\left[\left(\sqrt{|\xi|^2+c^2}\,t\right)^{\frac{1-k}{2}} H^{(2)}_{\frac{1-k}{2}}(\sqrt{|\xi|^2+c^2}\;t)\right]$$

will also be a fundamental solution of our problem. Then using (4.102) and (4.103) we obtain

$$G^k(t,x)=\frac{e^{\pm\frac{1}{2}\pi ni}\Gamma\left(\frac{n+|\gamma|-k+1}{2}\right)}{2^n\Gamma\left(\frac{1-k}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\,t^{1-k}(t^2-|x|^2-c^2\pm i0)_{\gamma}^{\frac{k-n-|\gamma|-1}{2}}.$$

Since $(t^2-|x|^2)^{\lambda}_{+,\gamma}$ has its support in the interior of the part of the sphere $S_1^+(n)$ when $x_1\geq 0,\ldots,x_n\geq 0$, we may conclude that in the case $k\neq -1,-3,-5,\ldots$ the generalized convolutions exist for arbitrary $\varphi(x)\in S'_{ev}$. However, in the case $k=-1,-3,-5,\ldots$ the fundamental solution is no longer concentrated within the part of the sphere $S_1^+(n)$. □

Corollary 1. *The solution $u\in S'_{ev}(\mathbb{R}^n_+)\times C^2_{ev}(0,\infty)$ of*

$$\left[(\triangle_\gamma)_x-(B_k)_t\right]u=0,$$
$$u(x,0;k)=f(x),\quad u_t(x,0;k)=0,\quad u=u(x,t;k)$$

for $k\neq -1,-3,-5,\ldots$ is unique and defined by the formula

$$u(x,t;k)=C(n,\gamma,k)\left(t^{1-k}(t^2-|x|^2)_+^{\frac{k-n-|\gamma|-1}{2}}*f(x)\right)_\gamma,$$

where

$$C(n,\gamma,k)=\frac{2^n\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

In the case when $k<0$ of (7.62)–(7.63) is not unique, when $k<0$ and $k\neq -1,-3,-5,\ldots$, the difference between two arbitrary solutions is always of the form

$$At^{1-k}u(t,x;2-k),\qquad A=const,\tag{7.75}$$

where $u(t,x;2-k)$ is solution of the Cauchy problem

$$\left[(\triangle_\gamma)_x-(B_{2-k})_t\right]u=0,$$
$$u(x,0;2-k)=\psi(x),\qquad u_t(x,0;2-k)=0,$$

where $\psi(x)$ is an arbitrary function or distribution belonging to S'_{ev}. When $k=-1,-3,-5,\ldots$, a nonunique solution of the Cauchy problem (7.62)–(7.63) will contain terms (7.75) and

$$\frac{e^{\pm\frac{1}{2}\pi ni}\Gamma\left(\frac{n+|\gamma|-k+1}{2}\right)}{2^n\Gamma\left(\frac{1-k}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\,t^{1-k}\left((t^2-|x|^2\pm i0)_{\gamma}^{\frac{k-n-|\gamma|-1}{2}}*f(x)\right)_\gamma.$$

Corollary 2. *The solution $u \in S'_{ev}(\mathbb{R}^n_+) \times C^2_{ev}(0,\infty)$ of the initial value problem for the singular Klein–Gordon equation*

$$\left[(\triangle_\gamma)_x - \frac{\partial^2}{\partial t^2}\right] v = c^2 v, \quad c>0, \quad v=v(x,t), \quad x\in\mathbb{R}^n_+, \quad t>0, \tag{7.76}$$

$$v(x,0)=f(x), \qquad v_t(x,0)=0, \qquad f(x)\in S'_{ev} \tag{7.77}$$

is

$$v(x,t)=\frac{2^n\sqrt{\pi}}{\Gamma\left(\frac{1-n-|\gamma|}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\times$$
$$\left(t(t^2-|x|^2)_{+,\gamma}^{-\frac{n+|\gamma|+1}{2}} j_{-\frac{n+|\gamma|+1}{2}}\left((t^2-|x|^2)_+^{\frac{1}{2}}\cdot c\right) * f(x)\right)_\gamma .$$

This solution was obtained by letting k tend to 0 in (7.64).

The Klein–Gordon equation

$$\left[\triangle_z - \frac{\partial^2}{\partial t^2}\right] v = c^2 v, \qquad v=v(z,t), \qquad z\in\mathbb{R}^N, \tag{7.78}$$

is the most frequently used wave equation for the description of particle dynamics in relativistic quantum mechanics. When function v is radially symmetric by some groups of variables $z_1,...,z_N$ in (7.78), we obtain (7.76) with a smaller number of spatial variables. In this case numbers γ_i, $i=1,..,n$, in (7.76) will be integer.

Corollary 3. *In the case $k>n+|\gamma|-1$ the integral in (7.64) converges in the usual sense and we obtain the unique classical solution of (7.62)–(7.63)*

$$u(x,t;k)=$$
$$A(n,\gamma,k)\,t^{1-k}\int\limits_0^t (t^2-r^2)^{\frac{k-n-|\gamma|-1}{2}} j_{\frac{k-n-|\gamma|-1}{2}}\left(c\sqrt{t^2-r^2}\right) r^{n+|\gamma|-1} M_r^\gamma[f(x)]dr,$$

$$A(n,\gamma,k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}.$$

Proof. For $k>n+|\gamma|-1$, passing to spherical coordinates, we obtain

$$u(x,t;k)=C(n,\gamma,k)\,t^{1-k}\times$$
$$\int\limits_{B_t^+(n)} (t^2-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} j_{\frac{k-n-|\gamma|-1}{2}}\left((t^2-|y|^2)^{\frac{1}{2}}\cdot c\right) {}^\gamma\mathbf{T}_x^y f(x)\, y^\gamma dy =$$
$$C(n,\gamma,k)\int\limits_{B_1^+(n)} (1-|y|^2)^{\frac{k-n-|\gamma|-1}{2}} j_{\frac{k-n-|\gamma|-1}{2}}\left((1-|y|^2)^{\frac{1}{2}}\cdot tc\right) {}^\gamma\mathbf{T}_x^{ty} f(x)\, y^\gamma dy=$$

$$C(n,\gamma,k)\int\limits_0^1 (1-r^2)^{\frac{k-n-|\gamma|-1}{2}}\, j_{\frac{k-n-|\gamma|-1}{2}}\left((1-r^2)^{\frac{1}{2}}\cdot tc\right) r^{n+|\gamma|-1} dr\times$$

$$\int\limits_{S_1^+(n)} {}^\gamma\mathbf{T}_x^{tr\theta} f(x)\,\theta^\gamma dS = \frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}\times$$

$$\int\limits_0^1 (1-r^2)^{\frac{k-n-|\gamma|-1}{2}}\, j_{\frac{k-n-|\gamma|-1}{2}}\left((1-r^2)^{\frac{1}{2}}\cdot tc\right) r^{n+|\gamma|-1} M_{rt}^\gamma[f(x)]dr =$$

$$\frac{2\Gamma\left(\frac{k+1}{2}\right) t^{1-k}}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}\int\limits_0^t (t^2-r^2)^{\frac{k-n-|\gamma|-1}{2}}\, j_{\frac{k-n-|\gamma|-1}{2}}\left((t^2-r^2)^{\frac{1}{2}}\cdot c\right)\times$$

$$r^{n+|\gamma|-1} M_r^\gamma[f(x)]dr. \qquad \square$$

Corollary 4. *In the case $k>n+|\gamma|-1$, the solution of*

$$\left[(\triangle_\gamma)_x-(B_k)_t\right]u=0,\quad k\in R,\quad u=u(x,t;k),\quad x\in R_+^n,\quad t>0, \tag{7.79}$$

$$u(x,0;k)=f(x),\qquad u_t(x,0;k)=0 \tag{7.80}$$

is unique and is given by

$$u(x,t;k)=A(n,\gamma,k)\,t^{1-k}\int\limits_0^t (t^2-r^2)^{\frac{k-n-|\gamma|-1}{2}} r^{n+|\gamma|-1} M_r^\gamma[f(x)]dr, \tag{7.81}$$

$$A(n,\gamma,k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)},$$

which coincides with (7.6).

Example 1. *Let us consider the Cauchy problem for $k>n+|\gamma|-1$,*

$$\left[(\triangle_\gamma)_x-(B_k)_t\right]u=c^2u,$$
$$u(x,0;k)=\mathbf{j}_\gamma(x;\xi),\qquad u_t(x,0;k)=0.$$

In this case the solution is unique and is gives by

$$u(x,t;k)=A(n,\gamma,k)\,t^{1-k}\times$$

$$\int\limits_0^t (t^2-r^2)^{\frac{k-n-|\gamma|-1}{2}}\, j_{\frac{k-n-|\gamma|-1}{2}}\left(c\sqrt{t^2-r^2}\right) r^{n+|\gamma|-1} M_r^\gamma \mathbf{j}_\gamma(x;\xi)dr,$$

$$A(n,\gamma,k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}.$$

For $M_r^\gamma \mathbf{j}_\gamma(x;\xi)$ we have formula (3.190), so we get

$$u(x,t;k)=\frac{2\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{k-n-|\gamma|+1}{2}\right)}\,t^{1-k}\,\mathbf{j}_\gamma(x;\xi)\times$$
$$\int\limits_0^t (t^2-r^2)^{\frac{k-n-|\gamma|-1}{2}}r^{n+|\gamma|-1}\,j_{\frac{k-n-|\gamma|-1}{2}}\left(c\sqrt{t^2-r^2}\right)j_{\frac{n+|\gamma|}{2}-1}(r|\xi|)dr=$$
$$\frac{2^{\frac{k-1}{2}}\Gamma\left(\frac{k+1}{2}\right)}{c^{\frac{k-n-|\gamma|-1}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}}\,t^{1-k}\,\mathbf{j}_\gamma(x;\xi)\times$$
$$\int\limits_0^t (t^2-r^2)^{\frac{k-n-|\gamma|-1}{4}}r^{\frac{n+|\gamma|}{2}}J_{\frac{k-n-|\gamma|-1}{2}}\left(c\sqrt{t^2-r^2}\right)J_{\frac{n+|\gamma|}{2}-1}(r|\xi|)dr.$$

Applying formula (2.12.35.2) from [456] of the form

$$\int\limits_0^t (t^2-x^2)^{m+\frac{\mu}{2}}x^{\nu+1+2l}J_\mu(c\sqrt{t^2-x^2})J_\nu(hx)dx=$$
$$t^{\mu+\nu-m-l+1}c^\mu h^\nu\left(\frac{\partial}{c\partial c}\right)^m\left(\frac{\partial}{h\partial h}\right)^l[(c^2+h^2)^{-\frac{\mu+\nu+m+l+1}{2}}J_{\mu+\nu+m+l+1}(t\sqrt{c^2+h^2})],$$
$$t>0,\qquad \operatorname{Re}\nu>-l-1,\qquad \operatorname{Re}\mu>-m-1,$$

we have $k=m=0$, $\nu=\frac{n+|\gamma|}{2}-1$, $\mu=\frac{k-n-|\gamma|-1}{2}$, $h=|\xi|$, and

$$\int\limits_0^t (t^2-r^2)^{\frac{k-n-|\gamma|-1}{4}}r^{\frac{n+|\gamma|}{2}}J_{\frac{k-n-|\gamma|-1}{2}}\left(c\sqrt{t^2-r^2}\right)J_{\frac{n+|\gamma|}{2}-1}(r|\xi|)dr=$$
$$\frac{t^{\frac{k-1}{2}}c^{\frac{k-n-|\gamma|-1}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}}{(\sqrt{c^2+|\xi|^2})^{\frac{k-1}{2}}}\,J_{\frac{k-1}{2}}(t\sqrt{c^2+|\xi|^2}).$$

Therefore,

$$u(x,t;k)=\mathbf{j}_\gamma(x;\xi)\,j_{\frac{k-1}{2}}(t\sqrt{c^2+|\xi|^2}).$$

7.2.2 Iterated ultrahyperbolic equation with Bessel operator

The classical ultrahyperbolic equation has the form

$$\Delta_x u = \Delta_y u, \qquad u = u(x, y), \qquad x \in \mathbb{R}^p, \qquad y \in \mathbb{R}^q. \tag{7.82}$$

Eq. (7.82) was studied by many authors (see [23,29,30,76,156,271,272,435,466]). For $p = 1$ or $q = 1$, (7.82) is the usual wave equation describing the dynamic development of many processes of classical and quantum physics. To equations of the form (7.82) for $p = q = 2$ lead, for example, the Hilbert problems of determining in a three-dimensional Cartesian space all metrics whose geodesics are straight lines (see [436]); the inverse diffraction problem in the study of the heterogeneity of the distribution of grains of polycrystalline materials; and the hyperspherical X-ray transformation, namely, the density functions of the crystallographic poles satisfy the ultrahyperbolic equation with the Laplace–Beltrami operator (see [418]). The case when in (7.82) $p > 2$ and $q > 2$ is important from a mathematical point of view thanks to the Asgeirsson theorem about the spherical mean (see [9], [75], p. 475, [155], p. 84, [162], p. 318, [170] I, p. 183). This theorem is a generalization of the mean value theorem for harmonic functions, as well as a generalization of the Green formula for a linear wave equation with constant coefficients.

Generally speaking, the initial problem for the ultrahyperbolic equation (7.82) is incorrect. In particular, in the general case, the solution of the initial problem for it either does not exist or is not unique, and if it is possible to find some solution, then the solution is unstable. However, in the article [76] it was shown that the initial problem for an ultrahyperbolic equation with a nonlocal constraint on codimensional hyperspaces has a unique global solution in the Sobolev space H^m. Thus, in this case, the initial problem for (7.82) is correct.

We will consider a generalization of Eq. (7.82) to the case when instead of every second derivative with respect to each variable the Bessel operator acts.

Let $n = p + q$, p and q are natural, $\gamma = (\gamma', \gamma'')$, $\gamma' = (\gamma_1, ..., \gamma_p)$, $\gamma'' = (\gamma_{p+1}, ..., \gamma_{p+q})$, $\gamma_i > 0$, $i = 1, ..., n$, $x' \in \mathbb{R}^p_+$, $x'' \in \mathbb{R}^q_+$, $x = (x', x'') \in \mathbb{R}^n_+$, $\mathbb{R}^n_+ = \mathbb{R}^p_+ \times \mathbb{R}^q_+$.

The **B-ultrahyperbolic equation** or singular ultrahyperbolic equation has the form

$$\Box_\gamma u = 0, \quad u = u(x), \tag{7.83}$$

where $\Box_\gamma$ is a homogeneous linear differential operator of the form

$$\Box_\gamma = (\Delta_{\gamma'})_{x'} - (\Delta_{\gamma''})_{x''} = B_{\gamma_1} + ... + B_{\gamma_p} - B_{\gamma_{p+1}} - ... - B_{\gamma_{p+q}},$$

$$(\Delta_{\gamma'})_{x'} = \sum_{i=1}^{p} (B_{\gamma_i})_{x_i},\ (\Delta_{\gamma''})_{x''} = \sum_{j=p+1}^{p+q} (B_{\gamma_j})_{x_j},\ B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i},\ i = 1, ..., n.$$

The **iterated B-ultrahyperbolic equation** we will call the equation of the form

$$\Box_\gamma^k u = f, \tag{7.84}$$

where $k \in \mathbb{N}$ and $f = f(x)$ is a suitable function.

In this subsection we find the fundamental solution to the equation $\Box_\gamma^k u = f$ using the results obtained for weighted generalized functions.

Let $x \in \mathbb{R}_+^n$, $n = p + q$, $p, q \in \mathbb{N}$. The fundamental solution to Eq. (7.84) is the weighted generalized function u such that

$$\Box_\gamma^k u = \delta_\gamma. \tag{7.85}$$

Note that the fundamental solutions for the hyperbolic and ultrahyperbolic equations with the Bessel operator applied only by one variable are obtained in [246,247].

Theorem 84. *Except when $n + |\gamma| = 2, 4, 6, \ldots$ and $k \geq \frac{n+|\gamma|}{2}$, the weighted generalized function*

$$u = (-1)^k \frac{e^{\pm i\frac{\pi(q+|\gamma''|)}{2}}\Gamma\left(\frac{n+|\gamma|}{2} - k\right)}{4^k (k-1)!|S_1^+(n)|_\gamma \Gamma\left(\frac{n+|\gamma|}{2} - 1\right)}(P \pm i0)_\gamma^{-\frac{n+|\gamma|}{2}+k} \tag{7.86}$$

is the fundamental solution to the equation $\Box_\gamma^k u = f$ in the sense (7.85). If $n + |\gamma| = 2, 4, 6, \ldots$ and $k \geq \frac{n+|\gamma|}{2}$, then the weighted generalized function $(P + i0)_\gamma^{-\frac{n+|\gamma|}{2}+k} = (P - i0)_\gamma^{-\frac{n+|\gamma|}{2}+k}$ is a solution to a homogeneous equation $\Box_\gamma^k u = 0$.

Proof. Using (4.61) we obtain

$$\Box_\gamma^k (P+i0)_\gamma^{\lambda+k} = 4^k(\lambda+1)\ldots(\lambda+k)\left(\lambda+\frac{n+|\gamma|}{2}\right)\ldots\left(\lambda+\frac{n+|\gamma|}{2}+k-1\right)(P+i0)_\gamma^\lambda.$$

Tending to the limit at $\lambda \to -\frac{n+|\gamma|}{2}$ in the last equality and using formula (4.74) for $k = 0$, we obtain

$$\begin{gathered}
\Box_\gamma^k (P+i0)_\gamma^{-\frac{n+|\gamma|}{2}+k} = \\
4^k\left(1 - \frac{n+|\gamma|}{2}\right)\ldots\left(k - \frac{n+|\gamma|}{2}\right)(k-1)! \lim_{\lambda \to -\frac{n+|\gamma|}{2}}\left(\lambda+\frac{n+|\gamma|}{2}\right)(P+i0)_\gamma^\lambda = \\
4^k\left(1 - \frac{n+|\gamma|}{2}\right)\ldots\left(k - \frac{n+|\gamma|}{2}\right)(k-1)! \operatorname*{res}_{\lambda=-\frac{n+|\gamma|}{2}}(P+i0)_\gamma^\lambda = \\
4^k\left(1 - \frac{n+|\gamma|}{2}\right)\ldots\left(k - \frac{n+|\gamma|}{2}\right)(k-1)! e^{-i\frac{\pi(q+|\gamma''|)}{2}}|S_1^+(n)|_\gamma \delta_\gamma(x).
\end{gathered}$$

If $n+|\gamma|$ is even and $k \geq \frac{n+|\gamma|}{2}$, then among the multipliers $\left(1 - \frac{n+|\gamma|}{2}\right)\ldots\left(k - \frac{n+|\gamma|}{2}\right)$ there is a zero and therefore $\Box_\gamma^k (P+i0)_\gamma^{-\frac{n+|\gamma|}{2}+k} = 0$ and $u = (P+i0)_\gamma^{-\frac{n+|\gamma|}{2}+k}$ is a solution to a homogeneous equation $\Box_\gamma^k u = 0$. For all other values $n + |\gamma|$ and k the

weighted generalized function

$$u=(-1)^k\frac{e^{i\frac{\pi(q+|\gamma''|)}{2}}\Gamma\left(\frac{n+|\gamma|}{2}-k\right)}{4^k(k-1)!|S_1^+(n)|_\gamma\Gamma\left(\frac{n+|\gamma|}{2}-1\right)}(P+i0)_\gamma^{-\frac{n+|\gamma|}{2}+k} \tag{7.87}$$

is the fundamental solution in the sense (7.85) to Eq. (7.84). In (7.87) it was used that

$$\left(1-\frac{n+|\gamma|}{2}\right)...\left(k-\frac{n+|\gamma|}{2}\right)=(-1)^k\left(\frac{n+|\gamma|}{2}-1\right)...\left(\frac{n+|\gamma|}{2}-k\right)$$
$$=(-1)^k\frac{\Gamma\left(\frac{n+|\gamma|}{2}-1\right)}{\Gamma\left(\frac{n+|\gamma|}{2}-k\right)}.$$

Similarly, using (4.75) it can be shown that if the number $n+|\gamma|$ is even and $k\geq\frac{n+|\gamma|}{2}$, then $u{=}(P-i0)_\gamma^{-\frac{n+|\gamma|}{2}+k}$ is a solution to a homogeneous equation $\Box_\gamma^k u=0$. For all other values $n+|\gamma|$ and k the weighted generalized function

$$u=(-1)^k\frac{e^{-i\frac{\pi(q+|\gamma''|)}{2}}\Gamma\left(\frac{n+|\gamma|}{2}-k\right)}{4^k(k-1)!|S_1^+(n)|_\gamma\Gamma\left(\frac{n+|\gamma|}{2}-1\right)}(P-i0)_\gamma^{-\frac{n+|\gamma|}{2}+k}$$

is the fundamental solution in the sense (7.85) to Eq. (7.84). □

7.2.3 Generalization of the Asgeirsson theorem

In this subsection we present the results generalizing the Asgeirsson theorem to the case of the B-ultrahyperbolic equation (7.83) (see [349,354]).

Let $u(x,y)\in C^2_{ev}(\mathbb{R}^n_+)$, $x=(x_1,...,x_{m'})$, $y=(y_1,...,y_{m''})$. Consider the spherical weighted means (3.183) taken on parts of surfaces of unit spheres $S_1^+(m')$ and $S_1^+(m'')$ in $\mathbb{R}^{m'}_+$ and $\mathbb{R}^{m''}_+$ by each of the groups and variables x and y centered at $y\in\overline{\mathbb{R}}^{m'}_+$ and $z\in\overline{\mathbb{R}}^{m''}_+$, respectively. For these weighted spherical means we introduce the notation

$$(M_r^{\gamma'}u)(x,r;y)=M_u^{\gamma'}(x,r;y)=\frac{1}{|S_1^+(m')|_{\gamma'}}\int\limits_{S_1^+(m')}{}^{\gamma'}\mathbf{T}x^{r\xi}u(x,y)\,\xi^{\gamma'}\,dS_\xi,$$

$$(M_s^{\gamma''}u)(x;y,s)=M_u^{\gamma''}(x;y,s)=\frac{1}{|S_1^+(m'')|_{\gamma''}}\int\limits_{S_1^+(m'')}{}^{\gamma''}\mathbf{T}y^{s\zeta}u(x,y)\,\zeta^{\gamma''}\,dS_\zeta.$$

So $M_u^{\gamma'}(x,r;y)$ is the weighted spherical mean of the function u in $\mathbb{R}^{m'}_+$ at constant $y=(y_1,\ldots,y_{m''})\in\overline{\mathbb{R}}^{m''}_+$, and $M_u^{\gamma''}(x;y,s)$ is the weighted spherical mean of the function u in $\mathbb{R}^{m''}_+$ at constant $x=(x_1,\ldots,x_{m'})\in\overline{\mathbb{R}}^{m'}_+$.

Let us define also the general weighted spherical mean of function u by (x, y) of the form

$$(M_r^{\gamma'} M_s^{\gamma''} u)(x,r;y,s) = U(x,r;y,s) =$$
$$\frac{1}{|S_1^+(m')|_{\gamma'}\,|S_1^+(m'')|_{\gamma''}} \int\limits_{S_1^+(m'')} \theta^{\gamma''}\,dS(\theta) \int\limits_{S_1^+(m')} {}^{\gamma'}\mathbf{T}_{x,y}^{r\xi,s\zeta} u(x,y)\xi^{\gamma'}\zeta^{\gamma''}\,dS(\xi).$$

It is obvious that

$$M_u^{\gamma'}(x,r;y) = U(x,r;y,0), \quad M_u^{\gamma''}(x;y,s) = U(x,0;y,s). \tag{7.88}$$

Theorem 85. *If*

$$m' + |\gamma'| = m'' + |\gamma''| \tag{7.89}$$

and the function $u(x', x'')$ satisfies the singular ultrahyperbolic equation

$$\Box_\gamma u = 0, \tag{7.90}$$

then

$$M_r^{\gamma'} M_s^{\gamma''} u = M_s^{\gamma'} M_r^{\gamma''} u\,. \tag{7.91}$$

Proof. Each of the weighted spherical means $\mu_{\gamma'}(y,z;r)$ and $\nu_{\gamma''}(y,z;r)$ of u satisfies Eq. (7.83). This gives two equalities for $\omega_{\gamma',\gamma''}(y,z;r,s)$:

$$\Delta_{\gamma'}\omega_{\gamma',\gamma''} = \sum_{i=1}^{m'} B_{y_i}\omega_{\gamma',\gamma''} = \frac{\partial^2\omega_{\gamma',\gamma''}}{\partial r^2} + \frac{m'+|\gamma'|-1}{r}\frac{\partial\omega_{\gamma',\gamma''}}{\partial r} \tag{7.92}$$

and

$$\Delta_{\gamma''}\omega_{\gamma',\gamma''} = \sum_{i=1}^{m''} B_{z_i}\omega_{\gamma',\gamma''} = \frac{\partial^2\omega_{\gamma',\gamma''}}{\partial s^2} + \frac{m''+|\gamma''|-1}{s}\frac{\partial\omega_{\gamma',\gamma''}}{\partial s}. \tag{7.93}$$

By virtue of property 5 of the weighted spherical average we get

$$\Delta_{\gamma'}\omega_{\gamma',\gamma''} = \left(\Delta_{\gamma'} M_r^{\gamma'} M_s^{\gamma''} u\right)(y,r;z,s) = \left(M_r^{\gamma'}\Delta_{\gamma'} M_s^{\gamma''} u\right)(y,r;z,s)\,. \tag{7.94}$$

In (7.94) operator $\Delta_{\gamma'}$ and $M_s^{\gamma''}$ are also permutable, since they act on different (non-intersecting) groups of variables. Therefore,

$$\Delta_{\gamma'}\omega_{\gamma',\gamma''} = \left(M_r^{\gamma'} M_s^{\gamma''}\Delta_{\gamma'} u\right)(y,r;z,s)\,.$$

Similarly, we find

$$\Delta_{\gamma''}\omega_{\gamma',\gamma''} = \left(M_r^{\gamma'} M_s^{\gamma''} \Delta_{\gamma''} u\right)(y, r; z, s).$$

According to (7.90),

$$\Delta_{\gamma'} u = \Delta_{\gamma''} u.$$

This implies the equality of the right parts in (7.92) and (7.93) are equal. Consequently, the double weighted spherical mean of the solution of a B-ultrahyperbolic equation satisfies the singular differential equation

$$\frac{\partial^2 \omega_{\gamma',\gamma''}}{\partial s^2} + \frac{m' + |\gamma'| - 1}{s}\frac{\partial \omega_{\gamma',\gamma''}}{\partial s} = \frac{\partial^2 \omega_{\gamma',\gamma''}}{\partial r^2} + \frac{m'' + |\gamma''| - 1}{r}\frac{\partial \omega_{\gamma',\gamma''}}{\partial r}. \tag{7.95}$$

Eq. (7.95) is Eq. (3.145). Besides,

$$\omega_{\gamma',\gamma''}(y, z; r, s)|_{s=0} = \mu_{\gamma'}(y, z; r) = f(r)$$

and

$$\left.\frac{\partial}{\partial s}\omega_{\gamma',\gamma''}(y, z; r, s)\right|_{s=0} = 0.$$

That gives

$$\omega_{\gamma',\gamma''}(y, z; r, s) = {}^{m+|\gamma|-1}T_r^s f(r) = {}^{m+|\gamma|-1}T_s^r f(s) = \omega_{\gamma',\gamma''}(y, z; s, r), \tag{7.96}$$

$${}^{m+|\gamma|-1}T_r^s f(r) =$$

$$\frac{\Gamma(m + |\gamma| - 1)}{2^{m+|\gamma|-3}\Gamma^2\left(\frac{m+|\gamma|-1}{2}\right)} \int_0^\pi f(\sqrt{r^2 - 2rs\cos\varphi + s^2}) \sin^{m+|\gamma|-2}\varphi d\varphi.$$

The equality ${}^{m+|\gamma|-1}T_r^s f(r) = {}^{m+|\gamma|-1}T_s^r f(s)$ follows from property 5 of the generalized translation. That gives $\omega_{\gamma',\gamma''}(y, z; r, s) = \omega_{\gamma',\gamma''}(z, y; s, r)$ or (7.91). □

Corollary 5. *The weighted spherical mean of the function u taken at constant y by z by the part of a sphere of radius r is equal to the weighted spherical mean of the function u taken at constant z by y by the part of a sphere of radius r:*

$$\mu_\gamma(y, z, r) = \nu_\gamma(y, z, r).$$

Proof. Since

$$\mu_\gamma(y, z, r) = \omega_\gamma(y, z; r, 0), \quad \nu_\gamma(y, z, r) = \omega_\gamma(y, z; 0, r),$$

the equality $\mu_\gamma(y, z, r) = \nu_\gamma(y, z, r)$ follows from (7.91). □

As a corollary of Theorem 85 follows a generalization of the classical Asgeirsson theorem (see [9,75,155] on the B-ultrahyperbolic equation (7.90)).

Theorem 86. *Let the function $u = u(x, y) \in C^2_{ev}$, $(n = m' + m'')$ be a solution of the B-ultrahyperbolic equation (7.90) and let the condition (7.89) be valid. Then the weighted spherical mean of the function $u(x, y)$ taken at constant $x \in \overline{\mathbb{R}}^{m'}_{+}$ in $\mathbb{R}^{m''}_{+}$ by the part of the sphere of radius r is equal to the weighted spherical mean of the function $u(x, y)$ taken at constant $y \in \overline{\mathbb{R}}^{m''}_{+}$ in $\mathbb{R}^{m'}_{+}$ by the part of the sphere of radius r:*

$$(M_u^{\gamma'})_x(x, y, r) = (M_u^{\gamma''})_y(x, y, r). \tag{7.97}$$

The inverse statement to this theorem is also true. It is the inverse Asgeirsson theorem for the B-ultrahyperbolic equation.

Theorem 87. *Let $u(x, y) \in C^2_{ev}(\mathbb{R}^{m'}_{+} \times \mathbb{R}^{m''}_{+})$, $n = m' + m''$ and let for every point $(x, y) \in \mathbb{R}^{m'}_{+} \times \mathbb{R}^{m''}_{+}$ and for any nonnegative r and s condition (7.97) be true. Then if (7.89) is valid, then the function $u(x, y)$ satisfies the B-ultrahyperbolic equation (7.83) in $\mathbb{R}^{m'}_{+} \times \mathbb{R}^{m''}_{+}$.*

In [170], p. 222, a clarification of the Asgeirsson theorem is given. This clarification is generalized to the case of the B-ultrahyperbolic equation.

Theorem 88. *Let $x \in \mathbb{R}^{m'}_{+}$, $y \in \mathbb{R}^{m''}_{+}$, let $u = u(x, y) \in C^2_{ev}(\mathbb{R}^{m'}_{+} \times \mathbb{R}^{m''}_{+})$ be a continuous in some neighborhood of the set $K = \{\theta \in \mathbb{R}^{m'}_{+}, \omega \in \mathbb{R}^{m''}_{+} : |\theta| + |\omega| = r\}$ solution of the B-ultrahyperbolic equation $(\Delta_{\gamma'})_x u = (\Delta_{\gamma''})_y u$, and let $m' + |\gamma'| = m'' + |\gamma''| \geq 3$. Then*

$$\frac{1}{|S_1(m')|_\gamma} \int\limits_{S_1^+(m')} u(r\theta; 0) \prod_{i=1}^{m'} \theta_i^{\gamma_i} \, dS_\theta = \frac{1}{|S_1(m'')|_\gamma} \int\limits_{S_1^+(m'')} u(0; r\omega) \prod_{i=1}^{m''} \omega_i^{\gamma_i} \, dS_\omega.$$

7.2.4 Descent method for the general Euler–Poisson–Darboux equation

Using the generalized Asgeirsson relations (7.97) found in the previous subsection, the method of descent of the solution of the general Euler–Poisson–Darboux equation is presented.

Theorem 89. *Let $0 < \delta < n + |\gamma|$, and consider the natural number $m \geq 1$ and the multi-index $(\gamma'_2, \ldots, \gamma'_m)$, $\gamma'_i > 0$, such that*

$$n + |\gamma| = m + \delta + \gamma'_2 + \ldots + \gamma'_m$$

exist. Then $u \in C^2_{ev}(\mathbb{R}^+_{n+1})$ is the solution to the problem

$$\left(\frac{\partial}{\partial t} + \frac{\delta}{t}\frac{\partial}{\partial t}\right) u(x, t) = \Delta_\gamma u(x, t), \quad \delta > 0, \tag{7.98}$$

$$u(x,0)=f(x), \quad u_t(x,0)=0, \tag{7.99}$$

where $f\in C_{ev}(\mathbb{R}_n^+)$ is given by the formula

$$u(x,t)=\frac{2}{B\left(\frac{n+|\gamma|}{2},\frac{\delta-n-|\gamma|+1}{2}\right)}\,\frac{1}{t^\delta}\,\frac{d}{dt}\int\limits_0^t s\,(t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}\,ds\times$$

$$\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr. \tag{7.100}$$

Proof. We use the traditional "descent method," appealing to a singular ultrahyperbolic equation. Namely, if the function $u(x,t)$ satisfies Eq. (7.98), then the function $\widetilde{u}(x,y)$, extended from $u(x,t)$ as a constant in the direction $t_2,\dots t_m$, where $y=(t,t_2,\dots t_m)$ obviously, also satisfies the equation

$$(\Delta_{\gamma'})_y\widetilde{u}(x,y)=(\Delta_\gamma)_x\widetilde{u}(x,y)\,,\quad y=(t,\,t_2,\,\dots\,,\,t_m)\in\mathbb{R}_m^+\,, \tag{7.101}$$

where $\gamma'=(\delta,\gamma_2',...,\gamma_m')$, $\gamma_i'>0$.

Let m and $\gamma'=(\delta,\gamma_2',...,\gamma_m')$ such that $n+|\gamma|=m+|\gamma'|$, where $|\gamma'|=\delta+\gamma_2'+...+\gamma_m'$. Then for $\widetilde{u}(x,y)$ we have (7.97)

$$\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{S_1^+(n)}{}^\gamma T_x^{r\xi}\widetilde{u}(x,y)\,\xi^\gamma\,dS(\xi)=\frac{1}{|S_1^+(m)|_{\gamma'}}\int\limits_{S_1^+(m)}{}^{\gamma'}T_y^{r\zeta}\widetilde{u}(x,y)\,\zeta^{\gamma'}\,dS(\zeta), \tag{7.102}$$

where $S_1^+(n)=\{\xi\in\mathbb{R}_n^+:|\xi|=1\}$ and $S_1^+(m)=\{\zeta\in\mathbb{R}_m^+:|\zeta|=1\}$. Since the function $\widetilde{u}(x,y)$ $(y=(t,t_2,\dots,t_m))$ is a function $u(x,t)$ extended as a constant by variables $t_2,\dots,t_m$, we have

$${}^{\gamma'}T_y^{r\zeta}\widetilde{u}(x,y)={}^\delta T_t^{r\zeta_1}\widetilde{u}(x,t,\,t_2,\,\dots\,,t_m)={}^\delta T_t^{r\zeta_1}u(x,t)\,\cdot\,1(t_2\,\dots\,,t_m).$$

We agree to write further ${}^\delta T_t^{r\zeta_1}u(x,t)$ instead of ${}^\delta T_t^{r\zeta_1}u(x,t)\cdot 1(t_2\,\dots\,,t_m)$. Taking into account that $T_0^t u(x,0)=T_t^0 u(x,t)=u(x,t)$, we obtain

$${}^{\gamma'}T_y^{r\zeta}u(x,y_1)|_{y_1=0}={}^\delta T_{y_1}^{r\zeta_1}u(x,y_1)|_{y_1=0}=u(x,r\zeta_1).$$

Therefore, setting $t=0$ in (7.102) and using the first condition in (7.85), we obtain

$$\frac{1}{|S_1^+(n)|_\gamma}\int\limits_{S_1^+(n)}{}^\gamma T_x^{r\xi}f(x)\,\xi^\gamma\,dS=\frac{1}{|S_1^+(m)|_{\gamma'}}\int\limits_{S_1^+(m)}u(x,r\zeta_1)\,\zeta^{\gamma'}\,dS.$$

Therefore, to find the unknown function $u=u(x,t)$ it remains to solve the integral equation

$$\frac{1}{|S_1^+(m)|_{\gamma'}}\int\limits_{S_1^+(m)} u(x,r\zeta_1)\,\zeta^{\gamma'}\,dS=M_r^\gamma f(x). \tag{7.103}$$

Multiplying both sides of equality (7.103) by $r^{n+|\gamma|-1}$ and integrating over r from 0 to s, we obtain

$$\frac{1}{|S_1^+(m)|_{\gamma'}}\int\limits_0^s r^{n+|\gamma|-1}dr\int\limits_{S_1^+(m)} u(x,r\zeta_1)\,\zeta^{\gamma'}\,dS=\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr.$$

Consider the left side of this equality, which we denote by J. Returning to the rectangular Cartesian coordinates by the formula $z=r\zeta$, we obtain

$$J=\frac{1}{|S_1^+(m)|_{\gamma'}}\int\limits_{B_s^+(m)} u(x,z_1)\,z^{\gamma'}\,dz\,,\left(=\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr\right),$$

where $B_s^+(m)$ is a part of a ball of radius s with center in the origin belonging to R_m^+. Given the continuity of the integrand, we can write it in the repetitive form

$$J=\frac{1}{|S_1^+(m)|_{\gamma'}}\int\limits_0^s u(x,z_1)\,z_1^\delta\,dz_1\int\limits_{\Omega^+} z_2^{\gamma_2'}\,\ldots\,z_m^{\gamma_m'}\,dz_2\ldots dz_m,$$

where $\Omega^+=\left\{(z_2,\,\ldots,\,z_m)\,:\,\sqrt{z_2^2+\ldots+z_m^2}<\sqrt{s^2-z_1^2}\right\}\in\mathbb{R}_{m-1}^+$.

The integral by the $(m-1)$-dimensional domain Ω^+ can be easily calculated by transition to spherical coordinates $z_2{=}\rho\,\theta_1,\,\ldots,\,z_m{=}\rho\,\theta_{m-1}$. We have

$$\int\limits_{\Omega^+} z_2^{\gamma_2'}\ldots z_m^{\gamma_m'}dz_2...dz_m=\int\limits_0^{\sqrt{s^2-z_1^2}}\rho^{m+\gamma_2'+\ldots+\gamma_m'-2}d\rho\int\limits_{S_1^+(m-1)}\theta_1^{\gamma_2'}...\theta_{m-1}^{\gamma_m'}dS. \tag{7.104}$$

The inner integral on the right side of equality (7.104) can be calculated by formula (1.107):

$$\int\limits_{S_1^+(m-1)}\theta_1^{\gamma_2'}...\theta_{m-1}^{\gamma_m'}dS=\frac{\Gamma\left(\frac{\gamma_2'+1}{2}\right)...\Gamma\left(\frac{\gamma_m'+1}{2}\right)}{2^{m-2}\Gamma\left(\frac{m-1+\gamma_2'+\ldots+\gamma_m'}{2}\right)}. \tag{7.105}$$

The external integral on the right side of equality (7.104) is easily found by the formula

$$\int_0^{\sqrt{s^2-z_1^2}} \rho^{m+\gamma_2'+...+\gamma_m'-2} d\rho = \frac{(s^2-z_1^2)^{\frac{m+\gamma_2'+...+\gamma_m'-1}{2}}}{m+\gamma_2'+...+\gamma_m'-1}. \tag{7.106}$$

Since $m+\gamma_2'+...+\gamma_m' = n+|\gamma|-\delta$, formulas (7.105) and (7.106) can be rewritten in the forms

$$\int_{S_1^+(m-1)} \theta_1^{\gamma_2'}...\theta_{m-1}^{\gamma_m'} dS = \frac{\Gamma\left(\frac{\gamma_2'+1}{2}\right)...\Gamma\left(\frac{\gamma_m'+1}{2}\right)}{2^{m-2}\Gamma\left(\frac{n+|\gamma|-\delta-1}{2}\right)} \tag{7.107}$$

and

$$\int_0^{\sqrt{s^2-z_1^2}} \rho^{m+\gamma_2'+...+\gamma_m'-2} d\rho = \frac{(s^2-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}}}{n+|\gamma|-\delta-1}. \tag{7.108}$$

Substituting (7.107) and (7.108) into (7.104), we get

$$\int_{\Omega^+} z_2^{\gamma_2'}...z_m^{\gamma_m'} dz_2...dz_m = \frac{1}{n+|\gamma|-\delta-1} \frac{\Gamma\left(\frac{\gamma_2'+1}{2}\right)...\Gamma\left(\frac{\gamma_m'+1}{2}\right)}{2^{m-2}\Gamma\left(\frac{n+|\gamma|-\delta-1}{2}\right)} (s^2-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}}. \tag{7.109}$$

Using formula (1.107), we obtain $|S_1^+(m)|_{\gamma'}$:

$$|S_1^+(m)|_{\gamma'} = \frac{\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{\gamma_2'+1}{2}\right)...\Gamma\left(\frac{\gamma_m'+1}{2}\right)}{2^{m-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}. \tag{7.110}$$

Applying (7.109) and (7.110), we get

$$\frac{1}{|S_1^+(m)|_{\gamma'}} \int_{B_s^+(m)} u(x,z_1)\, z^{\gamma'}\, d\zeta =$$

$$\frac{2\Gamma\left(\frac{n+|\gamma|}{2}\right)}{(n+|\gamma|-\delta-1)\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{n+|\gamma|-\delta-1}{2}\right)} \int_0^s (s^2-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}} u(x,z_1) z_1^{\delta} dz_1.$$

So we obtain

$$\int_0^s (s^2-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}} u(x,z_1) z_1^{\delta} dz_1 =$$

$$\frac{\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{n+|\gamma|-\delta+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr. \tag{7.111}$$

In (7.111) the formula $z\Gamma(z)=\Gamma(z+1)$ was used.

Equality (7.111) is the Abel equation with respect to $u(x,t)$ (see [494]). Multiplying both parts of (7.111) by $2s(t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}$ and integrating by s from 0 to t, we obtain

$$\int\limits_0^t 2s(t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}ds\int\limits_0^s (s^2-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}}u(x,z_1)z_1^\delta dz_1 =$$
$$\frac{2\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{n+|\gamma|-\delta+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^t s(t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}ds\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr.$$

In the left part, we change the order of integration:

$$\int\limits_0^t u(x,z_1)z_1^\delta dz_1\int\limits_{z_1}^t (t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}(s^2-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}}\,2sds =$$
$$\frac{2\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{n+|\gamma|-\delta+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^t s\,(t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}ds\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr. \tag{7.112}$$

In the inner integral on the left side of (7.112) replacing s^2 by h, we get

$$\int\limits_{z_1}^t (t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}(s^2-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}}\,2sds=\int\limits_{z_1^2}^{t^2}(t^2-h)^{\frac{\delta-n-|\gamma|-1}{2}}(h-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}}\,dh.$$

In the resulting integral, we introduce a new variable τ by the formula $h=\zeta_1^2+\tau(t^2-\zeta_1^2)$:

$$dh=(t^2-\zeta_1^2)d\tau,\quad t^2-h=(1-\tau)(t^2-\zeta_1^2),\quad h-\zeta_1^2=\tau(t^2-\zeta_1^2),$$
$$h=\zeta_1^2,\quad \tau=0;\qquad h=t^2,\quad \tau=1.$$

We have

$$\int\limits_{z_1}^t (t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}(s^2-z_1^2)^{\frac{n+|\gamma|-\delta-1}{2}}\,2sds=\int\limits_0^1 \tau^{\frac{n+|\gamma|-\delta-1}{2}}(1-\tau)^{\frac{\delta-n-|\gamma|-1}{2}}d\tau=$$

$$\Gamma\left(\frac{n+|\gamma|-\delta+1}{2}\right)\Gamma\left(\frac{\delta-n-|\gamma|+1}{2}\right).$$

Returning to (7.112), we can write

$$\Gamma\left(\frac{n+|\gamma|-\delta+1}{2}\right)\Gamma\left(\frac{\delta-n-|\gamma|+1}{2}\right)\int\limits_0^t u(x,z_1)z_1^\delta dz_1 =$$
$$\frac{2\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{n+|\gamma|-\delta+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^t s\,(t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}ds\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr$$

or

$$\int\limits_0^t u(x,z_1)z_1^\delta dz_1 =$$
$$\frac{2\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{\delta-n-|\gamma|+1}{2}\right)}\int\limits_0^t s\,(t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}ds\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr. \tag{7.113}$$

Differentiating both sides of equality (7.113) by t and dividing by t^δ we get

$$u(x,t)=$$
$$\frac{2\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{\delta-n-|\gamma|+1}{2}\right)}\frac{1}{t^\delta}\frac{d}{dt}\int\limits_0^t s\,(t^2-s^2)^{\frac{\delta-n-|\gamma|-1}{2}}ds\int\limits_0^s r^{n+|\gamma|-1}\,M_r^\gamma f(x)dr. \tag{7.114}$$

Application in (7.114) of the formula for the Euler beta function (1.9) completes the proof. □

7.3 Elliptic equations with Bessel operator

7.3.1 Weighted homogeneous distributions

Here, following the approach of Gelfand and Shapiro [178] (see also [170,177]), we study weighted homogeneous generalized functions. The research of homogeneous distributions is important because the fundamental solutions of many differential operators are homogeneous distributions. If a differential operator contains a Bessel operator, it is natural to use weighted instead of ordinary distributions. We also note that when we consider analytic continuation of weighted homogeneous distributions,

we thereby give a method for analytic continuation for more general distributions, since many functions in the neighborhood of singular points can be approximated by homogeneous ones. Further, we will apply these results to finding the fundamental solution to elliptic equations with Bessel operator.

For all $a > -1$, functions x_+^a and x_-^a on $\mathbb{R}$ are

$$x_+^a = \begin{cases} x^a & x > 0, \\ 0 & x \leq 0 \end{cases}$$

and

$$x_-^a = \begin{cases} 0 & x \geq 0, \\ |x|^a & x < 0. \end{cases}$$

Function x_+^a is locally integrable with the weight x^γ, and therefore determines the weighted distribution

$$(x_+^a, \varphi)_\gamma = \int\limits_0^\infty x^a \varphi(x) x^\gamma \, dx.$$

Similarly,

$$(x_-^a, \varphi)_\gamma = (-1)^{a+\gamma} \int\limits_{-\infty}^0 x^a \varphi(x) x^\gamma \, dx.$$

Functions x_+^a and x_-^a are connected by the equality

$$(x_-^a, \varphi)_\gamma = (x_+^a, \check{\varphi})_\gamma, \qquad \check{\varphi}(x) = \varphi(-x).$$

Function x_+^a is homogeneous of degree a for $a > -1$, i.e., the following equality for $t > 0$ is valid:

$$(x_+^a, \varphi)_\gamma = \int\limits_0^\infty x^a \varphi(x) x^\gamma \, dx = t^a \int\limits_0^\infty x^a \varphi(tx) t^{1+\gamma} x^\gamma \, dx = t^a (x_+^a, \varphi_t)_\gamma,$$

$$\varphi_t(x) = t^{1+\gamma} \varphi(tx).$$

Let us find $(x \pm i0)^a$:

$$(x \pm i0)^a = e^{a \ln(x \pm i0)} = e^{a \ln(x \pm i0)} = e^{a(\ln|x \pm i0| + i \arg(x \pm i0))} =$$

$$e^{a(\ln|x| + i \arg(x \pm i0))} = \begin{cases} x^a & x \geq 0, \\ e^{\pm i\pi a} |x|^a & x < 0 \end{cases} = x_+^a + e^{\pm i\pi a} x_-^a.$$

Therefore,

$$(x \pm i0)^a = x_+^a + e^{\pm i\pi a} x_-^a. \tag{7.115}$$

Definition 42. *Let function $u \in L^1_{loc,\gamma}(\mathbb{R}^n_+)$ be homogeneous of degree a, i.e., $u(tx) = t^a u(x)$ for $t > 0$. The weighted distribution u is homogeneous of degree a in $\mathbb{R}^n_+$ if the following equality is valid:*

$$(u, \varphi)_\gamma = t^a (u(x), \varphi_t(x))_\gamma, \tag{7.116}$$

where $\varphi_t(x) = t^{n+|\gamma|}\varphi(tx)$, $\varphi \in \overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^+_n)$. If u is a weighted distribution in $\overline{\mathbb{R}}{}^n_+$ and (7.116) is valid for all $\varphi \in \overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^+_n)$, then u is said to be homogeneous of degree a in $\overline{\mathbb{R}}{}^n_+$.

Let us comment on Definition 42. If the function $u \in L^1_{loc,\gamma}(\mathbb{R}^n_+)$ is homogeneous of degree a, i.e., $u(tx) = t^a u(x)$, for $t > 0$, then

$$(u(y), \varphi(y))_\gamma = \int\limits_{\mathbb{R}^n_+} u(y)\varphi(y)y^\gamma dy = \{y = tx, t > 0\}$$
$$= t^{n+|\gamma|+a} \int\limits_{\mathbb{R}^n_+} u(x)\varphi(tx)x^\gamma dx = t^a (u(x), \varphi_t(x))_\gamma,$$

where $\varphi_t(x) = t^{n+|\gamma|}\varphi(tx)$, $\varphi \in \overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^+_n)$. And vice versa, from the relation $(u(y), \varphi(y))_\gamma = t^a (u(x), \varphi_t(x))_\gamma$ it follows that u is homogeneous. If $a > -n - |\gamma|$, then u is integrable with weight x^γ in some neighborhood of zero, since in polar coordinates $x = r\omega$, $|\omega| = 1$, we have $dx = r^{n+|\gamma|-1}\omega^\gamma dr d\omega$ and

$$\int\limits_{U_\varepsilon(0)} |u(r\omega)|x^\gamma dx = \int\limits_0^\varepsilon r^{a+n+|\gamma|-1} dr \int\limits_{\{|\omega|=1\}^+} |u(\omega)|\omega^\gamma d\omega$$
$$= \frac{\varepsilon^{a+n+|\gamma|}}{a+n+|\gamma|} \int\limits_{\{|\omega|=1\}^+} |u(\omega)|\omega^\gamma d\omega < \infty.$$

The problem which we will discuss is the extension of weighted homogeneous distributions from $\mathbb{R}^+_n$ to $\overline{\mathbb{R}}{}^n_+$. First we prove the following theorem about homogeneity conditions for weighted homogeneous distributions.

Theorem 90. *The homogeneity conditions for weighted homogeneous distributions u of degree a*

$$(u, \varphi)_\gamma = t^a (u, \varphi_{n,\gamma,t})_\gamma, \quad \varphi_{n,\gamma,t}(x) = t^{n+|\gamma|}\varphi(tx), \quad \varphi(x) \in \overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^+_n) \tag{7.117}$$

and

$$(u, \psi)_\gamma = 0, \quad \psi \in \overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^+_n), \quad \int\limits_0^\infty r^{a+n+|\gamma|-1}\psi(rx)dr = 0 \tag{7.118}$$

are equivalent. In addition, for weighted homogeneous distributions u of degree a the formula

$$\sum_{k=1}^{n} x_k \frac{\partial u}{\partial x_k} = au \tag{7.119}$$

is valid.

Proof. Differentiating equality (7.117) by t we get

$$\begin{aligned} &at^{a-1}(u(x), t^{n+|\gamma|}\varphi(tx))_\gamma + \\ &t^a \left(u(x), (n+|\gamma|)t^{n+|\gamma|-1}\varphi(tx) + t^{n+|\gamma|-1}\frac{d\varphi(tx)}{dt} \right)_\gamma = 0. \end{aligned} \tag{7.120}$$

Since

$$\frac{d\varphi(tx)}{dt} = \sum_{k=1}^{n} \frac{\partial \varphi(tx)}{\partial x_k}\frac{d(tx_k)}{dt} = \sum_{k=1}^{n} x_k \frac{\partial \varphi(tx)}{\partial x_k},$$

equality (7.120) can be written in the form

$$\begin{aligned} &at^{a-1}(u(x), t^{n+|\gamma|}\varphi(tx))_\gamma + \\ &t^a \left(u(x), (n+|\gamma|)t^{n+|\gamma|-1}\varphi(tx) + t^{n+|\gamma|-1}\sum_{k=1}^{n} x_k \frac{\partial \varphi(tx)}{\partial x_k} \right)_\gamma = 0. \end{aligned} \tag{7.121}$$

Putting $t=1$ in (7.121), we obtain

$$(a+n+|\gamma|)(u,\varphi)_\gamma + (u,\lambda\varphi)_\gamma = 0, \tag{7.122}$$

where $\lambda = \sum\limits_{k=1}^{n} x_k \frac{\partial}{\partial x_k}$.

Let us consider the equation

$$(a+n+|\gamma|)\varphi(x) + \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}\varphi(x) = \psi(x) \tag{7.123}$$

and show that it has a solution in $\overset{\circ}{C}{}^{\infty}_{ev}(\mathbb{R}^+_n)$. Using in (7.123) spherical coordinates $x = r\omega$, we get

$$\frac{\partial}{\partial r}(r^{a+n+|\gamma|}\varphi(r\omega)) = \psi(r\omega)r^{a+n+|\gamma|-1}.$$

Indeed,

$$\frac{\partial}{\partial r}(r^{a+n+|\gamma|}\varphi(r\omega)) = (a+n+|\gamma|)r^{a+n+|\gamma|-1}\varphi(r\omega) + r^{a+n+|\gamma|}\frac{\partial \varphi(r\omega)}{\partial r} =$$

$$(a+n+|\gamma|)r^{a+n+|\gamma|-1}\varphi(r\omega)+r^{a+n+|\gamma|}\sum_{k=1}^{n}\frac{\partial\varphi(r\omega)}{\partial(r\omega_k)}\frac{\partial(r\omega_k)}{\partial r}=$$

$$(a+n+|\gamma|)r^{a+n+|\gamma|-1}\varphi(r\omega)+r^{a+n+|\gamma|-1}\sum_{k=1}^{n}r\omega_k\frac{\partial\varphi(r\omega)}{\partial(r\omega_k)}=$$

$$r^{a+n+|\gamma|-1}\left[(a+n+|\gamma|)\varphi(x)+\sum_{k=1}^{n}x_k\frac{\partial}{\partial x_k}\varphi(x)\right]=r^{a+n+|\gamma|-1}\psi(x).$$

Consequently, for

$$\psi=(a+n+|\gamma|)\varphi+\lambda\varphi, \tag{7.124}$$

the equality

$$\int\limits_0^\infty r^{a+n+|\gamma|-1}\psi(rx)dr=\int\limits_0^\infty\frac{\partial}{\partial r}(r^{a+n+|\gamma|}\varphi(r\omega))dr=r^{a+n+|\gamma|}\varphi(r\omega)\Big|_0^\infty=0$$

is true and $\psi\in\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}_n^+)$. So, from (7.124) and (7.122) equality (7.118) follows.

Now let us prove (7.119). For $(u,\lambda\varphi)_\gamma$ we have

$$(u,\lambda\varphi)_\gamma=\left(u(x),\sum_{k=1}^{n}x_k\frac{\partial\varphi(x)}{\partial x_k}\right)_\gamma=\sum_{k=1}^{n}\int\limits_{\mathbb{R}_+^n}u(x)x_k\frac{\partial\varphi(x)}{\partial x_k}x^\gamma dx=$$

$$\sum_{k=1}^{n}\int\limits_{\mathbb{R}_+^{n-1}}x_1^{\gamma_1}...x_{k-1}^{\gamma_{k-1}}x_{k+1}^{\gamma_{k+1}}...x_n^{\gamma_n}dx_1...dx_{k-1}dx_{k+1}...dx_n\left[\int\limits_0^\infty u(x)\frac{\partial\varphi(x)}{\partial x_k}x_k^{\gamma_k+1}dx_k\right]. \tag{7.125}$$

We apply the integration formula by parts to the integral over x_k:

$$\int\limits_0^\infty u(x)\frac{\partial\varphi(x)}{\partial x_k}x_k^{\gamma_k+1}dx_k=\left\{U=u(x)x_k^{\gamma_k+1},dV=\frac{\partial\varphi(x)}{\partial x_k}dx_k\right\}=$$

$$u(x)\varphi(x)x_k^{\gamma_k+1}\Big|_0^\infty-\int\limits_0^\infty\left[x_k\frac{\partial u}{\partial x_k}+(1+\gamma_k)u(x)\right]\varphi(x)x_k^{\gamma_k}dx_k=$$

$$-\int\limits_0^\infty x_k\frac{\partial u}{\partial x_k}\,\varphi(x)x_k^{\gamma_k}dx_k-(1+\gamma_k)\int\limits_0^\infty u(x)\varphi(x)x_k^{\gamma_k}dx_k.$$

Summing by k from 1 to n and returning to the integral (7.125), we get

$$(u,\lambda\varphi)_\gamma=-(\lambda,\varphi)_\gamma-(n+|\gamma|)(u,\varphi)_\gamma. \tag{7.126}$$

Substituting (7.126) in (7.122), we obtain

$$(a+n+|\gamma|)(u(x),\varphi(x))_\gamma-\left(\sum_{k=1}^n x_k\frac{\partial u}{\partial x_k},\varphi(x)\right)_\gamma-(n+|\gamma|)(u(x),\varphi(x))_\gamma=0$$

or

$$a(u(x),\varphi(x))_\gamma=\left(\sum_{k=1}^n x_k\frac{\partial u}{\partial x_k},\varphi(x)\right)_\gamma,$$

which gives (7.119). □

7.3.2 Extension of the weighted homogeneous distributions

Theorem 91. *Let $u\in\mathcal{D}'_{ev}(\mathbb{R}^n_+)$ be weighted homogeneous distributions u of degree a. If $a\neq k$, $k\in\mathbb{Z}$, $k\leq -n-|\gamma|$, then u has a unique weighted extension $u^*\in\mathcal{D}'_{ev}(\overline{\mathbb{R}}^n_+)$ homogeneous of degree a. If $a\neq 1-n-|\gamma|$, then $(B_{\gamma_j}u)^*=B_{\gamma_j}u^*$. The map $u\to u^*$ is continuous.*

Proof. We first prove the existence of a weighted distribution $u^*\in\mathcal{D}'_{ev}(\overline{\mathbb{R}}^n_+)$, homogeneous of degree a, which is an extension of $u\in\mathcal{D}'_{ev}(\mathbb{R}^n_+)$.

If u is a function and $\varphi\in\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$, then using spherical coordinates $x=r\omega$ we get

$$(u,\varphi)_\gamma=\int\limits_{\mathbb{R}^n_+}u(x)\varphi(x)x^\gamma dx=\int\limits_0^\infty\int\limits_{\{|\omega|=1\}^+}u(\omega)\varphi(r\omega)r^{a+n+|\gamma|-1}\omega^\gamma drd\omega.$$

Based on this equality, we introduce the one-dimensional distribution

$$(R_a\varphi)(x)=(t_+^{a+n+|\gamma|-1},\varphi(tx)),\qquad \varphi\in\overset{\circ}{C}{}^\infty_{ev}(\overline{\mathbb{R}}^n_+). \tag{7.127}$$

The function $R_a\varphi$ is homogeneous of degree $-n-|\gamma|-a$, i.e., $(R_a\varphi)(bx)=b^{-n-|\gamma|-a}(R_a\varphi)(x)$. Indeed,

$$(R_a\varphi)(bx)=(t_+^{a+n+|\gamma|-1},\varphi(btx))=\int\limits_0^\infty t^{a+n+|\gamma|-1}\varphi(btx)dt=\{bt=y\}=$$

$$b^{-n-|\gamma|-a}\int\limits_0^\infty y^{a+n+|\gamma|-1}\varphi(yx)dy=b^{-n-|\gamma|-a}(R_a\varphi)(x).$$

From [170] it follows that R_a is a continuous map from $\overset{\circ}{C}{}^\infty_{ev}(K)$ to $\overset{\circ}{C}{}^\infty_{ev}(\mathbb{R}^n_+)$ for every compact set $K\subset\mathbb{R}^n_+$.

We choose a fixed function $\psi \in \overset{\circ}{C}{}^{\infty}_{ev}(\mathbb{R}^n_+)$ such that

$$\int\limits_0^\infty \psi(tx)\frac{dt}{t} = 1, \qquad x \neq 0.$$

Then $\psi R_a\varphi \in \overset{\circ}{C}{}^{\infty}_{ev}(\mathbb{R}^n_+)$ and

$$R_a(\psi R_a\varphi)(x) = \int\limits_0^\infty t^{a+n+|\gamma|-1}\psi(tx)(R_a\varphi)(tx)dt =$$

$$(R_a\varphi)(x)\int\limits_0^\infty \psi(tx)\frac{dt}{t} = (R_a\varphi)(x).$$

So, $u(\psi R_a\varphi)$ is always independent of ψ and $u(\psi R_a\varphi) = u(\varphi)$ if $\varphi \in \overset{\circ}{C}{}^{\infty}_{ev}(\mathbb{R}^n_+)$. Thus

$$(u^*, \varphi)_\gamma = (u, \psi R_a\varphi)_\gamma, \qquad \varphi \in \overset{\circ}{C}{}^{\infty}_{ev}(\overline{\mathbb{R}}{}^n_+)$$

defines a distribution u^* in $\overline{\mathbb{R}}{}^n_+$ which extends u. The map $u \to u^*$ is continuous. Since

$$(R_a\varphi_{n,\gamma,t})(x) = (r_+^{a+n+|\gamma|-1}, t^{n+|\gamma|}\varphi(rtx)) = t^{-a}R_a\varphi(x),$$

that gives homogeneity of u^*. Finally we note that $(B_{\gamma_j}u)^* - B_{\gamma_j}u^*$ is homogeneous of degree $a-2$ and supported by 0, so it must be zero. This completes the proof of the theorem. □

7.3.3 Weighted fundamental solution of the Laplace–Bessel operator

Weighted fundamental solutions are very important in the study of existence and regularity of solutions of differential equations with Bessel operators.

Definition 43. *A distribution $E \in \mathcal{D}'_{ev}(\mathbb{R}^n_+)$ is called a* ***weighted fundamental solution*** *of the differential operator with Bessel operators $L = \sum\limits_{i=1}^m a_i B_{\gamma_i}$ with constant (complex) coefficients if $LE = \delta_\gamma$.*

Theorem 92. *Let $u_1, ..., u_n \in \mathcal{D}'_{ev}(\mathbb{R}^n_+)$ all be homogeneous of degree $2-n-|\gamma|$ in $\mathbb{R}^n_+$ and satisfy the condition $\sum\limits_{j=1}^n B_{\gamma_j}u_j = 0$. Then*

$$\sum_{j=1}^{n} B_{\gamma_j} u_j^* = c\delta_\gamma,$$

where c is some constant.

Proof. The distribution $\sum\limits_{j=1}^{n} B_{\gamma_j} u_j^*$ is homogeneous of degree $-n-|\gamma|$ and supported by 0, so

$$\sum_{j=1}^{n} B_{\gamma_j} u_j^* = c\delta_\gamma$$

for some constant c. $\square$

Lemma 19. *Let $x \in \mathbb{R}_+^n$, $n > 1$, and*

$$E(x) = \begin{cases} \frac{1}{|S_1^+(n)|_\gamma} \ln|x| & n+|\gamma| = 2, \\ \frac{|x|^{2-n-|\gamma|}}{(2-n-|\gamma|)|S_1^+(n)|_\gamma} & n+|\gamma| > 2, \end{cases}$$

where $|S_1^+(n)|_\gamma$ is defined by (1.107). Then for $|x| > \varepsilon\ \forall\varepsilon > 0$

$$\triangle_\gamma E(x) = 0.$$

Proof. Let us consider $n+|\gamma| > 2$. We have

$$\triangle_\gamma E(x) = \sum_{j=1}^{n} B_{\gamma_j} E(x) = \sum_{j=1}^{n} \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} x_j^{\gamma_j} \frac{\partial}{\partial x_j} E(x) =$$

$$\frac{1}{(2-n-|\gamma|)|S_n^+|_\gamma} \sum_{j=1}^{n} \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} x_j^{\gamma_j} \frac{\partial}{\partial x_j} |x|^{2-n-|\gamma|} =$$

$$\frac{1}{(2-n-|\gamma|)|S_n^+|_\gamma} \sum_{j=1}^{n} \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} x_j^{\gamma_j} \frac{(2-n-|\gamma|)}{2} |x|^{-n-|\gamma|} 2x_j =$$

$$\frac{1}{|S_n^+|_\gamma} \sum_{j=1}^{n} \frac{1}{x_j^{\gamma_j}} \frac{\partial}{\partial x_j} |x|^{-n-|\gamma|} x_j^{1+\gamma_j} =$$

$$\frac{1}{|S_n^+|_\gamma} \sum_{j=1}^{n} \frac{1}{x_j^{\gamma_j}} \left[\frac{(-n-|\gamma|)}{2} |x|^{-n-|\gamma|-2} 2x_j^{2+\gamma_j} + (1+\gamma_j)|x|^{-n-|\gamma|} x_j^{\gamma_j} \right] =$$

$$\frac{1}{|S_n^+|_\gamma} \sum_{j=1}^{n} [(-n-|\gamma|)|x|^{-n-|\gamma|-2} x_j^2 + (1+\gamma_j)|x|^{-n-|\gamma|}] =$$

$$\frac{1}{|S_n^+|_\gamma} [(-n-|\gamma|)|x|^{-n-|\gamma|} + (n+|\gamma|)|x|^{-n-|\gamma|}] = 0.$$

We now consider the case $n+|\gamma|=2$. We get

$$\triangle_\gamma E(x)=\sum_{j=1}^n B_{\gamma_j}E(x)=\sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)=$$

$$\frac{1}{|S_n^+|_\gamma}\sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}\ln|x|=\frac{1}{|S_n^+|_\gamma}\sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}|x|^{-2}x_j^{1+\gamma_j}=$$

$$\frac{1}{|S_n^+|_\gamma}\sum_{j=1}^n \frac{1}{x_j^{\gamma_j}}[-2|x|^{-4}x_j^{2+\gamma_j}+(1+\gamma_j)|x|^{-2}x_j^{\gamma_j}]=$$

$$\frac{1}{|S_n^+|_\gamma}\sum_{j=1}^n[-2|x|^{-4}x_j^2+(1+\gamma_j)|x|^{-2}]=$$

$$\frac{1}{|S_n^+|_\gamma}[-2|x|^{-2}+(n+|\gamma|)|x|^{-2}]=0. \qquad \square$$

Theorem 93. *Let $x\in\mathbb{R}_n^+$, $n>1$, and*

$$E(x)=\begin{cases}\frac{1}{|S_2^+|_\gamma}\ln|x| & n+|\gamma|=2,\\ \frac{|x|^{2-n-|\gamma|}}{(2-n-|\gamma|)|S_n^+|_\gamma} & n+|\gamma|>2.\end{cases}$$

Then $B_{\gamma_j}E\in L^1_{loc,\gamma}(\mathbb{R}^n_+)$ and

$$\triangle_\gamma E=\delta_\gamma.$$

Proof. First let us prove that $B_{\gamma_j}E\in L^1_{loc,\gamma}(\mathbb{R}^n_+)$. For $\varphi\in \overset{\circ}{C}{}^\infty_{ev}(\overline{\mathbb{R}}{}^n_+)$, we have

$$(B_{\gamma_j}E,\varphi)_\gamma=\int\limits_{\mathbb{R}^n_+}B_{\gamma_j}E(x)\;\varphi(x)x^\gamma dx=\int\limits_{\mathbb{R}^n_+}\left(\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)\right)\varphi(x)x^\gamma dx=$$

$$\int\limits_{\mathbb{R}^{n-1}_+}x_1^{\gamma_1}...x_{j-1}^{\gamma_{j-1}}x_{j+1}^{\gamma_{j+1}}...x_n^{\gamma_n}dx_1...dx_{j-1}dx_{j+1}...dx_n\int\limits_0^\infty\left(\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)\right)\varphi(x)dx_j.$$

Integrating by parts by x_j we obtain

$$\int\limits_0^\infty(B_{\gamma_j}E(x))\,\varphi(x)x_j^{\gamma_j}dx_j=\int\limits_0^\infty\left(\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)\right)\varphi(x)x_j^{\gamma_j}dx_j=$$

$$\int\limits_0^\infty\left(\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)\right)\varphi(x)dx_j=\left\{U=\varphi(x),dV=\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)dx_j\right\}=$$

$$x_j^{\gamma_j}\frac{\partial}{\partial x_j}E(x)\varphi(x)\Big|_0^\infty-\int\limits_0^\infty x_j^{\gamma_j}\left(\frac{\partial}{\partial x_j}E(x)\right)\frac{\partial}{\partial x_j}\varphi(x)dx_j=$$

$$-\int\limits_0^\infty\left(\frac{\partial}{\partial x_j}E(x)\right)x_j^{\gamma_j}\frac{\partial}{\partial x_j}\varphi(x)dx_j=$$

$$\left\{U=x_j^{\gamma_j}\frac{\partial}{\partial x_j}\varphi(x),dV=\frac{\partial}{\partial x_j}E(x)dx_j\right\}=$$

$$-x_j^{\gamma_j}\frac{\partial}{\partial x_j}\varphi(x)E(x)\Big|_0^\infty+\int\limits_0^\infty\left(\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}\varphi(x)\right)E(x)dx_j=$$

$$\int\limits_0^\infty\left(\frac{1}{x_j^{\gamma_j}}\frac{\partial}{\partial x_j}x_j^{\gamma_j}\frac{\partial}{\partial x_j}\varphi(x)\right)E(x)x_j^{\gamma_j}dx_j=\int\limits_0^\infty(B_{\gamma_j}\varphi(x))\,E(x)x_j^{\gamma_j}dx_j.$$

Therefore,

$$(B_{\gamma_j}E,\varphi)_\gamma=(E,B_{\gamma_j}\varphi)_\gamma=\lim_{\varepsilon\to0}\int\limits_{\{|x|>\varepsilon\}^+}(B_{\gamma_j}\varphi(x))\,E(x)x^\gamma dx=$$

$$\lim_{\varepsilon\to0}\left[\int\limits_{\mathbb{R}_+^n}(B_{\gamma_j}\varphi(x))\,E(x)x^\gamma dx-\int\limits_{\{|x|\le\varepsilon\}^+}(B_{\gamma_j}\varphi(x))\,E(x)x^\gamma dx\right].$$

Using formula (1.99) we obtain

$$(B_{\gamma_j}E,\varphi)_\gamma=\int\limits_{\mathbb{R}_n^+}(B_{\gamma_j}\varphi(x))\,E(x)x^\gamma dx-\lim_{\varepsilon\to0}\int\limits_{\{|x|=\varepsilon\}^+}\frac{\partial\varphi(x)}{\partial x_j}\,E(x)\cos(\vec{\nu},\vec{e_j})x^\gamma dS,$$

where $\vec{\nu}$ is the direction of the outer normal to the boundary $\{|x|=\varepsilon\}^+$ and $\vec{e_j}$ is the direction of axis Ox_j. Since $\cos(\vec{\nu},\vec{e_j})=\frac{x_j}{|x|}$, we have

$$(B_{\gamma_j}E,\varphi)_\gamma=\int\limits_{\mathbb{R}_+^n}(B_{\gamma_j}\varphi(x))\,E(x)x^\gamma dx-\lim_{\varepsilon\to0}\int\limits_{\{|x|=\varepsilon\}^+}\frac{\partial\varphi(x)}{\partial x_j}\,E(x)\frac{x_j}{|x|}x^\gamma dS.$$

Moreover, we have

$$\lim_{\varepsilon\to0}\int\limits_{\{|x|=\varepsilon\}^+}\frac{\partial\varphi(x)}{\partial x_j}\,E(x)\frac{x_j}{|x|}x^\gamma dS=0$$

and thus

$$(B_{\gamma_j} E,\varphi)_\gamma=\int\limits_{\mathbb{R}^n_+}(B_{\gamma_j} E(x))\,\varphi(x)x^\gamma\,dx,$$

i.e., the weighted distribution $B_{\gamma_j} E$ defined by a function $B_{\gamma_j} E(x)$ locally integrable with the weight x^γ.

Since for $|x|>\varepsilon$, $\forall\varepsilon>0$, the equality

$$\triangle_\gamma E(x)=0$$

holds, using again formula (1.99) we obtain

$$(\triangle_\gamma E,\varphi)_\gamma=(E,\triangle_\gamma\varphi)_\gamma=\lim_{\varepsilon\to 0}\int\limits_{\{|x|>\varepsilon\}^+}E(x)(\triangle_\gamma\varphi(x))x^\gamma\,dx=$$

$$\lim_{\varepsilon\to 0}\int\limits_{\{|x|>\varepsilon\}^+}[E(x)(\triangle_\gamma\varphi(x))-(\triangle_\gamma E(x))\varphi(x)]x^\gamma\,dx=$$

$$\lim_{\varepsilon\to 0}\int\limits_{\{|x|=\varepsilon\}^+}\left(E(x)\frac{\partial\varphi(x)}{\partial\vec{\nu}}-\varphi(x)\frac{\partial E(x)}{\partial\vec{\nu}}\right)x^\gamma\,dS=\varphi(0).$$

This result also follows from Theorem 92. □

7.3.4 The Dirichlet problem for an elliptic singular equation

In this subsection we consider the second order singular elliptic equations of the form

$$\sum_{i=1}^n\left(\frac{\partial^2 u}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i}\right)=b^2u. \tag{7.128}$$

For $b>0$ we find an analytical solution to the Dirichlet problem for this equation. A procedure for obtaining its solution is proposed based on the use of a modified Bessel function of the second kind.

Elliptic equations with the Bessel operator occur in modern models of mathematical physics. Methods for solving problems for elliptic equations with the Bessel operator have been sufficiently developed by A. Weinstein [592,594] and I. A. Kipriyanov [242]. In the paper of M. B. Kapilevic [217] the theory of degenerate elliptic differential equations of Bessel class were considered. Here following [216] we obtain the solution to the Dirichlet problem for Eq. (7.128) and get some properties of that solution.

Theorem 94. *Let $\tau(x')=\tau(x_1,...,x_{n-1})\in C^2_{ev}$ be a bounded function, $\gamma_n<1$. The solution to the Dirichlet problem*

$$(\triangle_\gamma)_x u=b^2u,\qquad b>0, \tag{7.129}$$

$$u(x_1,x_2,...,x_{n-1},0)=\tau(x_1,x_2,...,x_{n-1}), \qquad u_{x_i}(x)|_{x_i=0}=0 \tag{7.130}$$

is

$$u(x)=C(n,\gamma)\int\limits_{\mathbb{R}^{n-1}_+}[\,{}^{\gamma'}\mathbf{T}^{x_n y'}_{x'}\tau](x')\,(1+|y'|)^{\frac{\gamma_n-n-|\gamma'|}{2}}\,\widetilde{K}_{\frac{n+|\gamma'|-\gamma_n}{2}}(bx_n\sqrt{1+|y'|})(y')^{\gamma'}dy', \tag{7.131}$$

where $(y')^{\gamma'}=\prod\limits_{i=1}^{n-1}y_i^{\gamma_i}$,

$$C(n,\gamma)=\frac{2^{n-1}\Gamma\left(\frac{n+|\gamma'|-\gamma_n}{2}\right)}{\Gamma\left(\frac{1-\gamma_n}{2}\right)\prod\limits_{i=1}^{n-1}\Gamma\left(\frac{\gamma_i+1}{2}\right)}, \qquad \widetilde{K}_\nu(r)=\frac{2^{1-\nu}}{\Gamma(\nu)}r^\nu K_\nu(r),$$

such that $\widetilde{K}_\nu(0)=1$ *when* $\nu>0$.

Proof. Let us show that a solution to Eq. (7.129) is

$$u=x_n^{1-\gamma_n}k_{\frac{n+|\gamma'|-\gamma_n}{2}}(b|x|)=$$
$$2^{\frac{n+|\gamma'|-\gamma_n}{2}}\Gamma\left(\frac{n+|\gamma'|-\gamma_n}{2}+1\right)x_n^{1-\gamma_n}|x|^{\frac{\gamma_n-n-|\gamma'|}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}}(b|x|)=$$
$$2^{\frac{n+|\gamma'|-\gamma_n}{2}}\Gamma\left(\frac{n+|\gamma'|-\gamma_n}{2}+1\right)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}}\times$$
$$K_{\frac{n+|\gamma'|-\gamma_n}{2}}\left(b\sqrt{x_1^2+...+x_n^2}\right).$$

Taking into account that (see [591], p. 96, formula (6))

$$\frac{\partial}{\partial z}z^{-\nu}K_\nu(z)=-z^{-\nu}K_{\nu+1}(z),$$

we can easily check that u is a solution to (7.129). When $i=1,...,n-1$, we have for $\frac{\partial u}{\partial x_i}$, $\frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i}$, and $\frac{\partial^2 u}{\partial x_i^2}$

$$\frac{\partial u}{\partial x_i}=-bx_iC(n,\gamma)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\times$$
$$\left(b\sqrt{x_1^2+...+x_n^2}\right),$$
$$\frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i}=-b\gamma_iC(n,\gamma)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\times$$
$$\left(b\sqrt{x_1^2+...+x_n^2}\right),$$

$$\frac{\partial^2 u}{\partial x_i^2} = -bC(n,\gamma)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\times$$
$$\left(b\sqrt{x_1^2+...+x_n^2}\right)+$$
$$b^2x_i^2C(n,\gamma)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-1}K_{\frac{n+|\gamma'|-\gamma_n}{2}+2}\left(b\sqrt{x_1^2+...+x_n^2}\right).$$

Considering that $B_{\gamma_i}u = \frac{\partial^2 u}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial u}{\partial x_i}$ and summing by i from 1 to $n-1$, we obtain

$$\sum_{i=1}^{n-1} B_{\gamma_i}u = -b(n-1+|\gamma'|)C(n,\gamma)\times$$
$$x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\left(b\sqrt{x_1^2+...+x_n^2}\right)+$$
$$b^2|x'|^2C(n,\gamma)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-1}K_{\frac{n+|\gamma'|-\gamma_n}{2}+2}\left(b\sqrt{x_1^2+...+x_n^2}\right).$$

Now let us find $\frac{\partial u}{\partial x_n}$, $\frac{\gamma_n}{x_n}\frac{\partial u}{\partial x_n}$, and $\frac{\partial^2 u}{\partial x_n^2}$:

$$\frac{\partial u}{\partial x_n} = C(n,\gamma)(1-\gamma_n)x_n^{-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}}K_{\frac{n+|\gamma'|-\gamma_n}{2}}\left(b\sqrt{x_1^2+...+x_n^2}\right)-$$
$$C(n,\gamma)bx_n^{2-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\left(b\sqrt{x_1^2+...+x_n^2}\right),$$
$$\frac{\gamma_n}{x_n}\frac{\partial u}{\partial x_n} =$$
$$C(n,\gamma)\gamma_n(1-\gamma_n)x_n^{-1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}}K_{\frac{n+|\gamma'|-\gamma_n}{2}}\left(b\sqrt{x_1^2+...+x_n^2}\right)-$$
$$C(n,\gamma)b\gamma_n x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\left(b\sqrt{x_1^2+...+x_n^2}\right),$$
$$\frac{\partial^2 u}{\partial x_n^2} =$$
$$C(n,\gamma)(-\gamma_n)(1-\gamma_n)x_n^{-1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}}K_{\frac{n+|\gamma'|-\gamma_n}{2}}\left(b\sqrt{x_1^2+...+x_n^2}\right)-$$
$$C(n,\gamma)b(1-\gamma_n)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\left(b\sqrt{x_1^2+...+x_n^2}\right)-$$
$$C(n,\gamma)b(2-\gamma_n)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\left(b\sqrt{x_1^2+...+x_n^2}\right)+$$
$$C(n,\gamma)b^2x_n^{3-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-1}K_{\frac{n+|\gamma'|-\gamma_n}{2}+2}\left(b\sqrt{x_1^2+...+x_n^2}\right).$$

Considering that $B_{\gamma_n}u=\frac{\partial^2 u}{\partial x_n^2}+\frac{\gamma_n}{x_n}\frac{\partial u}{\partial x_n}$, we obtain

$$B_{\gamma_n}u=$$
$$(\gamma_n-3)C(n,\gamma)bx_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\left(b\sqrt{x_1^2+...+x_n^2}\right)+$$
$$C(n,\gamma)b^2x_n^{3-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-1}K_{\frac{n+|\gamma'|-\gamma_n}{2}+2}\left(b\sqrt{x_1^2+...+x_n^2}\right).$$

Using the recurrence formula $K_{\nu+1}(z)-\frac{2\nu}{z}K_\nu(z)=K_{\nu-1}(z)$ (see [591]), we get

$$\Delta_\gamma u=\sum_{i=1}^{n-1}B_{\gamma_i}u+B_{\gamma_n}u=-b(n+|\gamma'|-\gamma_n+2)C(n,\gamma)x_n^{1-\gamma_n}\times$$
$$(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}-\frac{1}{2}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+1}\left(b\sqrt{x_1^2+...+x_n^2}\right)+$$
$$C(n,\gamma)b^2x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}}K_{\frac{n+|\gamma'|-\gamma_n}{2}+2}\left(b\sqrt{x_1^2+...+x_n^2}\right)=$$
$$b^2C(n,\gamma)x_n^{1-\gamma_n}(x_1^2+...+x_n^2)^{\frac{\gamma_n-n-|\gamma'|}{4}}K_{\frac{n+|\gamma'|-\gamma_n}{2}}\left(b\sqrt{x_1^2+...+x_n^2}\right)=$$
$$b^2x_n^{1-\gamma_n}k_{\frac{n+|\gamma'|-\gamma_n}{2}}\left(b\sqrt{x_1^2+...+x_n^2}\right).$$

So $u=x_n^{1-\gamma_n}k_{\frac{n+|\gamma'|-\gamma_n}{2}}(br)$ is a solution to (7.129).

Since ${}^{\gamma'}\mathbf{T}_{x'}^{y'}(\Delta_\gamma)_x=(\Delta_\gamma)_x\,{}^{\gamma'}\mathbf{T}_{x'}^{y'}$, $x_n^{1-\gamma_n}[{}^{\gamma'}\mathbf{T}_{x'}^{y'}k_{\frac{n+|\gamma'|-\gamma_n}{2}}](b|x|)$ is also a solution to (7.129). Moreover, if $g(x')=g(x_1,...,x_{n-1})$ is a twice continuously differentiable, bounded function such that $g_{x_i}(x')|_{x_i=0}=0$ for $i=1,...,n-1$, then $x_n^{1-\gamma_n}g(y')[{}^{\gamma'}\mathbf{T}_{x'}^{y'}k_{\frac{n+|\gamma'|-\gamma_n}{2}}](b|x|)(y')^{\gamma'}$ is also a solution to (7.129). Integrating by y', the function

$$u(x)=x_n^{1-\gamma_n}\int\limits_{\mathbb{R}_+^{n-1}}g(y')[{}^{\gamma'}\mathbf{T}_{x'}^{y'}k_{\frac{n+|\gamma'|-\gamma_n}{2}}](b|x|)(y')^{\gamma'}dy'=$$
$$x_n^{1-\gamma_n}\int\limits_{\mathbb{R}_+^{n-1}}[{}^{\gamma'}\mathbf{T}_{x'}^{y'}g](x')k_{\frac{n+|\gamma'|-\gamma_n}{2}}(b\sqrt{|y'|^2+x_n^2})(y')^{\gamma'}dy'$$

satisfies Eq. (7.129). Changing variables $y'\to x_ny'$, we get

$$u(x)=\int\limits_{\mathbb{R}_+^{n-1}}[{}^{\gamma'}\mathbf{T}_{x'}^{x_ny'}g](x')\,(1+|y'|^2)^{\frac{\gamma_n-n-|\gamma'|}{2}}\widetilde{K}_{\frac{n+|\gamma'|-\gamma_n}{2}}(bx_n\sqrt{1+|y'|^2})(y')^{\gamma'}dy'. \tag{7.132}$$

Putting $x_n = 0$, we obtain

$$\tau(x') = g(x') \int\limits_{\mathbb{R}_+^{n-1}} (1 + |y'|^2)^{\frac{\gamma_n - n - |\gamma'|}{2}} (y')^{\gamma'} dy' = \{y' = \sigma\rho\} =$$

$$g(x') \int\limits_0^\infty (1 + \rho^2)^{\frac{\gamma_n - n - |\gamma'|}{2}} \rho^{n+|\gamma'|-2} d\rho \int\limits_{S_1^+(n-1)} \sigma^{\gamma'} dS =$$

$$g(x') \frac{\prod\limits_{i=1}^{n-1} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)} \int\limits_0^\infty (1 + \rho^2)^{\frac{\gamma_n - n - |\gamma'|}{2}} \rho^{n+|\gamma'|-2} d\rho = \frac{1}{C(n, \gamma)} g(x')$$

or $g(x') = C(n, \gamma)\tau(x')$, where $C(n, \gamma) = \dfrac{2^{n-1}\Gamma\left(\frac{n+|\gamma'|-\gamma_n}{2}\right)}{\Gamma\left(\frac{1-\gamma_n}{2}\right) \prod\limits_{i=1}^{n-1} \Gamma\left(\frac{\gamma_i+1}{2}\right)}$, $\gamma_n < 1$. Substituting the expression for $g(x')$ into equality (7.132) gives (7.131).

Since $K_\nu(r)$ is exponentially decaying when $r \to \infty$, the integral (7.131) is uniformly and absolutely convergent. □

Theorem 95. *Let $\tau(x') = \tau(x_1, ..., x_{n-1}) \in C_{ev}^2$ be a bounded function, $\gamma_n < 1$,*

$$C(n, \gamma) = \frac{2^{n-1}\Gamma\left(\frac{n+|\gamma'|-\gamma_n}{2}\right)}{\Gamma\left(\frac{1-\gamma_n}{2}\right) \prod\limits_{i=1}^{n-1} \Gamma\left(\frac{\gamma_i+1}{2}\right)}, \qquad \widetilde{K}_\nu(r) = \frac{2^{1-\nu}}{\Gamma(\nu)} r^\nu K_\nu(r).$$

The function

$$u(x) = C(n,\gamma) \int\limits_{\mathbb{R}_+^{n-1}} [\gamma' \mathbf{T}_{x'}^{x_n y'} \tau](x') \, (1+|y'|)^{\frac{\gamma_n - n - |\gamma'|}{2}} \widetilde{K}_{\frac{n+|\gamma'|-\gamma_n}{2}}(b x_n \sqrt{1+|y'|})(y')^{\gamma'} dy' \tag{7.133}$$

is a function from C_{ev}^2 bounded at the orthant $\mathbb{R}_+^n$ and vanishes when $x_n \to \infty$.

Proof. Let $|\tau(x')| \leq M$ for $0 < x_i \leq \infty$, $i = 1, ..., n-1$. Then using property (3.149) of generalized translation, we obtain

$$|u(x)| \leq M \cdot C(n, \gamma) \int\limits_{\mathbb{R}_+^{n-1}} (1 + |y'|)^{\frac{\gamma_n - n - |\gamma'|}{2}} \widetilde{K}_{\frac{n+|\gamma'|-\gamma_n}{2}}(b x_n \sqrt{1 + |y'|})(y')^{\gamma'} dy'.$$

Passing to spherical coordinates $y' = \sigma\rho$, we get

$$\int\limits_{\mathbb{R}_+^{n-1}} (1+|y'|^2)^{\frac{\gamma_n-n-|\gamma'|}{2}} \widetilde{K}_{\frac{n+|\gamma'|-\gamma_n}{2}}(bx_n\sqrt{1+|y'|^2})(y')^{\gamma'}dy' =$$

$$\int\limits_0^\infty (1+\rho^2)^{\frac{\gamma_n-n-|\gamma'|}{2}} \widetilde{K}_{\frac{n+|\gamma'|-\gamma_n}{2}}(bx_n\sqrt{1+\rho^2})\rho^{n+|\gamma'|-2}d\rho \int\limits_{S_1^+(n-1)} \sigma^{\gamma'}dS =$$

$$\frac{\prod\limits_{i=1}^{n-1}\Gamma\left(\frac{\gamma_i+1}{2}\right)2^{1-\frac{n+|\gamma'|-\gamma_n}{2}}}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)\Gamma\left(\frac{n+|\gamma'|-\gamma_n}{2}\right)}(bx_n)^{\frac{n+|\gamma'|-\gamma_n}{2}}\times$$

$$\int\limits_0^\infty (1+\rho^2)^{\frac{\gamma_n-n-|\gamma'|}{4}} K_{\frac{n+|\gamma'|-\gamma_n}{2}}(bx_n\sqrt{1+\rho^2})\rho^{n+|\gamma'|-2}d\rho.$$

Let us find the integral using formula (2.13.1.2) from [456]:

$$I = \int\limits_0^\infty (1+\rho^2)^{\frac{\gamma_n-n-|\gamma'|}{4}} K_{\frac{n+|\gamma'|-\gamma_n}{2}}(bx_n\sqrt{1+\rho^2})\rho^{n+|\gamma'|-2}d\rho =$$

$$\{\sqrt{1+\rho^2}=z\} = \int\limits_1^\infty z^{\frac{\gamma_n-n-|\gamma'|}{2}+1} K_{\frac{n+|\gamma'|-\gamma_n}{2}}(bx_n z)(z^2-1)^{\frac{n+|\gamma'|-3}{2}}dz =$$

$$(bx_n)^{\frac{1-n-|\gamma'|}{2}} 2^{\frac{n+|\gamma'|-3}{2}}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right) K_{\frac{1-\gamma_n}{2}}(bx_n).$$

So

$$|u(x)| \le M\cdot C(n,\gamma)\frac{\prod\limits_{i=1}^{n-1}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-\frac{\gamma_n+3}{2}}\Gamma\left(\frac{n+|\gamma'|-\gamma_n}{2}\right)}(bx_n)^{\frac{1-\gamma_n}{2}}K_{\frac{1-\gamma_n}{2}}(bx_n) =$$

$$M\frac{2^{\frac{\gamma_n+1}{2}}}{\Gamma\left(\frac{1-\gamma_n}{2}\right)}(bx_n)^{\frac{1-\gamma_n}{2}}K_{\frac{1-\gamma_n}{2}}(bx_n) = M\,\widetilde{K}_{\frac{1-\gamma_n}{2}}(bx_n),$$

which gives

$$|u(x)| \le M$$

since $\widetilde{K}_{\frac{1-\gamma_n}{2}}(bx_n) \le 1$ for $x_n > 0$. Considering that

$$K_\nu(r) = \sqrt{\frac{\pi}{2r}}e^{-r}[1+O(r^{-1})]$$

for large r, we get $u(x) \to 0$ when $x_n \to \infty$. The fact that $u \in C^2_{ev}$ follows from the properties of integral (7.133) and from the fact that $\tau(x') \in C^2_{ev}$. So the theorem is proved. □

A classical solution to the Dirichlet problem for a singular second order linear elliptic partial differential equation has been obtained. This equation contains, for example, the Tricomi equation in the upper half-plane, which arises in the study of aerodynamics (see [385]). The Keldysh equation arising in modeling weak shock reflection at a wedge is a particular case of the studied equation (see [171,434]). One more particular case of (7.128) is the equation of Weinstein generalized axially symmetric potential theory, which arises in the study of fluid dynamics and elasticity [592, 594]. Finally, Eq. (7.128) generalizes the Schrödinger equation with a singular potential, which arises in quantum mechanics (see [24]). There are a lot of open problems for Eq. (7.128), for example, the principle of extremum, the principle of Hopf, the principle of Zaremba–Giraud, and other quality properties of the solution to (7.128).

Applications of transmutations to different problems

8

8.1 Inverse problems and applications of Buschman–Erdélyi transmutations

8.1.1 *Inverse problems*

In this subsection we would like to pay attention to deep connection of inverse problems and transmutation theory.

From historical perspectives transmutation methods formed a basis to solve initial inverse problems for Sturm–Lioville operators (see [322,323,327]). In this case the inverse problem by spectral function data or the inverse problem by scattering data leads by applying transmutations to similar integral equations, namely famous Marchenko and Gelfand–Levitan equations (see [52,60,376]). These are equations for transmutation kernels, and after finding them transmutation kernels on diagonals recover potentials from initial equations by simple formulas. The same technique based on transmutations also works for perturbed Bessel equations, Dirac system, and some other general problems (see [60,376,567,590]).

Although Marchenko and Levitan–Gelfand equations lead to many important results and applications in different fields these equations have some restrictions which were by and by recognized. In short the restrictions are due to 1) computational difficulties in gathering data and numeric solving of these equations; 2) lower effectiveness for multidimensional problems. So for many years another competing methods for solving inverse problems for Sturm–Liouville equations were proposed, which avoid the direct usage of Marchenko and Levitan–Gelfand equations. Among them are:

- method of M. G. Krein, cf. [296–301];
- method of A. N. Tikhonov, cf. [8,181,566,567];
- method of spectral mappings of Z. L. Leybenzon, cf. [310–312,603,604],
- method of V. V. Kravchenko, cf. [279,280].

The above mentioned methods together with Marchenko and Levitan–Gelfand equations form a powerful ground for solving different theoretical and applied inverse problems.

For the connection of Riesz potential theory to inverse problem see [162,449]. Namely, hyperbolic Riesz potential is the negative real power of the hyperbolic operator (10.44). Such operator inversion problem is closely related to the determination of a function from its weighted integral over Lorentzian spheres or Lorentzian weighted spherical mean (for non-weighted case see [164]). A problem of finding weighted spherical mean $(M_r^\gamma)f$ (see formula (3.183)) from f is well posed, if it has a unique solution and small variations in f lead to small variations in $(M_r^\gamma)f$. Otherwise the

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problem is ill-posed (see [302]). For the weighted spherical mean $(M_r^\gamma)f$ we have very simple formula for finding function f if we know $(M_r^\gamma)f$. It is $f = \lim\limits_{r\to 0}(M_r^\gamma)f$ (see [350]). But for the Lorentzian weighted spherical mean the inverse problem is not simple at all. In [164] for the determination of a function from its integral over Lorentzian spheres the hyperbolic Riesz potential was used. In this chapter we obtain an inverse operator for the mixed hyperbolic Riesz B-potential which generalizes the classical, used in [164]. Thus, it will be possible to solve the problem of finding a function from its Lorentzian weighted spherical mean.

8.1.2 Copson lemma

Consider the partial differential equation with two variables on the plane

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{2\alpha}{x}\frac{\partial u(x,y)}{\partial x} = \frac{\partial^2 u(x,y)}{\partial y^2} + \frac{2\beta}{y}\frac{\partial u(x,y)}{\partial y}$$

(this is the Euler–Poisson–Darboux equation or the B-hyperbolic one in Kipriyanov's terminology) for $x>0$, $y>0$, and $\beta>\alpha>0$ with boundary conditions on the characteristics

$$u(x,0) = f(x), \quad u(0,y) = g(y), \quad f(0) = g(0).$$

It is supposed that the solution $u(x,y)$ is continuously differentiable in the closed first quadrant and has second derivatives in this open quadrant, and boundary functions $f(x)$, $g(y)$ are differentiable.

Then if the solution exists, the following formulas hold true:

$$\frac{\partial u}{\partial y} = 0,\ y = 0,\ \frac{\partial u}{\partial x} = 0,\ x = 0, \tag{8.1}$$

$$2^\beta \Gamma\left(\beta + \frac{1}{2}\right)\int\limits_0^1 f(xt)t^{\alpha+\beta+1}\left(1-t^2\right)^{\frac{\beta-1}{2}} P_{-\alpha}^{1-\beta} t\, dt = \tag{8.2}$$

$$2^\alpha \Gamma\left(\alpha + \frac{1}{2}\right)\int\limits_0^1 g(xt)t^{\alpha+\beta+1}\left(1-t^2\right)^{\frac{\alpha-1}{2}} P_{-\beta}^{1-\alpha} t\, dt.$$

Therefore,

$$g(y) = \frac{2\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(\beta-\alpha)} y^{1-2\beta}\int\limits_0^y x^{2\alpha-1} f(x)(y^2-x^2)^{\beta-\alpha-1} x\, dx, \tag{8.3}$$

where $P_\nu^\mu(z)$ is the Legendre function of the first kind [532].

So the main conclusion from the Copson lemma is that the data on characteristics cannot be taken arbitrary; these functions must be connected by the Buschman–Erdélyi operators of the first kind (for a more detailed consideration, cf. [532]).

8.1.3 Norm estimates and embedding theorems in Kipriyanov spaces

Consider a set of functions $\mathbb{D}(0,\infty)$ such that if $f(x)\in\mathbb{D}(0,\infty)$, then $f(x)\in C^\infty(0,\infty)$ and $f(x)$ tends to zero at infinity. On this set, define the seminorms

$$\|f\|_{h_2^\alpha} = \|D_-^\alpha f\|_{L_2(0,\infty)}, \tag{8.4}$$

$$\|f\|_{\widehat{h}_2^\alpha} = \|x^\alpha\left(-\frac{1}{x}\frac{d}{dx}\right)^\alpha f\|_{L_2(0,\infty)}, \tag{8.5}$$

where D_-^α is the Riemann–Liouville fractional integro-differentiation, the operator in (8.5) is defined by

$$\left(-\frac{1}{x}\frac{d}{dx}\right)^\beta = 2^\beta I_{-;\,2,\,0}^{-\beta} x^{-2\beta}, \tag{8.6}$$

$I_{-;2,\,0}^{-\beta}$ is the Erdélyi–Kober operator, and $\alpha\in\mathbb{R}$. For $\beta=n\in\mathbb{N}_0$ expression (8.6) reduces to classical derivatives.

Theorem 96. *Let $f(x)\in\mathbb{D}(0,\infty)$. Then the following formulas are valid:*

$$D_-^\alpha f = {}_1S_-^{\alpha-1} x^\alpha\left(-\frac{1}{x}\frac{d}{dx}\right)^\alpha f, \tag{8.7}$$

$$x^\alpha\left(-\frac{1}{x}\frac{d}{dx}\right)^\alpha f = {}_1P_-^{\alpha-1} D_-^\alpha f. \tag{8.8}$$

So the Buschman–Erdélyi transmutations of zero order smoothness for $\alpha\in\mathbb{N}$ link differential operators in seminorm definitions (8.4) and (8.5).

Theorem 97. *Let $f(x)\in\mathbb{D}(0,\infty)$. Then the following inequalities hold true for seminorms:*

$$\|f\|_{h_2^\alpha} \le \max(1,\sqrt{1+\sin\pi\alpha})\|f\|_{\widehat{h}_2^\alpha}, \tag{8.9}$$

$$\|f\|_{\widehat{h}_2^\alpha} \le \frac{1}{\min(1,\sqrt{1+\sin\pi\alpha})}\|f\|_{h_2^\alpha}, \tag{8.10}$$

where α is any real number except $\alpha\ne -\frac{1}{2}+2k,\ k\in\mathbb{Z}$.

The constants in inequalities (8.9) and (8.10) are not greater than 1, which will be used below. If $\sin\pi\alpha=-1$ or $\alpha=-\frac{1}{2}+2k,\ k\in\mathbb{Z}$, then the estimate (8.10) is not valid.

Define on $\mathbb{D}(0,\infty)$ the Sobolev norm

$$\|f\|_{W_2^\alpha} = \|f\|_{L_2(0,\infty)} + \|f\|_{h_2^\alpha}. \tag{8.11}$$

Define one more norm,

$$\|f\|_{\widehat{W}_2^\alpha} = \|f\|_{L_2(0,\infty)} + \|f\|_{\widehat{h}_2^\alpha}. \tag{8.12}$$

Define the spaces W_2^α, $\widehat{W}_2^\alpha$ as closures of $D(0,\infty)$ in (8.11) or (8.12), respectively.

Theorem 98. *(a) For all $\alpha\in\mathbb{R}$ the space $\widehat{W}_2^\alpha$ is continuously imbedded in W_2^α; moreover,*

$$\|f\|_{W_2^\alpha}\le A_1\|f\|_{\widehat{W}_2^\alpha}, \tag{8.13}$$

with $A_1=\max(1,\sqrt{1+\sin\pi\alpha})$.

(b) Let $\sin\pi\alpha\neq -1$ or $\alpha\neq -\frac{1}{2}+2k,\ k\in\mathbb{Z}$. Then the inverse embedding of W_2^α in $\widehat{W}_2^\alpha$ is valid; moreover,

$$\|f\|_{\widehat{W}_2^\alpha}\le A_2\|f\|_{W_2^\alpha}, \tag{8.14}$$

with $A_2=1/\min(1,\sqrt{1+\sin\pi\alpha})$.

(c) Let $\sin\pi\alpha\neq -1$. Then the spaces W_2^α and $\widehat{W}_2^\alpha$ are isomorphic with equivalent norms.

(d) The constants in embedding inequalities (8.13) and (8.14) are sharp.

In fact this theorem is a direct corollary of the results on boundedness and norm estimates in L_2 of the Buschman–Erdélyi transmutations of zero order smoothness. In the same manner, from the unitarity of these operators the following theorem follows.

Theorem 99. *The norms*

$$\|f\|_{W_2^\alpha}=\sum_{j=0}^{s}\|D_-^j f\|_{L_2}, \tag{8.15}$$

$$\|f\|_{\widehat{W}_2^\alpha}=\sum_{j=0}^{s}\left\|x^j\left(-\frac{1}{x}\frac{d}{dx}\right)^j f\right\|_{L_2} \tag{8.16}$$

are equivalent for integer $s\in\mathbb{Z}$. Moreover, each term in (8.15) equals an appropriate term in (8.16) of the same index j.

I. Kipriyanov introduced in [243] function spaces which essentially influenced the theory of partial differential equations with Bessel operators and, in more general sense, the theory of singular and degenerate equations. These spaces are defined in the following way. First we consider a subset of even functions in $\mathbb{D}(0,\infty)$ with all zero derivatives of odd orders at $x=0$. We denote this set as $\mathbb{D}_c(0,\infty)$ and equip it with a norm

$$\|f\|_{\widetilde{W}_{2,k}^s}=\|f\|_{L_{2,k}}+\|B_k^{\frac{s}{2}}\|_{L_{2,k}}, \tag{8.17}$$

where s is an even natural number and $B_k^{s/2}$ is an iteration of the Bessel operator. We define the Kipriyanov spaces for even s as a closure of $D_c(0,\infty)$ in the norm (8.17). It is a known fact that a norm equivalent to (8.17) may be defined by [243]

$$\|f\|_{\widetilde{W}_{2,k}^s}=\|f\|_{L_{2,k}}+\left\|x^s\left(-\frac{1}{x}\frac{d}{dx}\right)^s f\right\|_{L_{2,k}}. \tag{8.18}$$

So the norm $\widetilde{W}^s_{2,\,k}$ may be defined for all s. Essentially this approach is the same as in [243]. Another approach is based on usage of the Hankel transform. Below we adopt the norm (8.18) for the space $\widetilde{W}^s_{2,k}$.

We define the weighted Sobolev norm by

$$\|f\|_{W^s_{2,k}} = \|f\|_{L_{2,k}} + \|D^s_- f\|_{L_{2,k}} \tag{8.19}$$

and a space $W^s_{2,\,k}$ as a closure of $\mathbb{D}_c(0,\infty)$ in this norm.

Theorem 100. *(a) Let $k \neq -n,\ n \in \mathbb{N}$. Then the space $\widetilde{W}^s_{2,\,k}$ is continuously embedded into $W^s_{2,\,k}$, and there exists a constant $A_3 > 0$ such that*

$$\|f\|_{W^s_{2,k}} \leq A_3 \|f\|_{\widetilde{W}^s_{2,k}}. \tag{8.20}$$

(b) Let $k+s \neq -2m_1 - 1,\ k-s \neq -2m_2 - 2,\ m_1 \in \mathbb{N}_0,\ m_2 \in \mathbb{N}_0$. Then the inverse embedding holds true of $W^s_{2,\,k}$ into $\widetilde{W}^s_{2,\,k}$, and there exists a constant $A_4 > 0$ such that

$$\|f\|_{\widetilde{W}^s_{2,k}} \leq A_4 \|f\|_{W^s_{2,k}}. \tag{8.21}$$

(c) If the abovementioned conditions are not valid, then the embedding theorems under consideration fail.

Corollary 15. *Let the following conditions hold true: $k \neq -n,\ n \in \mathbb{N},\ k+s \neq -2m_1 - 1,\ m_1 \in \mathbb{N}_0,\ k-s \neq -2m_2 - 2,\ m_2 \in \mathbb{N}_0$. Then the Kipriyanov spaces may be defined as closure of $D_c(0,\infty)$ in the weighted Sobolev norm* (8.19).

Corollary 16. *The sharp constants in embedding theorems* (8.20) *and* (8.21) *are*

$$A_3 = \max(1, \|{}_1S^{s-1}_-\|_{L_{2,k}}),\ A_4 = \max(1, \|{}_1P^{s-1}_-\|_{L_{2,k}}).$$

It is obvious that the theorem above and its corollaries are direct consequences of estimates for the Buschman–Erdélyi transmutations. The sharp constants in embedding theorems (8.20) and (8.21) are also direct consequences of estimates for the Buschman–Erdélyi transmutations of zero order smoothness. Estimates in $L_{p,\alpha}$ allow to consider embedding theorems for the general Sobolev and Kipriyanov spaces.

So by applying the Buschman–Erdélyi transmutations of zero order smoothness, we received an answer to a problem which for a long time was discussed in "folklore": Are the Kipriyanov spaces isomorphic to power weighted Sobolev spaces or not? Of course we investigated just the simplest case; the results can be generalized to other seminorms, higher dimensions, and bounded domains, but the principal idea is clear. All that disparages neither the essential role nor the necessity of applications of the Kipriyanov spaces in the theory of partial differential equations in any sense.

The importance of Kipriyanov spaces is a special case of the following general principle of L. Kudryavtsev:

"*Every equation must be investigated in its own space!*"

The embedding theorems proved in this section may be applied to direct transfer of known solution estimates for B-elliptic equations in Kipriyanov spaces (cf. [242,243]) to new estimates in weighted Sobolev spaces. It is a direct consequence of boundedness and transmutation properties of the Buschman–Erdélyi transmutations.

8.1.4 Other applications of Buschman–Erdélyi operators

First let us show how to apply Buschman–Erdélyi operators to the Radon transform.

It was proved by Ludwig in [342] that the Radon transform in terms of spherical harmonics acts in every harmonics at radial components as Buschman–Erdélyi operators. Let us formulate this result.

Theorem 101. (Ludwig theorem, [162,342]) *Let the function $f(x)$ be expanded in $\mathbb{R}^n$ by spherical harmonics*

$$f(x) = \sum_{k,l} f_{k,l}(r) Y_{k,l}(\theta). \tag{8.22}$$

Then the Radon transform of this function may be calculated as another series in spherical harmonics,

$$Rf(x) = g(r,\theta) = \sum_{k,l} g_{k,l}(r) Y_{k,l}(\theta), \tag{8.23}$$

$$g_{k,l}(r) = A(n) \int\limits_r^\infty \left(1 - \frac{s^2}{r^2}\right)^{\frac{n-3}{2}} C_l^{\frac{n-2}{2}}\left(\frac{s}{r}\right) f_{k,l}(r) r^{n-2}\, ds, \tag{8.24}$$

where $A(n)$ is some known constant and $C_l^{\frac{n-2}{2}}\left(\frac{s}{r}\right)$ is the Gegenbauer function (see [19]). The inverse formula is also valid of representing values $f_{k,l}(r)$ via $g_{k,l}(r)$.

The Gegenbauer function may be easily reduced to the Legendre function (see [19]). So the Ludwig formula (8.24) reduces the Radon transform in terms of spherical harmonics series and up to unimportant power and constant terms to Buschman–Erdélyi operators of the first kind.

Exactly, this formula in dimension two was developed by Cormack as the first step to the Nobel Prize. Special cases of Ludwig's formula proved in 1966 are for any special spherical harmonics and in the simplest case on pure radial functions; in

this case it is reduced to Sonine–Poisson–Delsarte transmutations of Erdélyi–Kober type. Besides the fact that such formulas are known for about half a century they are rediscovered still... As consequences of the above connections, the results may be proved for integral representations, norm estimates, and inversion formulas for the Radon transform via Buschman–Erdélyi operators. In particular, it makes clear that different kinds of inversion formulas for the Radon transform are at the same time inversion formulas for the Buschman–Erdélyi transmutations of the first kind, and vice versa. A useful reference for this approach is [80].

Now let us consider an application of the Buschman–Erdélyi transmutations for estimation of generalized Hardy operators. Unitarity of the shifted Hardy operators (5.30) was proved in [305]. It is interesting that the Hardy operators naturally arise in transmutation theory. We use Theorem 7 with integer parameter, which guarantees the unitarity for finding more unitary in $L_2(0,\infty)$ integral operators of very simple form.

Theorem 102. *The following are pairs of unitary mutually inverse integral operators in $L_2(0,\infty)$:*

$$U_3 f = f + \int_0^x f(y)\,\frac{dy}{y},\ U_4 f = f + \frac{1}{x}\int_x^\infty f(y)\,dy,$$

$$U_5 f = f + 3x\int_0^x f(y)\,\frac{dy}{y^2},\ U_6 f = f - \frac{3}{x^2}\int_0^x y f(y)\,dy,$$

$$U_7 f = f + \frac{3}{x^2}\int_x^\infty y f(y)\,dy,\ U_8 f = f - 3x\int_x^\infty f(y)\frac{dy}{y^2},$$

$$U_9 f = f + \frac{1}{2}\int_0^x \left(\frac{15x^2}{y^3} - \frac{3}{y}\right) f(y)\,dy,$$

$$U_{10} f = f + \frac{1}{2}\int_x^\infty \left(\frac{15y^2}{x^3} - \frac{3}{x}\right) f(y)\,dy.$$

Next we consider an application of the Buschman–Erdélyi transmutations in the works of V. Katrakhov. Namely, he found a new approach for boundary value problems for elliptic equations with strong singularities of infinite order. For example, for the Poisson equation he studied problems with solutions of arbitrary growth. At singular point he proposed a new kind of boundary condition: the K-trace. His results are based on the constant usage of Buschman–Erdélyi transmutations of the first kind for the definition of norms, solution estimates, and correctness proofs [225, 227].

Finally, we briefly discuss applications of the Buschman–Erdélyi transmutations to Dunkl operators. In recent years the Dunkl operators were thoroughly studied. These are difference–differentiation operators consisting of combinations of classical derivatives and finite differences. In higher dimensions, the Dunkl operators are defined by symmetry and reflection groups. For this class there are many results on transmutations which are of Sonine–Poisson–Delsarte and Buschman–Erdélyi types (cf. [560] and references therein).

8.2 Applications of the transmutation method to estimates of the solutions for differential equations with variable coefficients and the problem of E. M. Landis

8.2.1 Applications of the transmutations method to the perturbed Bessel equation with a potential

The problem of constructing an integral formula for solutions to the differential equation with certain asymptotics is considered,

$$B_\alpha g(x) - q(x)g(x) = \lambda^2 g(x), \tag{8.25}$$

where B_α is a Bessel operator, which in this subsection is convenient for us to define in the following form:

$$B_\alpha g = g''(x) + \frac{2\alpha}{x} g'(x), \qquad \alpha > 0. \tag{8.26}$$

This problem is solved by the transmutation method. To do this, it is enough to construct a pair of mutually inverse transformation operators, the first of which is S_α, of the form

$$S_\alpha h(x) = h(x) + \int\limits_x^\infty S(x,t)h(t)\,dt, \tag{8.27}$$

which intertwines the operators $B_\alpha - q(x)$ and B_α by the formula

$$S_\alpha(B_\alpha - q(x))h = B_\alpha S_\alpha h. \tag{8.28}$$

The second operator P_α, inverse to the first, should be constructed as integral with the kernel $P(x,t)$,

$$P_\alpha h(x) = h(x) + \int\limits_x^\infty P(x,t)h(t)\,dt, \tag{8.29}$$

and act by the formula

$$P_\alpha B_\alpha h = (B_\alpha - q(x))P_\alpha h,$$

where $h \in C^2(0,\infty)$ and such that integrals in (8.27) and (8.28) converge.

As a result, on solutions to the differential equation (8.25) the function $S_\alpha u = v$ will be expressed through solutions of the unperturbed equation obtained by discarding the term with potential in (8.25), that is, in fact, through the Bessel functions. The function $u = P_\alpha v$ will be a solution to the initial perturbed equation (8.25). In this case, the integral representation will be obtained for the solution (8.29) with an explicit kernel description $P(x,t)$. This technique reflects one of the main applications of transmutations. It is the expression of solutions of more complex differential equations through similar simpler ones, which has already been noted several times. Also note that the same pair of mutually inverse transmutations allows one to obtain both representations of solutions to the differential equation (8.25) with spectral parameter. Besides, we can present the solution in the homogeneous equation case

$$B_\alpha h(x) - q(x)h(x) = 0.$$

Moreover, if we consider the problem of finding a representation for the solutions to the perturbed equation (8.25), then we can skip the construction of the direct transmutation operator and go straight to the construction of the inverse and find the integral representation for the desired solution of the form (8.29).

An original technique for solving such problems was developed by V. V. Stashevskaya [557,558], which allowed her to include singular potentials with an extremely accurate estimate at zero $|q(x)| \le cx^{-3/2+\varepsilon}$, $\varepsilon > 0$, for entire α. This technique, based on the application of generalized Paley–Wiener theorems, has been widely developed and recognized. The case of continuous q for $\alpha > 0$ was considered in papers of A. S. Sokhin [546–549] and also in the paper [586] of V. Ya. Volk. Moreover, Povzner type transformation operators with integration over a finite interval were constructed in the works of V. V. Stashevskaya and V. Ya. Volk and Levin types with integration over an infinite interval were constructed in the works of A. S. Sokhin. Further, we propose a new modified method that allows one to combine both of these approaches.

Among many works on obtaining representations of solutions for the perturbed Bessel equation (8.25)–(8.26), we note those in which the solution is sought in the form of series of a special form. These are the works of A. Fitouhi et al. [66,145] and works of V. V. Kravchenko et al. [45,57,278,281,282,285,287–292]. A critical analysis of a number of results on this problem was recently presented in [169].

However, in many mathematical and physical problems it is necessary to consider strongly singular potentials, for example, admitting an arbitrary power singularity at zero. Here we formulate results about the integral representation of solutions of equations with similar singular potentials. From the potential only a majority is required by a certain function summed at infinity. In particular, $q = x^{-2}$ is the singular potential, $q = x^{-2-\varepsilon}$, $\varepsilon > 0$, is the strongly singular potential, and $q = e^{-\alpha x}/x$ are the Yukawa potentials of the Bargman and Batman–Shadan types [497] and a number of others. Moreover, no additional conditions of the type $q(x)$ impose fast oscillations at the

origin or constant sign, which allows one to study attractive and repulsive potentials using a single method.

It should be noted that operators constructed in this book are transmutations of a special kind that differ from previously known ones in some details. Prior to this, only cases of the same limits were considered (both types $[0; a]$ and $[a; \infty]$) in the basic integral equation for the kernel of the transmutation operator. Here it is shown that the case of various limits in the main integral equation can be considered. It is this arrangement of limits that made it possible to cover a wider class of potentials with singularities at zero. In addition, in comparison with the arguments on the model of the classical work of B. M. Levitan [317], we are making an improvement to this scheme. The Green function used in the proofs, as it turned out, can be expressed not only through the general Gauss hypergeometric function, but also more specifically through the Legendre function, which depends on a smaller number of parameters, which allows us to get rid of the indefinite constants in the estimates from previous works.

Due to the limited volume of the book, this subsection only presents the statement of the problem, a summary of the main results, and consequences without proof (for a detailed exposition, see [233,521,528–530]).

8.2.2 The solution of the basic integral equation for the kernel of the transmutation operator

Let us introduce variables and functions by formulas

$$\begin{gathered}\xi = \frac{t+x}{2},\ \eta = \frac{t-x}{2},\ \xi \geq \eta > 0,\\ K(x,t) = \left(\frac{x}{t}\right)^{\alpha} P(x,t),\ u(\xi,\eta) = K(\xi-\eta, \xi+\eta),\end{gathered} \tag{8.30}$$

where $P(x,t)$ is a kernel from (8.29). Denote $\nu = \alpha - 1$. So in order to justify the submission (8.29) for solution to Eq. (8.25) it is enough to define a function $u(\xi,\eta)$. It is known [546–549] that if a twice continuously differentiable solution $u(\xi,\eta)$ to the integral equation

$$u(\xi,\eta) = -\frac{1}{2}\int\limits_{\xi}^{\infty} R_\nu(s,0;\xi,\eta) q(s)\,ds - \int\limits_{\xi}^{\infty} ds \int\limits_{0}^{\eta} q(s+\tau) R_\nu(s,\tau;\xi,\eta) u(s,\tau)\,d\tau$$

under conditions $0 < \tau < \eta < \xi < s$ exists, then the function $P(x,t)$ is determined by (8.30) through this solution $u(\xi,\eta)$. The function $R_\nu = R_{\alpha-1}$ is the Riemann function that arises when solving a certain Goursat problem for a singular hyperbolic equation

$$\frac{\partial^2 u(\xi,\eta)}{\partial\xi\partial\eta} + \frac{4\alpha(\alpha-1)\xi\eta}{(\xi^2-\eta^2)^2} u(\xi,\eta) = q(\xi+\eta) u(\xi,\eta).$$

This function is known explicitly (see [546–549]) and has a form

$$R_\nu = \left(\frac{s^2-\eta^2}{s^2-\tau^2}\cdot\frac{\xi^2-\tau^2}{\xi^2-\eta^2}\right)^\nu {}_2F_1\left(-\nu,-\nu;1;\frac{s^2-\xi^2}{s^2-\eta^2}\cdot\frac{\eta^2-\tau^2}{\xi^2-\tau^2}\right), \tag{8.31}$$

where ${}_2F_1$ is the hypergeometric function (1.33). This expression is simplified in [233], where it is shown that the Riemann function in this case is expressed in terms of the Legendre function by the formula

$$R_\nu(s,\tau,\xi,\eta) = P_\nu\left(\frac{1+A}{1-A}\right),\ A=\frac{\eta^2-\tau^2}{\xi^2-\tau^2}\cdot\frac{s^2-\xi^2}{s^2-\eta^2}. \tag{8.32}$$

The main content of this subsection is the following result.

Theorem 103. *Let function $q(r)\in C^1(0,\infty)$ satisfy the condition*

$$|q(s+\tau)|\le|p(s)|,\ \forall s,\forall\tau,\ 0<\tau<s,\ \int\limits_\xi^\infty |p(t)|\,dt<\infty, \forall\xi>0. \tag{8.33}$$

Then there exists an integral representation of the form (8.29) whose kernel satisfies the estimate

$$|P(r,t)|\le\left(\frac{t}{r}\right)^\alpha \frac{1}{2}\int\limits_{\frac{t+r}{2}}^\infty P_{\alpha-1}\left(\frac{y^2(t^2+r^2)-(t^2-r^2)}{2try^2}\right)|p(y)|\,dy\times$$

$$\exp\left[\left(\frac{t-r}{2}\right)\frac{1}{2}\int\limits_{\frac{t+r}{2}}^\infty P_{\alpha-1}\left(\frac{y^2(t^2+r^2)-(t^2-r^2)}{2try^2}\right)|p(y)|\,dy\right].$$

In this case, the kernel of the transmutation operator $P(x,t)$ and the solution to (8.25) are twice continuously differentiable on $(0,\infty)$ functions in their arguments.

We list the classes of potentials for which the conditions in (8.33) are satisfied. If $|q(x)|$ monotonously decreases, then it is possible to accept $p(x)=|q(x)|$. For potentials with an arbitrary singularity at the origin and increasing at $0<x<M$ (for example, Coulomb potential $q=-1/x$), which are trimmed by zero at infinity, $q(x)=0,\ x>M$, we can take $p(x)=|q(M)|,\ x<M,\ p(x)=0,\ x\ge M$. Potentials with estimate $q(x+\tau)\le c|q(x)|=|p(x)|$ also will satisfies the conditions in (8.33). The possibility of such strengthening of Theorem 8.25 was indicated by V. V. Katrakhov. In particular, these conditions are satisfied by the following potentials encountered in applications: strongly singular potential with a power feature of the form $q(x)=x^{-2-\varepsilon}$, different Bargman potentials

$$q_1(x)=-\frac{e^{-ax}}{(1+\beta e^{-ax})^2},\ q_2(x)=\frac{c_2}{(1+c_3x)^2},\ q_3(x)=\frac{c_4}{ch^2(c_5x)},$$

and Yukawa

$$q_4(x) = -\frac{e^{-ax}}{x}, \; q_5(x) = \int_x^\infty e^{-at} \, dc(t).$$

(see, for example, [497]).

Remark 14. *In fact, in the proof of the above theorem, an explicit form of the Riemann function* (8.32) *is not required. Only the existence of the Riemann function, its positivity, and some special property of monotonicity are used. These facts are quite general, so the results can be extended to a fairly wide class of differential equations.*

The estimate from Theorem 103 for potentials of a general form can be transformed into a grosser, but also more visible one.

Theorem 104. *Let the conditions of Theorem 103 be satisfied. Then the kernel of the transmutation operator $P(x,t)$ satisfies the estimate*

$$|P(x,t)| \leq \frac{1}{2}\left(\frac{t}{x}\right)^{\alpha} P_{\alpha-1}\left(\frac{t^2+x^2}{2tx}\right) \times$$
$$\int_x^\infty |p(y)|\,dy \, \exp\left[\frac{1}{2}\left(\frac{t-x}{2}\right) P_{\alpha-1}\left(\frac{t^2+x^2}{2tx}\right)\int_x^\infty |p(y)|\,dy\right].$$

Note that for $x \to 0$ the kernel of the integral representation can have an exponential singularity.

For a class of potentials with a power singularity of the form

$$q(x) = x^{-(2\beta+1)}, \; \beta > 0, \tag{8.34}$$

obtained estimates can be simplified without reducing their accuracy. The restriction on β is caused by the condition of summability at infinity.

Theorem 105. *Consider the potential of the form* (8.34)*. Then Theorem 103 performed with evaluation*

$$|P(x,t)| \leq \left(\frac{t}{x}\right)^{\alpha} \frac{\Gamma(\beta)4^{\beta-1}}{(t^2-x^2)^{\beta}} \cdot P_{\alpha-1}^{-\beta}\left(\frac{t^2+x^2}{2tx}\right) \times$$
$$\exp\left[\left(\frac{t-x}{x}\right)\frac{\Gamma(\beta)4^{\beta-1}}{(t^2-x^2)^{\beta}} P_{\alpha-1}^{-\beta}\left(\frac{t^2+x^2}{2tx}\right)\right],$$

where $P_\nu^\mu(\cdot)$ is a Legendre function, β is defined by (8.34)*, and α is defined by* (8.26)*.*

Note that this estimate is obtained after quite lengthy calculations using the famous Slater–Marichev [361] theorem, which helps to calculate the necessary integrals in terms of hypergeometric functions after they are reduced to the Mellin convolution.

The simplest such estimate was obtained in [233] for the potential $q(x) = cx^{-2}$, for which $\beta = \dfrac{1}{2}$. As follows from [20], in this case the Legendre function $P_\nu^{-\frac{1}{2}}(z)$ can be expressed through elementary functions. Therefore, the corresponding estimate can be expressed in terms of elementary functions.

Another potential for which the obtained estimate can be simplified and expressed in terms of elementary functions is a potential of the form $q(x) = x^{-(2\beta+1)}$, when the parameters are related by the relation $\beta = \alpha - 1$.

Corollary 17. *Let the relation between the parameters $\beta = \alpha - 1$ be true. Then the estimate from Theorem 103 takes the form*

$$|P(x,t)| \le \left(\frac{t}{x}\right)^{\beta+1} \frac{2^{\beta-2}}{\beta} \left[\frac{t^2+x^2}{2tx}\right]^{\beta} \exp\left[\left(\frac{t-x}{2}\right) \frac{2^{\beta-2}}{\beta} \left[\frac{t^2+x^2}{2tr}\right]^{\beta}\right] =$$
$$\frac{1}{4\beta} \frac{1}{x^{2\beta+1}} (t^2+x^2)^{\beta} \exp\left[\frac{2^{\beta-2}}{\beta} \left(\frac{t-x}{2}\right) \left(\frac{t^2+x^2}{2tx}\right)^{\beta}\right]. \tag{8.35}$$

Let us note that for $\alpha = 0$ in (8.25)–(8.29), Theorem 103 reduces to well-known estimates for the kernel of the integral representation of Jost solutions for the Sturm–Liouville equation.

The above technique is fully transferred to the problem of constructing nonclassical operators of generalized translation. This problem is essentially equivalent to expressing solutions to the equation

$$B_{\alpha,x} u(x,y) - q(x)u(x,y) = B_{\beta,y} u(x,y) \tag{8.36}$$

through solutions of the unperturbed Euler–Poisson–Darboux equation (or wave equation in the nonsingular case) in the presence of additional conditions ensuring correctness. Such representations are obtained already from the existence of transmutation operators and have been studied for the nonsingular case ($\alpha = \beta = 0$) in [316,317, 319] as a consequence of the theory of generalized translation (see also [375]). An interesting original technique for obtaining such representations was also developed in the nonsingular case in the works of A. V. Borovskikh [33,34]. The presented results imply the integral representations of a certain subclass of solutions of Eq. (8.36) in the general singular case for sufficiently arbitrary potentials with singularities at the origin. Moreover, the estimates for the solutions do not contain any indefinite constants, and for the kernels of integral representations the integrals are written in explicit form equations that they satisfy.

8.2.3 Application of the method of transmutation operators to the problem of E. M. Landis

In a paper of E. M. Landis [307], the following problem is posed: Prove that the solution of the stationary Schrödinger equation with bounded potential of the form

$$\Delta u(x) - q(x)u(x) = 0, \; x \in \mathbb{R}^n, \; |x| \ge R_0 > 0, \tag{8.37}$$

$$|q(x)| \leq \lambda^2,\ \lambda > 0,\ u(x) \in C^2(|x| \geq R_0),$$

satisfying the estimate

$$|u(x)| \leq \text{const} \cdot e^{-(\lambda+\varepsilon)|x|},\ \varepsilon > 0,$$

is identically equal to zero.

V. Z. Meshkov (see [379,380]) gave a counterexample to this problem. Also the existence of counterexamples with solutions that are complex functions was proved. Moreover, it was shown that if we strengthen the estimate in the hypothesis of E. M. Landis to

$$|u(x)| \leq \text{const} \cdot e^{-(\lambda+\varepsilon)|x|^{4/3}},\ \varepsilon > 0,$$

then the answer will be "such nonzero solutions do not exist." Recently, interest in these results has not disappeared. Topics related to the hypothesis of E. M. Landis and the results of V. Z. Meshkova are actively developed by leading mathematicians in the field of differential equations, such as J. Bourgein, K. Koenig, and several others (see [35,79,236,237,481]). The main question remains the study of the hypothesis of E. M. Landis for real solutions, and the answer to this question has not yet been obtained. In connection with the foregoing, it seems reasonable to name the following text the Landis–Meshkov problem.

Landis–Meshkov problem. *Is it true that for given domains D and positive functions $r(x), s(x)$ only the zero classical solution of the stationary Schrödinger equation*

$$\Delta u(x) - q(x)u(x) = 0,\quad x \in \mathbb{R}^n,\quad |q(x)| \leq r(x), \tag{8.38}$$

satisfies the estimate

$$|u(x)| \leq s(x)? \tag{8.39}$$

Let D be the exterior of some circle, $q(x) = \lambda^2$, $s(x) = e^{-(\lambda+\varepsilon)|x|}$, $\varepsilon > 0$. Then from the results of V. Z. Meshkov, the negative answer to this problem in the case of complex solutions follows. If D is the exterior of some circle, $q(x) = \lambda^2$, $s(x) = e^{-(\lambda+\varepsilon)|x|^{4/3}}$, $\varepsilon > 0$, the positive answer in this problem in the case of complex solutions has place. For real valued solutions, even in these particular cases, the answers are unknown.

Further, we show that despite the general negative solution of V. Z. Meshkov for the initial statement of the problem of E. M. Landis for some classes of potentials, the problem is solved positively for real solutions. In this case, the method of transmutation operators of a special form is used [518–520].

Further, this problem is solved for the case of a potential that depends on only one variable: $q(x) = q(x_i)$, where $1 \leq i \leq n$. Further, for definiteness it is considered that $i = 1$. This case is a particular case of Eq. (8.37):

$$\Delta u - q(x_1)u = 0. \tag{8.40}$$

Here the potential $q(x_1)$ is bounded by an arbitrary nondecreasing function. The solution is based on the use of transmutation operators reducing Eq. (8.40) to the Laplace equation.

Let the conditions of the problem (8.37) be satisfied in semispace $x_1 \geq R_0$ and let the functions in (8.37) be invariant under the change of variables $z = x_1 - R_0$. Therefore, we will consider the problem (8.37) in the half-space $z \geq 0$ or, keeping the previous notation for the variable x_1, $x_1 \geq 0$. It will be proved that the solution to the problem (8.37) is zero in the half-space $x_1 \geq 0$, and then by virtue of Calderon's theorem on the uniqueness of continuation (see [382], Chapter 6, p. 14), such a solution is identically zero in the space $\mathbb{R}^n$.

Denote by $T(\delta)$ the set of functions satisfying in part of the space $\mathbb{R}^n_+$ the following conditions:

$$u(x) \in C^2\left(\mathbb{R}^n_+\right), \tag{8.41}$$

$$|u(x)| \leq c_1 e^{-\delta|x|}, \ \delta > 0, \tag{8.42}$$

$$\left|\frac{\partial u}{\partial x_1}\right| \leq c_2 e^{-\delta|x|}. \tag{8.43}$$

Let us construct for functions from $T(\lambda+\varepsilon)$ the transmutation operator of the form (see [518–520])

$$Su(x) = u(x) + \int\limits_{x_1}^{\infty} K(x_1, t) u(t, x^1)\, dt, \tag{8.44}$$

so that the following equality holds:

$$S\left(\frac{\partial^2 u}{\partial x_1^2} - q(x_1)u\right) = \frac{\partial^2}{\partial x_1^2} Su, \ |q(x_1)| \leq \lambda^2, \tag{8.45}$$

where $(x_1, x^1) {=} (x_1, x_2, \ldots, x_n)$. Substitution of expression (8.44) into formula (8.45) leads to the equalities

$$\frac{\partial^2 K}{\partial t^2} - \frac{\partial^2 K}{\partial x_1^2} = q(t) K, \tag{8.46}$$

$$3\frac{\partial K(x_1, x_1)}{\partial x_1} = q(x_1), \tag{8.47}$$

$$\lim_{t\to\infty} K(x_1, t)\frac{\partial u(t, x^1)}{\partial t} - \lim_{t\to\infty} \frac{\partial K(x_1, t)}{\partial t} u(t, x^1) = 0. \tag{8.48}$$

Performing a standard variable change $w = \dfrac{t + x_1}{2}$, $v = \frac{t - x_1}{2}$, we reduce the system (8.46)–(8.47) to a simpler (satisfying conditions (8.48) on solutions of the prob-

lem (8.37) will be shown later)

$$\frac{\partial^2 K}{\partial w \partial v} = q(w+v)K, \tag{8.49}$$

$$K(w,0) = \frac{1}{3}\int\limits_0^w q(s)\,ds. \tag{8.50}$$

The problem (8.49)–(8.50) is a consequence of the integral equation

$$K(w,v) = \frac{1}{3}\int\limits_0^w q(s)\,ds + \int\limits_0^w d\alpha \int\limits_0^v q(\alpha+\beta)K(\alpha,\beta)\,d\beta, \tag{8.51}$$

$$|q| \le \lambda^2, \quad w \ge v \ge 0.$$

Eq. (8.51) differs from that usually used when considering transformation operators on an infinite interval of the integral equation by changing the integration region from the semiaxis (w,∞) to the interval $(0,w)$, which implies an exponential kernel growth $K(x_1,t)$. It is further proved that such a kernel exists and a transmutation operator with such a kernel (8.44) is defined on the set $T(\lambda+\varepsilon)$. The possibility to reduce problem (8.46)–(8.48) to nonequivalent integral equations follows from the underdetermination of the Cauchy problem (8.49)–(8.50).

It should be noted that the integral equation (8.51) can be solved in a wider region without the limitations of $w \ge v$, otherwise the kernel will not be defined under the signs of the integrals. The proof of the existence of a solution in this wider field is carried out in the same way as the proof below. The nuance in proving the existence of a solution to the integral equation (8.51) usually does not pay attention to this (remark of A. V. Borovskikh).

Lemma 20. *There is a unique continuous solution to Eq.* (8.51)*, satisfying the inequality*

$$|K(w,v)| \le \frac{\lambda}{3}\sqrt{\frac{w}{v}}\, I_1\left(2\lambda\sqrt{wv}\right), \tag{8.52}$$

where $I_1(x)$ is a modified Bessel function. Moreover, on the valid potential $q(x_1) \equiv \lambda^2$ in (8.52) *an equal sign is reached.*

Remark 15. *Further, the symbol c denotes an absolute positive constant, the value of which does not play a role.*

Proof. We introduce the notation

$$K_0(w,v) = \frac{1}{3}\int\limits_0^w q(s)\,ds,$$

$$PK(w,v)=\int\limits_0^w d\alpha\int\limits_0^v q(\alpha+\beta)K(\alpha+\beta)\,d\beta.$$

Then Eq. (8.51) can be written as $K=K_0+PK$. We are looking for its solution in the form of the Neumann series

$$K=K_0+PK_0+P^2K_0+\dots. \tag{8.53}$$

For the terms of the series (8.53), taking into account the condition $|q(x_1)|\le\lambda^2$, we get

$$\left|P^nK_0(w_0v)\right|\le\frac{1}{3}\left(\lambda^2\right)^{n+1}\frac{w^{n+1}}{(n+1)!}\frac{v^n}{n!},\ n=0,1,2,\dots. \tag{8.54}$$

Application of the representation of $I_1(x)$ as a series

$$I_1(x)=\sum_{k=0}^{\infty}\frac{(x/2)^{2k+1}}{k!(k+1)!}$$

gives inequality (8.52).

The estimate (8.52) is exact, since for $q(x_1)\equiv\lambda^2$, the inequalities in (8.54) turn into equalities for all integers $n\ge0$. The lemma is proved. □

Lemma 21. *In terms of the variables x_1,t, the estimate*

$$|K(x_1,t)|\le c\,t\,e^{\lambda t}$$

is valid.

Proof. Let us consider the inequality

$$\left|\frac{1}{x}I_1(x)\right|\le c\,e^x,\ x\ge0.$$

To verify the truth of this inequality it is necessary to parse cases (i) $x\ge1$ and (ii) $0\le x\le1$ and use the well-known asymptotics of the function $I_1(x)$ when $x\to\infty$ and $x\to+0$ (see [21]). Hence, using the obvious inequalities

$$\frac{x_1+t}{2}\le t,\ 2\sqrt{wv}=\sqrt{t^2-x_1^2}\le t$$

and estimate (8.52) follows the statement of the lemma. □

It follows from the lemma that expression (8.44) is defined on functions from $T(\lambda+\varepsilon)$. We show that expression (8.44) gives the transmutation operator $T(\lambda+\varepsilon)$.

To do this, it remains to check the relation (8.48). From $u(x) \in T(\lambda + \varepsilon)$ and from Lemma 21 it follows that

$$\lim_{t \to \infty} K(x_1, t) \frac{\partial u(t, x^1)}{\partial t} = 0.$$

Therefore, it remains to prove that if $u(x) \in T(\lambda + \varepsilon)$, then

$$\lim_{t \to \infty} \frac{\partial K(x_1, t)}{\partial t} u(x_1, t) = 0.$$

The last relation follows from the estimate

$$\left| \frac{\partial K(x_1, t)}{\partial t} \right| \leq c\, t\, e^{\lambda t}. \tag{8.55}$$

To prove inequality (8.55), we need to go over to the variables w, v and, using the already established estimates for the kernel $K(x_1, t)$, estimate the derivatives $\frac{\partial K}{\partial w}$, $\frac{\partial K}{\partial v}$, differentiating Eq. (8.51). Since

$$\frac{\partial K}{\partial t} = \frac{1}{2} \left(\frac{\partial K}{\partial w} + \frac{\partial K}{\partial v} \right),$$

we get (8.55).

8.2.4 The solution to the E. M. Landis problem belongs to $T(\lambda + \varepsilon)$

We show that any solution to the problem (8.37) belongs to $T(\lambda + \varepsilon)$ and, therefore, the operator (8.44) is defined on such solutions. To do this, let us verify the condition (8.43).

Lemma 22. *Let the function $u(x) \in C^2(|x| \geq R_0)$ be a solution to the problem* (8.37). *Then there is a constant $c > 0$ such that*

$$\left| \frac{\partial u}{\partial x_1} \right| \leq c\, e^{-(\lambda+\varepsilon)|x|}.$$

Proof. By a priori Schauder estimates, in a closed ball $B(x, 1)$ of unit radius centered at x, $|x| \geq R_0 + 1$, we have (see [384], Theorem 33, II)

$$u_1 \leq c \left(u_{1,\lambda_1}{}^{\frac{1}{1+\lambda_1}} \cdot u_0{}^{\frac{\lambda_1}{\lambda_1+1}} + u_0 \right),$$

where

$$u_0 = \|u(x)\|_{C^0(B(x,1))}, \;\; u_1 = \|u(x)\|_{C^1(B(x,1))}$$

and u_{1,λ_1} is the sum of the Hölder coefficients of the function $u(x)$ and its derivatives of the first order $\frac{\partial u}{\partial x_i}$, $1 \le i \le n$. It follows that

$$\left|\frac{\partial u(x)}{\partial x_1}\right| \le c\left(u_{1,\lambda_1}{}^{\frac{1}{1+\lambda_1}} \cdot u_0{}^{\frac{\lambda_1}{\lambda_1+1}} + u_0\right). \tag{8.56}$$

Note that since all the conditions (see [384], Statement 33 V) are satisfied, the constant c in formula (8.56) does not depend on x.

From Morrey's results (see [382], Theorem 39, IV) the following estimate for u_{1,λ_1} follows:

$$u_{1,\lambda_1} \le c\left[\|u\|_{L_2(B(x,1))} + \|q\,u\|_{L_2(B(x,1))}\right]. \tag{8.57}$$

The constant in (8.57) does not depend on x. From the conditions $|q(x_1)| \le \lambda^2$, using the mean value theorem, we obtain from (8.57)

$$u_{1,\lambda_1} \le c\left(\int\limits_{B(x,1)} |u(y)|^2\,dy\right)^{1/2} \le c^1\,e^{-(\lambda+\varepsilon)|x|}.$$

Substituting the last inequality into (8.56), we obtain

$$\left|\frac{\partial u}{\partial x_1}\right| \le c\left[\left(e^{-(\lambda+\varepsilon)|x|}\right)^{\frac{1}{1+\lambda}+\frac{\lambda}{1+\lambda}} + e^{-(\lambda+\varepsilon)|x|}\right] \le c\,e^{-(\lambda+\varepsilon)|x|}.$$

Thus, the required inequality is established for $|x_1| \ge R_0 + 1$. Since the set $R_0 \le |x| \le R_0 + 1$ is compact in $\mathbb{R}^n$, this inequality is true for $|x| \ge R_0$. Lemma 22 is proved. □

Changing the coordinate again $z = x_1 - R_0$, we obtain that Lemma 22 is valid in the half-space $x_1 \ge 0$ (we will redesignate z by x_1).

We apply the operator S to Eq. (8.40). From the identity (8.45) and the permutation of S with the derivatives $\frac{\partial^2 u}{\partial x_i^2}$, $2 \le i \le n$, we obtain that in the half-space $\mathbb{R}^n_+$

$$S\,(\Delta u - q(x_1)u) = \Delta S u = 0.$$

Denoting Su by v from (8.44), (8.51), we obtain that if $u(x) \in C^2\left(\mathbb{R}^n_+\right)$, $q(x) \in C\left(\mathbb{R}^n_+\right)$, then $v(x) \in C^2\left(\mathbb{R}^n_+\right)$. Let us show that $v(x)$ decreases exponentially in $\mathbb{R}^n_+$ when $|x| \to \infty$ and therefore is equal to zero.

Lemma 23. *Let $u(x) \in T\,(\lambda+\varepsilon)$. Then for $x \in \mathbb{R}^n_+$*

$$|v| = |Su| \le c\,|x|\,e^{-\varepsilon|x|},\ \varepsilon > 0.$$

Proof. From (8.44) and Lemma 22, we obtain

$$|Su| \le |u(x)| + \int_{x_1}^{\infty} t\, e^{\lambda t} c\, e^{-(\lambda+\varepsilon)\sqrt{t^2+|x^1|^2}}\, dt \le$$
$$c\left(e^{-(\lambda+\varepsilon)|x|} + \int_{x_1}^{\infty} t\, e^{-(\lambda+\varepsilon)\sqrt{t^2+|x^1|^2}}\, dt\right).$$

Calculating the integral by changing variables $y = \sqrt{t^2 + |x^1|^2}$ with subsequent integration in parts, we obtain the required estimate. The lemma is proved. □

So $v(x) = 0$ in $\mathbb{R}^n_+$. Define on $T\,(\lambda+\varepsilon)$ the inverse to S operator P by the formula

$$Pu(x) = u(x) + \int_{x_1}^{\infty} N(x_1, t) u(t, x^1)\, dt.$$

Then for the kernel $N\,(x_1, t)$, statements of Lemmas 20–22 are valid. In addition, if $Su \in T\,(\lambda+\varepsilon)$, then

$$PSu(x) = u(x). \tag{8.58}$$

Since obviously $0 \in T\,(\lambda+\varepsilon)$, applying (8.58) to both parts of $Su = 0$ we obtain that $u = 0$ in $\mathbb{R}^n_+$. It was shown above that this implies $u \equiv 0$ in all $\mathbb{R}^n$.

Remark 16. *Consideration of the part of space $\mathbb{R}^n_+$ used in the proof is necessary because expression (8.44) is not defined in the area obtained by the intersection of the ball $|x| \le R_0$ and the infinite half cylinder $\{|x^1| \le R_0,\ |x_1| \le R_0\}$.*

The above reasoning leads to a theorem.

Theorem 106. *Any solution $u(x) \in C^2\,(|x| > R_0)$ to the stationary Schrödinger equation with bounded potential*

$$\Delta u(x) - q(x_1)u = 0,\ x \in \mathbb{R}^n,\ |x| \ge R_0 > 0,$$
$$q(x_1) \in C\,(|x| \ge R_0)\,,\ |q(x_1)| \le \lambda^2,\ \lambda > 0,$$

satisfying the estimate

$$|u(x)| \le \mathrm{const}\, e^{-(\lambda+\varepsilon)|x|},\ \varepsilon > 0,$$

is the identity zero.

The used technique of transmutation operators allows us to strengthen the result. We denote by $L_{2,\,loc}\,(x_1 \ge R_0)$ the set of functions for which for any $x_1 \ge R_0$ the

integral $\int\limits_{R_0}^{x_1} \psi^2(s)\,ds$ is finite. Suppose further that a nonnegative function $g(x)$ is given. Let for $g(x)$ the integral $\int\limits_{x_1}^{\infty} t\,g(t,x^1)\,dt = p(x)$ be finite for any $x_1 \geq R_0$ and for some constant $\alpha > 0$

$$|p(x)| \leq c \cdot \exp\left(-\alpha|x|^{\delta}\right), \ \delta > 0.$$

Then, according to the scheme of proof of the previous theorem, the following fact can be established.

Theorem 107. *Let $\psi(x_1) \in L_{2,\,loc}\,(x_1 \geq R_0)$, let $\psi(x_1)$ be a nondecreasing function, and let function $g(x)$ satisfy the above requirements. Then any solution to the equation*

$$\Delta u(x) - q(x_1)u = 0, \ x \in \mathbb{R}^n, \ |x| \geq R_0 > 0,$$
$$|q(x_1)| \leq \psi^2(x_1),$$

for which

$$\psi(x_1)|u(x)| \leq \mathrm{const}\, e^{-\psi(x_1)|x|} g(x), \ g(x) \geq 0,$$

is the identity zero.

Under the conditions of Theorem 106 we need to set $g(x) = e^{-\varepsilon|x|}$. An example of another suitable function $g(x)$ is the function $g(x) = \exp\left(-\varepsilon|x|^{\delta}\right)$, $0 < \delta < 1$. This case is also an example of the generalized Landis–Meshkov problem (8.38)–(8.39).

In a similar way, the case of potential depending only on the radial variable can be considered. The answer in the original statement of the E. M. Landis problem after passing to spherical coordinates is also positive (see [518–520]).

It is possible to consider generalizations of the E. M. Landis problem to the case of more general differential equations and the corresponding estimates of the growth of solutions. For example, it is of interest to study the questions posed for the nonlinear p-Laplacian equation [113,329].

8.3 Applications of transmutations to perturbed Bessel and one-dimensional Schrödinger equations

The integral representations of solutions of one differential equation with singularities in the coefficients containing a Bessel operator perturbed by some potential are considered in this section. The existence of integral representations of a certain type for the indicated solutions is proved by the method of successive approximations using transmutation operators. In this case, potentials with strong singularities at the origin are allowed. The Riemann function is expressed not through the general hypergeometric function, but more specifically through the Legendre function, which avoids unknown constants in the estimates.

8.3.1 Formulation of the problem

Consider the problem of constructing an integral representation of a certain kind for solutions of a differential equation

$$B_\alpha u(x) - q(x)u(x) = 0, \tag{8.59}$$

where B_α is the Bessel operator (9.1) of the form

$$B_\alpha u = u''(x) + \frac{2\alpha}{x} u'(x), \ \alpha > 0. \tag{8.60}$$

This problem is solved by the method of transmutation operators. To do this, it suffices to construct a transmutation operator P_α of Poisson type (see (3.120)) of the form

$$P_\alpha u(x) = u(x) + \int\limits_x^\infty P(x,t)u(t)\,dt, \tag{8.61}$$

with a kernel $P(x,t)$ which intertwines the operators B_α and $B_\alpha - q(x)$ by the formula

$$B_\alpha P_\alpha u = P_\alpha (B_\alpha - q(x))u, \tag{8.62}$$

where $u \in C^2(0,\infty)$. As a result, we obtain a formula expressing solutions of Eq. (8.59) with a spectral parameter of the form

$$B_\alpha u(x) - q(x)u(x) = \lambda^2 u(x)$$

through solutions of the unperturbed equation, that is, through Bessel functions. In this case, the spectral parameter λ does not affect the form of the linear transmutation operators whose kernels are independent of it. This approach reflects one of the applications of transmutation operators, that is, the expression of solutions of more complex differential equations through similar simpler ones.

The theory of transmutation operators is an important branch of modern mathematics that has numerous applications (see [51–53,234,323,376,537]). The possibility of representing the form (8.61) with a sufficiently "good" kernel P for a wide class of potentials $q(x)$ lies at the heart of classical methods for solving inverse problems of the quantum theory of scattering [172,497]. For the Sturm–Liouville equations, transformation operators of the form (8.61) were first constructed by B. Ya. Levin (see [313,314]).

The transmutation operators for the Bessel operator of the Sonine and Poisson type were introduced by Delsarte [83]. In Russian their theory was firstly presented and developed in the famous work of B. M. Levitan [317]. Then, in a number of papers, transmutation operators with the property (8.62) for variable potentials were also considered (see [284,291]). Moreover, the inverse to (8.61) Sonine type transmutation operators S_α, which satisfy the intertwining relation, is simultaneously considered,

$$S_\alpha(B_\alpha - q(x))u = B_\alpha S_\alpha u,$$

on suitable functions.

An original method for constructing transmutation operators for the perturbed Bessel equation on the semiaxis was developed by V. V. Stashevskaya [557,558], which allowed it to include singular potentials with an estimate of zero $|q(x)| \leq cx^{-3/2+\varepsilon},\ \varepsilon > 0$, for integer α. This technique was further widely developed. The case of continuous $q, \alpha > 0$ was considered in detail in the works of A. S. Sokhin [546–549], as well as a number of other authors (see for more details [234,537]). Transmutation operators with "bad" potentials, such distributions from a certain class were considered in [172]. V. V. Kravchenko developed a special method for representing the kernels of transformation operators in the form of series (the Spectral Parameter Power Series method, SPPS) [276,283]. This method turned out to be well adapted for numerical solution of applied problems and computer modeling, including direct and inverse spectral problems [279,280]. Results for some special cases of the considered problem of constructing transformation operators (8.61)–(8.62) were published in [233,526].

In many mathematical and physical problems, it is necessary to consider strongly singular potentials, for example, admitting an arbitrary power singularity at zero. In this section, we formulate results on the integral representation of solutions of equations with similar singular potentials. From the potential only a majority by a certain function summed at infinity is required. In particular, the class of admissible potentials includes the singular potential $q = x^{-2}$, the strongly singular potential with a power singularity $q = x^{-2-\varepsilon}$, $\varepsilon > 0$, the potentials of Yukawa type $q = e^{-\alpha x}/x$, Bargman and Batman–Shadan potentials [497], and a number of others. Moreover, it is not necessary to require any additional conditions for a $q(x)$ such as fast oscillations at the origin or constant sign. This allows us to study attractive and repulsive potentials using a single method. Exemption from limiting conditions at zero is the advantage of considering Levin type transmutation operators (8.61).

In this section, the main object of study is the integral equation for the kernel of the transformation operator (8.61). After reducing the problem to an integral equation, the existence and uniqueness of the solution and its necessary smoothness are proved. Estimates of the solution are obtained in terms of the parameter and potential of the original equation (8.59). Such estimates are expressed using special Legendre functions. For a particular class of potential type potentials, simpler estimates are obtained. Here a technique based on the application of the Riemann function for the Euler–Poisson–Darboux equation was used. Estimates of integrals were obtained using the Mellin transform and the Slater–Marichev theorem.

It should be noted that a special kind of transmutation operators are constructed in this section. These operators differ from previously known operators by some details. Prior to this, only cases of identical limits (both types of $[0; a]$ and $[a; \infty]$) in the main integral equation for the kernel of the transmutation operator were considered. Here it is shown that cases of various limits can be considered in the main integral equation. This arrangement of limits made it possible to cover a wider class of potentials with singularities at zero. In addition, in comparison with the arguments on the model of

the classical work of B. M. Levitan [317], we are making some improvements to this scheme. As it turned out, the Riemann function used in the proof can be expressed not only in terms of the general Gaussian hypergeometric function with three parameters, but also more specifically in terms of the Legendre function with two parameters, which allows one to get rid of the indefinite constants in the estimates from previous papers.

8.3.2 Solution of the basic integral equation for the kernel of a transmutation operator

We introduce new variables and functions using the following formulas:

$$\begin{aligned} &\xi = \frac{t+x}{2}, \ \eta = \frac{t-x}{2}, \ \xi \geq \eta > 0, \\ &K(x,t) = \left(\frac{x}{t}\right)^{\alpha} P(x,t), \ w(\xi,\eta) = K(\xi-\eta, \xi+\eta). \end{aligned} \tag{8.63}$$

Let $\nu = \alpha - 1$. In order to justify the representation (8.61) for solution to (8.59) it is sufficient to define the function $w(\xi, \eta)$. It is known (see [233,526]) that if there exists a twice continuously differentiable solution $w(\xi, \eta)$ of the integral equation

$$w(\xi,\eta) = -\frac{1}{2}\int\limits_{\xi}^{\infty} R_\nu(s,0;\xi,\eta) q(s)\,ds - \int\limits_{\xi}^{\infty} ds \int\limits_{0}^{\eta} q(s+\tau) R_\nu(s,\tau;\xi,\eta) w(s,\tau)\,d\tau,$$

for $0 < \tau < \eta < \xi < s$, then the function $P(x,t)$ is determined by the formulas in (8.63) using $w(\xi,\eta)$. The function $R_\nu = R_{\alpha-1}$ is the Riemann function that arises when solving a certain Goursat problem for a singular inhomogeneous hyperbolic equation of the form (one of the forms of the Euler–Poisson–Darboux equation)

$$\frac{\partial^2 w(\xi,\eta)}{\partial\xi\partial\eta} + \frac{4\alpha(\alpha-1)\xi\eta}{(\xi^2-\eta^2)^2} w(\xi,\eta) = f(\xi,\eta),$$

which in our case is converted to the form

$$\frac{\partial^2 w(\xi,\eta)}{\partial\xi\partial\eta} + \frac{4\alpha(\alpha-1)\xi\eta}{(\xi^2-\eta^2)^2} w(\xi,\eta) = q(\xi+\eta) w(\xi,\eta).$$

This Riemann function is known explicitly and has a form (see [317])

$$R_\nu = \left(\frac{s^2-\eta^2}{s^2-\tau^2}\cdot\frac{\xi^2-\tau^2}{\xi^2-\eta^2}\right)^{\nu} {}_2F_1\left(-\nu,-\nu;1;\frac{s^2-\xi^2}{s^2-\eta^2}\cdot\frac{\eta^2-\tau^2}{\xi^2-\tau^2}\right). \tag{8.64}$$

This expression is simplified in [526], where it is shown that the Riemann function in this case is expressed in terms of the Legendre function by the formula

$$R_\nu(s,\tau,\xi,\eta) = P_\nu\left(\frac{1+A}{1-A}\right), \ A = \frac{\eta^2-\tau^2}{\xi^2-\tau^2}\cdot\frac{s^2-\xi^2}{s^2-\eta^2}. \tag{8.65}$$

Theorem 108. *Let function* $q(r) \in C^1(0,\infty)$ *satisfy the conditions*

$$|q(s+\tau)| \le |p(s)|, \ \forall s, \forall \tau, \ 0<\tau<s, \ \int\limits_{\xi}^{\infty} |p(t)|\, dt < \infty, \forall \xi > 0. \tag{8}$$

Then there exists an integral representation of the form (8.61) whose kernel satisfies the estimate

$$|P(r,t)| \le \left(\frac{t}{r}\right)^{\alpha} \frac{1}{2} \int\limits_{\frac{t+r}{2}}^{\infty} P_{\alpha-1}\left(\frac{y^2(t^2+r^2)-(t^2-r^2)}{2try^2}\right)|p(y)|\, dy \cdot$$

$$\exp\left[\left(\frac{t-r}{2}\right)\frac{1}{2}\int\limits_{\frac{t+r}{2}}^{\infty} P_{\alpha-1}\left(\frac{y^2(t^2+r^2)-(t^2-r^2)}{2try^2}\right)|p(y)|\, dy\right].$$

Moreover, the kernel of the transmutation operator $P(x,t)$ *and the solution of Eq. (8.59) are functions that are twice continuously differentiable on* $(0,\infty)$ *with respect to their arguments.*

We break the proof of Theorem 108 into some lemmas.

Let us introduce the notation

$$I_q(\xi,\eta) = \frac{1}{2}\int\limits_{\xi}^{\infty} R_\nu(y,0;\xi,\eta)|p(y)|\, dy =$$

$$\frac{1}{2}\int\limits_{\xi}^{\infty} P_\nu\left(\frac{y^2(\xi^2+\eta^2)-2\xi^2\eta^2}{y^2(\xi^2-\eta^2)}\right)|p(y)|\, dy, \tag{8.66}$$

$$w_0(\xi,\eta) = -\frac{1}{2}\int\limits_{\xi}^{\infty} R_\nu(s,0;\xi,\eta)|p(s)|\, ds,$$

$$Aw_0(\xi,\eta) = -\int\limits_{\xi}^{\infty} ds \int\limits_{0}^{\eta} q(s+\tau)R_\nu(s,\tau;\xi,\eta)u_0(s,\tau)\, d\tau.$$

Let us prove the uniform convergence of the operator Neumann series

$$\sum_{k=0}^{\infty} A^k w_0(\xi,y) \tag{8.67}$$

and the possibility of its double differentiation.

Lemma 24. *The estimate*

$$|w_0(\xi,\eta)| \leq I_q(\xi,\eta)$$

is valid

The proof follows immediately from the definition (8.66).

Lemma 25. *Let* $0 < \tau < \eta < \xi < s$. *Then the inequality*

$$I_q(s,t) \leq I_q(\xi,\eta) \tag{8.68}$$

is valid.

Proof. By the statement of the lemma we have $0 < \tau < \eta < \xi < s < y$. We show that then

$$\frac{\tau^2}{s^2}\cdot\frac{(y^2-s^2)}{(y^2-\tau^2)} \leq \frac{\eta^2}{\xi^2}\cdot\frac{(y^2-\xi^2)}{(y^2-\eta^2)}\ (\leq 1).$$

Indeed, this inequality is equivalent to

$$\tau^2\xi^2(y^2-s^2)(y^2-\eta^2) \leq \eta^2 s^2(y^2-\xi^2)(y^2-\tau^2),$$

which is obvious, since each of the factors on the left does not exceed the corresponding factor on the right. Further, we consider for $0 < x < 1$ the function

$$f(x) = \frac{1+x}{1-x} \geq 1,\ \ f'(x) = \frac{2}{(1-x)^2} > 0,\ \ 0 < x < 1.$$

Therefore, this function increases in x. Therefore,

$$\frac{1+\frac{\tau^2}{s^2}\cdot\frac{(y^2-s^2)}{(y^2-\tau^2)}}{1-\frac{\tau^2}{s^2}\cdot\frac{(y^2-s^2)}{(y^2-\tau^2)}} \leq \frac{1+\frac{\eta^2}{\xi^2}\cdot\frac{(y^2-\xi^2)}{(y^2-\eta^2)}}{1-\frac{\eta^2}{\xi^2}\cdot\frac{(y^2-\xi^2)}{(y^2-\eta^2)}}.$$

The Legendre function $P_\nu(x)$ for $x \in (1,\infty)$, $\nu > -1$, monotonously increases, and in addition, $P_\nu(x) > 1$. So

$$P_\nu\left(\frac{1+\frac{\tau^2}{s^2}\cdot\frac{(y^2-s^2)}{(y^2-\tau^2)}}{1-\frac{\tau^2}{s^2}\cdot\frac{(y^2-s^2)}{(y^2-\tau^2)}}\right) \leq P_\nu\left(\frac{1+\frac{\eta^2}{\xi^2}\cdot\frac{(y^2-\xi^2)}{(y^2-\eta^2)}}{1-\frac{\eta^2}{\xi^2}\cdot\frac{(y^2-\xi^2)}{(y^2-\eta^2)}}\right).$$

The last inequality can be written differently, i.e.,

$$P_\nu\left(\frac{y^2(s^2+\tau^2)-2s^2\tau^2}{y^2(s^2-\tau^2)}\right) \leq P_\nu\left(\frac{y^2(\xi^2+\eta^2)-2\xi^2\eta^2}{y^2(\xi^2-\eta^2)}\right).$$

Note that we have actually proved the inequality for the Riemann function

$$R_\nu(y,0;s,\tau) \le R_\nu(y,0;\xi,\eta), \tag{12}$$

when $0<\tau<\eta<\xi<s<y$.

From the calculations we obtain the estimate

$$I_q(s,\tau) = \frac{1}{2}\int\limits_s^\infty R_\nu(y,0;s,\tau)|p(y)|\,dy \le \frac{1}{2}\int\limits_\xi^\infty R_\nu(y,0;s,\tau)|p(y)|\,dy.$$

Replacing the lower limit of integration of s by $\xi<s$, we can only increase the value of the integral, since the Riemann function is positive, $R_\nu>0$. As a result, we arrive at the estimate (8.68). □

Lemma 26. *For the n-th member of the Neumann series (8.67), the estimate*

$$|w_n(\xi,\eta)| \le I_q(\xi,\eta)\cdot\frac{[\eta I_q(\xi,\eta)]^n}{n!} \tag{8.69}$$

is valid.

Proof. We apply the method of mathematical induction. For $n=0$, inequality (8.69) reduces to the already proved inequality from Lemma 24. Let (8.69) be valid for some $n=k$. Then for the next member of the Neumann series we get

$$|w_{k+1}(\xi,\eta)| \le |\int\limits_\xi^\infty ds\int\limits_0^\eta R_\nu(s,\tau;\xi,\eta)w_k(s,\tau)q(s+\tau)\,d\tau| \le$$

$$\int\limits_\xi^\infty ds\int\limits_0^\eta R_\nu(s,\tau;\xi,\eta)|q(s+\tau)|I_q(s,\tau)\frac{[\eta I_q(s,\tau)]^k}{k!}\,d\tau.$$

Repeating the arguments of the previous lemma, we obtain

$$R_\nu(s,\tau;\xi,\eta) \le R_\nu(s,0;\xi,\eta), \tag{8.70}$$

since

$$R_\nu(s,\tau;\xi,\eta) = P_\nu\left(\frac{1+A}{1-A}\right),\ \ A=\frac{\eta^2-\tau^2}{\xi^2-\tau^2}\cdot\frac{s^2-\xi^2}{s^2-\eta^2},$$

and the maximum value for A is achieved when $\tau=0$. Taking into account inequality (8.70) and the alleged inequality (8.69), we arrive at the estimate

$$|w_{k+1}(\xi,\eta)| \le I_q(\xi,\eta)\frac{[\tau I_q(\xi,\eta)]^k}{k!}\cdot\int\limits_\xi^\infty R_\nu(s,0;\xi,\eta)\int\limits_0^\eta |q(s+\tau)|\tau^k\,d\tau\,ds.$$

We consider potentials for which the inequality $|q(s+\tau)| \le |p(s)|,\ 0 < \tau < s$. Finally we get

$$|w_{k+1}(\xi,\eta)| \le I_q(\xi,\eta) \frac{[I_q(\xi,\eta)]^{k+1}}{k!} \cdot \frac{\eta^{k+1}}{(k+1)},$$

which gives estimate (8.69) for all n. □

Now we complete the proof of Theorem 108. Summarizing the estimates in (8.69), we obtain that the Neumann series converges uniformly in the domain $0 < \eta < \xi$ and its sum is some continuous function satisfying the inequality

$$|w(\xi,\eta)| \le I_q(\xi,\eta) \exp[\eta \cdot I_q(\xi,\eta)]. \tag{8.71}$$

It follows from (8.71) that we could prove the convergence of series (8.67) for the integrable potential q, which can be approximated by continuous potentials.

Returning to the functions K and P, we obtain the inequalities

$$|K(x,t)| \le I_q\left(\frac{t+x}{2}, \frac{t-x}{2}\right) \exp\left[\left(\frac{t-x}{2}\right) I_q\left(\frac{t+x}{2}, \frac{t-x}{2}\right)\right],$$
$$|P(x,t)| \le \left(\frac{t}{x}\right)^{\alpha} I_q\left(\frac{t+x}{2}, \frac{t-x}{2}\right) \exp\left[\left(\frac{t-x}{2}\right) I_q\left(\frac{t+x}{2}, \frac{t-x}{2}\right)\right].$$

We convert the value I_q included in the estimates

$$I_q\left(\frac{t+x}{2}, \frac{t-x}{2}\right) = \frac{1}{2} \int\limits_{\frac{t+x}{2}}^{\infty} P_{\alpha-1}\left(\frac{y^2(t^2+x^2)-(t^2-x^2)}{2txy^2}\right) |p(y)|\, dy.$$

Thus, we arrive at estimate (8.66).

To complete the proof of Theorem 108, it remains to justify the existence of second continuous derivatives of the function $P(x,t)$ with respect to the variables x and t under the condition $q \in C^1(x>0)$. Obviously, this is equivalent to the existence of second continuous derivatives of the function $u(\xi,\eta)$ with respect to the variables ξ, η. The proof of the last statement is carried out according to the above model by the method of successive approximations and completely repeats the corresponding fragment of the proof from [526].

The theorem is proved.

We list the classes of potentials for which conditions (10.18) are satisfied. If $|q(x)|$ is monotonically decreasing, then we can take $p(x) = |q(x)|$. For potentials with an arbitrary singularity at the origin and increasing for $0 < x < M$ (for example, Coulomb potential $q = -\frac{1}{x}$), which are cut off by zero at infinity, $q(x) = 0,\ x > M$, we can take $p(x) = |q(M)|,\ x < M,\ p(x) = 0,\ x\ eq M$. Potentials with the estimate $q(x+\tau) \le c|q(x)| = |p(x)|$ will also satisfy condition (10.18). V. V. Katrakhov drew

the attention of one of the authors to the possibility of a similar strengthening of Theorem 108.

In particular, the following potentials, encountered in applications, satisfy the above conditions: a strongly singular potential with a power-law singularity of the form $q(x) = x^{-2-\varepsilon}$, various Bargman potentials

$$q_1(x) = -\frac{e^{-ax}}{(1+\beta e^{-ax})^2}, \qquad q_2(x) = \frac{c_2}{(1+c_3x)^2}, \qquad q_3(x) = \frac{c_4}{\mathrm{ch}^2(c_5x)},$$

and Yukawa

$$q_4(x) = -\frac{e^{-ax}}{x}, \qquad q_5(x) = \int_x^\infty e^{-at}\,dc(t)$$

(see, for example, [497]).

Remark 17. *In fact, in the proof of the above theorem, an explicit form of the Riemann function (8.65) is not needed. We use only the existence of the Riemann function, its positivity, and some special property of monotonicity (8.70). These facts are quite general, so the results can be extended to a fairly wide class of differential equations.*

The estimate from Theorem 108 for potentials of a general form can be transformed into a less accurate, but more visible one.

Theorem 109. *Let the conditions of Theorem 108 be satisfied. Then the kernel of the transmutation operator $P(x,t)$ satisfies the estimate*

$$|P(x,t)| \le \frac{1}{2}\left(\frac{t}{x}\right)^{\alpha} P_{\alpha-1}\left(\frac{t^2+x^2}{2tx}\right)\int_x^\infty |p(y)|\,dy \times$$
$$\exp\left[\frac{1}{2}\left(\frac{t-x}{2}\right)P_{\alpha-1}\left(\frac{t^2+x^2}{2tx}\right)\int_x^\infty |p(y)|\,dy\right].$$

Note that for $x \to 0$ the kernel of the integral representation can have an exponential singularity.

8.3.3 Estimates for the case of a power singular at zero potential

For a class of potentials with a power singularity of the form

$$q(x) = x^{-(2\beta+1)}, \ \beta > 0, \tag{8.72}$$

obtained estimates can be simplified without reducing their accuracy. The restriction on β is caused by the condition of integrability at infinity.

Theorem 110. *Consider a potential of the form (8.72). Then Theorem 108 holds with the estimate*

$$|P(x,t)| \le \left(\frac{t}{x}\right)^{\alpha} \frac{\Gamma(\beta)4^{\beta-1}}{(t^2-x^2)^{\beta}} \cdot P_{\alpha-1}^{-\beta}\left(\frac{t^2+x^2}{2tx}\right) \times$$
$$\exp\left[\left(\frac{t-x}{x}\right)\frac{\Gamma(\beta)4^{\beta-1}}{(t^2-x^2)^{\beta}} P_{\alpha-1}^{-\beta}\left(\frac{t^2+x^2}{2tx}\right)\right],$$

where $P_{\nu}^{\mu}(\cdot)$ is the Legendre function (1.42)*, the value β is determined from (8.72), and the value α is determined from (8.60).*

Anticipating the proof, we note that this estimate is obtained after rather lengthy calculations using the famous Slater–Marichev theorem [361], which helps to calculate the necessary integrals in terms of hypergeometric functions after they are reduced to the Mellin convolution.

For this class of potentials, we simplify estimate (8.71), which constitutes the content of Theorem 108, without reducing its accuracy. For this, the value I_q included in estimate (8.71) will be calculated explicitly. We divide the proof of Theorem 110 into two lemmas.

Lemma 27. *For a potential of the form (8.72), we have the relation*

$$I_q(\xi,\eta) = \frac{1}{4\xi^{2\beta}} \int_0^1 P_{\nu}(2\alpha z+1)(1-z)^{\beta-1}\,dz, \tag{8.73}$$

where P_{ν} is the Legendre function, $\alpha = \eta^2/(\xi^2-\eta^2)$.

Proof. Let us consider

$$I_q(\xi,\eta) = \frac{1}{2}\int_{\xi}^{\infty} P_{\nu}\left(\frac{t^2(\xi^2+\eta^2)-2\xi^2\eta^2}{t^2(\xi^2-\eta^2)}\right)\frac{dt}{t^{2\beta+1}}.$$

We carry out the change of variables, denoting the argument of the Legendre function by x,

$$x = \frac{t^2(\xi^2+\eta^2)-2\xi^2\eta^2}{t^2(\xi^2-\eta^2)}, \quad dx = \frac{4\xi^2\eta^2}{t^3(\xi^2-\eta^2)}dt.$$

With such a change of variables, the numbers of integration become new limits

$$1, \ 1+\frac{2\eta^2}{\xi^2-\eta^2} = \frac{\xi^2+\eta^2}{\xi^2-\eta^2} = B > 1,$$

and the variable t will have the form

$$t = \xi\eta\left(\frac{2}{\xi^2+\eta^2-x(\xi^2-\eta^2)}\right)^{\frac{1}{2}}.$$

This leads to the following expression for I_q:

$$I_q(\xi,\eta)=\frac{1}{2}\int\limits_1^B P_\nu(x)\frac{t^3(\xi^2-\eta^2)}{4\xi^2\eta^2}\frac{dt}{t^{2\beta+1}}=$$

$$\frac{1}{2}\int\limits_1^B P_\nu(x)\left[\frac{\xi^2-\eta^2}{4\xi^2\eta^2}\right]\cdot\left[\frac{\xi^2+\eta^2-x(\xi^2-\eta^2)}{2\xi^2\eta^2}\right]^{\beta-1}dx.$$

In the last integral, we make another change of variables by the formula

$$z=(x-1)\frac{\xi^2-\eta^2}{2\eta^2},\quad \left(dz=\frac{\xi^2-\eta^2}{2\eta^2}\right)dx.$$

Then we obtain

$$I_q(\xi,\eta)=\frac{1}{2}\left(\frac{\xi^2-\eta^2}{4\xi^2\eta^2}\right)\int\limits_0^1 P_\nu(2\alpha z+1)\frac{2\eta^2}{\xi^2-\eta^2}\times$$

$$\left[\frac{\xi^2+\eta^2-(\xi^2-\eta^2)\left(\frac{2\eta^2}{\xi^2-\eta^2}z+1\right)}{2\xi^2\eta^2}\right]^{\beta-1}dz=$$

$$\frac{1}{4\xi^{2\beta}}\int\limits_0^1 P_\nu(2\alpha z+1)(1-z)^{\beta-1}\,dz,$$

where $\alpha=\eta^2/(\xi^2-\eta^2)$. We get (8.73).

□

Lemma 28. *Let the conditions $a>0$, $\beta>0$ be satisfied. Then the following formula is valid:*

$$\int\limits_0^1 P_\nu(2\alpha x+1)(1-x)^{\beta-1}\,dx=\Gamma(\beta)\left[\frac{1+\alpha}{\alpha}\right]^{\frac{\beta}{2}}P_\nu^{-\beta}(2\alpha+1). \tag{8.74}$$

Proof. In the proof we will use notation and a technique based on the Slater–Marichev theorem [361].

In the integral from (8.74), we change the variables $t=1/x$. We get

$$\int\limits_0^1 P_\nu(2\alpha x+1)(1-x)^{\beta-1}\,dx=\int\limits_1^\infty P_\nu\left(2\frac{\alpha}{t}+1\right)(t-1)^{\beta-1}t^{-\beta}\frac{dt}{t}=$$

$$\int_0^\infty P_\nu\left(2\frac{\alpha}{t}+1\right)(t-1)_+^{\beta-1}t^{-\beta}\frac{dt}{t}=I(\alpha),$$

where the notation for the truncated power function is used, $x_+^\lambda = x^\lambda$ when $x>0$, and $x_+^\lambda=0$ when $x\le 0$. We apply the Mellin transform (see Definition 11) with respect to the variable α, $(\alpha>0)$ to the function $I(\alpha)$. Using the Mellin convolution theorem (see [361]), we obtain

$$M[I(\alpha)](s)=M[P_\nu(2x+1)](s)\cdot M[x^{-\beta}(x-1)_+^{\beta-1}](s).$$

Using formulas (6)(1), (4), and (2)(4) from [361], we obtain

$$M[I(\alpha)](s)=-\frac{\sin\pi\nu}{\pi}\frac{\Gamma(s)\Gamma(-\nu-s)\Gamma(1+\nu-s)\Gamma(\beta)\Gamma(1-s)}{\Gamma(1-s)\Gamma(1+\beta-s)}=$$
$$-\frac{\sin\pi\nu}{\pi}\Gamma(\beta)\,\Gamma\left[\begin{matrix}s, & -\nu-s, & 1+\nu-s\\ & 1+\beta-s & \end{matrix}\right],$$

where the Slater designation is used for the ratio of the products of gamma functions. In the notation of the Slater–Marichev theorem, we have

$$(a)=(0),\ (b)=(-\nu,1+\nu),\ (c)=\varnothing,\ (d)=(1+\beta),$$
$$A=1,\ B=2,\ C=0,\ D=1.$$

Using the Slater–Marichev theorem, we obtain formulas for $I(\alpha)$ for $0<\alpha<1$:

$$I(\alpha)=-\frac{\sin\pi\nu}{\pi}\frac{\Gamma(1+\nu)\Gamma(-\nu)}{\Gamma(1+\beta)}{}_2F_1(-\nu,1+\nu;1+\beta;-\alpha)=$$
$$\Gamma(\beta)\alpha^{-\frac{\beta}{2}}(1+\alpha)^{\frac{\beta}{2}}P_\nu^{-\beta}(1+2\alpha), \tag{8.75}$$

where formula (3) from [19], p. 126, and the identity for gamma functions (see [19])

$$\Gamma(-\nu)=\frac{\pi}{\nu\,\Gamma(\nu)\sin\pi\nu}$$

are used. For $\alpha\ge 1$ we get another expression:

$$I(\alpha)=-\frac{\sin\pi\nu}{\pi}\Gamma(\beta)\times$$
$$\left\{\alpha^\nu\,\Gamma\left[\begin{matrix}1+\nu+\nu, & -\nu\\ 1+\beta+\nu & \end{matrix}\right]{}_2F_1\left(-\nu,1-1-\beta-\nu;1-1-\nu-\nu;-\frac{1}{\alpha}\right)+\right.$$
$$\left.\alpha^{-1-\nu}\Gamma\left[\begin{matrix}-\nu-1-\nu, & 1+\nu\\ 1+\beta-1-\nu & \end{matrix}\right]{}_2F_1\left(1+\nu,1-1-\beta+1+\nu;1+\nu;-\frac{1}{\alpha}\right)\right\}=$$
$$-\frac{\sin\pi\nu}{\pi}\Gamma(\beta)\cdot\left\{\alpha^\nu\frac{\Gamma(2\nu+1)\Gamma(-\nu)}{\Gamma(1+\beta+\nu)}{}_2F_1\left(-\nu,-\beta-\nu;-2\nu;-\frac{1}{\alpha}\right)+\right.$$

$$\alpha^{-1-\nu}\frac{\Gamma(-1-2\nu)\Gamma(1+\nu)}{\Gamma(\beta-\nu)}\,{}_2F_1\left(1+\nu,1+\nu-\beta;1+\nu;-\frac{1}{\alpha}\right)\Bigg\}.$$

But from [19], p. 131, formula (19), it follows that equality is obtained for $I(\alpha)$ when $0<\alpha<1$ and $\alpha\geq 1$ coincide.

From (8.75) it follows that we obtained the desired formula (1.101); however, its conclusion is not completely rigorous, since we did not check the legality of applying the Mellin transform and the validity conditions of the Slater–Marichev theorem (which is, in our case, rather complicated). However, now we can apply the Mellin transform to both sides of the obtained formal equality (1.101). As a result, we prove that for $\alpha>0$, $\beta>0$, relation (1.101) is an identity. □

As a consequence, we now obtain the necessary estimate and we prove Theorem 110.

The simplest similar estimate was obtained in [526] for the potential $q(x)=cx^{-2}$, for which $\beta=\frac{1}{2}$. As follows from [19], in this case the Legendre function $P_\nu^{-\frac{1}{2}}(z)$ can be expressed in terms of elementary functions. Therefore, the corresponding estimate can be expressed in terms of elementary functions.

Another potential for which the obtained estimate can be simplified and expressed in terms of elementary functions is a potential of the form $q(x)=x^{-(2\beta+1)}$, when the parameters are related by the relation $\beta=\alpha-1$.

Corollary 18. *Let the relation between the parameters $\beta=\alpha-1$. Then the estimate of Theorem 110 has a form*

$$|P(x,t)|\leq\left(\frac{t}{x}\right)^{\beta+1}\frac{2^{\beta-2}}{\beta}\left[\frac{t^2+x^2}{2tx}\right]^{\beta}\cdot\exp\left[\left(\frac{t-x}{2}\right)\frac{2^{\beta-2}}{\beta}\left[\frac{t^2+x^2}{2tr}\right]^{\beta}\right]=$$
$$\frac{1}{4\beta}\frac{1}{x^{2\beta+1}}(t^2+x^2)^{\beta}\exp\left[\frac{2^{\beta-2}}{\beta}\left(\frac{t-x}{2}\right)\left(\frac{t^2+x^2}{2tx}\right)^{\beta}\right]. \tag{8.76}$$

Proof. In this case, we transform the estimate from Theorem 110 as follows:

$$\frac{\Gamma(\beta)4^{\beta-1}}{(t^2-x^2)^{\beta}}P_\beta^{-\beta}\left(\frac{t^2+x^2}{2tx}\right)=\frac{\Gamma(\beta)4^{\beta-1}}{(t^2-x^2)^{\beta}}\frac{2^{-\beta}}{\Gamma(\beta+1)}\left[\left(\frac{t^2+x^2}{2tx}\right)^2-1\right]^{\frac{\beta}{2}}=$$
$$\frac{2^{\beta-2}}{\beta}\frac{1}{(t^2-x^2)^{\beta}}\frac{(t^2-x^2)^{\beta}(t^2+x^2)^{\beta}}{(2tx)^{\beta}}=\frac{2^{\beta-2}}{\beta}\left[\frac{t^2+x^2}{2tx}\right]^{\beta},$$

where the transformations used the formula (see [19])

$$P_\nu^{-\nu}(z)=\frac{2^{-\nu}}{\Gamma(\nu+1)}(z^2-1)^{\frac{\nu}{2}},\ z>1.$$

Therefore, the inequality for the kernel with $\beta=\alpha-1$ takes the form (8.76). □

Note that for $\alpha = 0$ in formulas (8.59)–(8.60), Theorem 108 reduces to well-known estimates for the kernel of the Levin type integral representation for the Sturm–Liouville equation.

The above technique is fully transferred to the problem of constructing nonclassical operators of generalized translation. This problem is essentially equivalent to expressing solutions to the equation

$$B_{\alpha,x}u(x,y) - q(x)u(x,y) = B_{\beta,y}u(x,y) \tag{8.77}$$

through solutions of the unperturbed Euler–Poisson–Darboux equation with Bessel operators in each variable (in the nonsingular case, the wave) in the presence of additional conditions that ensure correctness.

Such representations are obtained already from the fact of the existence of transmutation operators and were studied for the nonsingular case ($\alpha = \beta = 0$) in [316,321] as a consequence of the generalized translation theory. An interesting original technique for obtaining such representations was also developed in the nonsingular case in the papers of A. V. Borovskikh [34]. From the results of this subsection, integral representations of a certain subclass of solutions of Eq. (8.77) in the general singular case for sufficiently arbitrary potentials with singularities at the origin of coordinates follow. Moreover, the estimates for the solutions do not contain any indefinite constants, and for the kernels of integral representations the integral equations are written in explicit form, which they satisfy.

8.3.4 Asymptotically exact inequalities for Legendre functions

In this subsection we show how one can use formula (1.101) to establish an asymptotically exact lower bound for Legendre functions. This assessment, in our opinion, is of independent interest.

Consider the obvious inequalities ($\alpha > 0$)

$$\int_0^1 P_\nu(1+2\alpha x)(1-x)^{-\frac{1}{2}}\,dx \le \int_0^1 P_\nu(1+2\alpha x)x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx \le$$

$$P_\nu(1+2\alpha)\int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx.$$

The second integral is known, and, as follows, for example, from [19], p. 225, it is calculated by the formula

$$I = \pi^{\frac{1}{2}}\,\Gamma\left(\frac{1}{2}\right)\left\{P_\nu(\sqrt{1+\alpha})\right\}^2.$$

Therefore, the inequality

$$\sqrt{\pi}\,\alpha^{-\frac{1}{4}}(\alpha+1)^{-\frac{1}{4}}\,P_\nu^{-\frac{1}{2}}(2\alpha+1) \le \pi\left\{P_\nu(\sqrt{1+\alpha})\right\}^2$$

is valid. Expressing the Legendre function in elementary functions, we finally get

$$\left\{P_\nu(\sqrt{1+\alpha})\right\}^2 \geq \frac{1}{\pi(2\nu+1)\sqrt{x}}\left\{\left(\sqrt{x}+\sqrt{x+1}\right)^{2\nu+1}-\left(\sqrt{x}+\sqrt{x+1}\right)^{-2\nu-1}\right\}.$$

The proved inequality gives the correct asymptotics for all parameters included in it, as follows from the asymptotic formula given in [448], p. 107,

$$\left[P_\nu(\sqrt{1+x})\right]^2 \sim \frac{1}{2\pi\nu}\frac{\left(\sqrt{x}+\sqrt{x+1}\right)^{2\nu+1}}{\sqrt{x}}.$$

The second of the considered integral inequalities also leads to an interesting estimate

$$\left\{P_\nu(\sqrt{1+\alpha})\right\}^2 \leq P_\nu(1+2\alpha).$$

In connection with the last inequalities presented, we note that the Legendre functions are associated with complete and incomplete elliptic Legendre integrals of three kinds; these two classes of special functions are expressed through each other at certain values. On the other hand, Legendre functions are particular cases of the Gauss hypergeometric function. Various inequalities for elliptic Legendre integrals are considered in [221,222], and inequalities for hypergeometric functions are considered in [223,224].

8.4 Iterated spherical mean in the computed tomography problem

Spherical means have numerous applications in theoretical mathematics and its applications. In the classic books [75,155,170], various applications of spherical means to the theory of partial differential equations, including elliptic, hyperbolic, and ultrahyperbolic types, are given. In addition, spherical means are the object of study of integral geometry with the application of research results to tomography [163], for example photoacoustics [95] and diffraction tomography [18]. The close connection of spherical means with the Fourier transform and Riesz potentials is also known (see [162]). Representation of solutions of various partial differential equations using spherical means is also related to the theory of transmutation operators [234].

In this section we consider the weighted spherical mean (3.183) and obtain identities for iterated weighted spherical means, which are necessary to obtain explicit formulas when restoring a function from its weighted spherical means. In addition, using weighted spherical means we give the formula for representing the function through the generalized translation operators (3.144) and the Hankel transform (12). Such formulas are used in various applied problems of tomography and integral geometry.

8.4.1 Iterated weighted spherical mean and its properties

The iterated weighted spherical mean has the form (see [536])

$$I_f^\gamma(x;\lambda,\mu)=I^\gamma_{\lambda,\mu}f(x)=M_\lambda^\gamma M_\mu^\gamma f(x)=$$
$$\frac{1}{|S_1^+(n)|^2_\gamma}\int\limits_{S_1^+(n)}\int\limits_{S_1^+(n)}{}^\gamma T_x^{\lambda\zeta}\,{}^\gamma T_x^{\mu\xi}[f(x)]\zeta^\gamma\xi^\gamma dS(\xi)dS(\zeta),$$

where M_λ^γ and M_μ^γ are weighted spherical means (3.183).

Using the permutation property of the generalized translation (3.150), we obtain that the iterated weighted spherical mean is symmetric with respect to

$$I_f^\gamma(x;\lambda,\mu)=I_f^\gamma(x;\mu,\lambda).$$

It is clear that

$$I_f^\gamma(x;\lambda,0)=I_f^\gamma(x;0,\lambda)=M_\lambda^\gamma f(x)$$

and

$$I_f^\gamma(x;0,0)=f(x).$$

Following [155], p. 73, we prove the equality expressing the iterated spherical mean $I_f^\gamma(x;\lambda,\mu)$ through a single integral from the weighted spherical mean $M_r^\gamma[f(x)]$.

Theorem 111. *Let $f\in L_1^\gamma$. Then*

$$I_f^\gamma(x;\lambda,\mu)={}^\nu T_\mu^\lambda M_f^\gamma(x;\mu),\tag{8.78}$$

where $\nu=n+|\gamma|-1$. The following formulas are also true:

$$I_f^\gamma(x;\lambda,\mu)=\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\frac{1}{(2\lambda\mu)^{n+|\gamma|-2}}\times$$
$$\int\limits_{\lambda-\mu}^{\lambda+\mu}\left((\lambda^2-(r-\mu)^2)((r+\mu)^2-\lambda^2)\right)^{\frac{n+|\gamma|-3}{2}}M_r^\gamma[f(x)]\,r\,dr,\tag{8.79}$$

$$I_f^\gamma\left(x;\frac{\beta-\alpha}{2},\frac{\beta+\alpha}{2}\right)=$$
$$\frac{\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\frac{2^{n+|\gamma|-1}}{(\beta^2-\alpha^2)^{n+|\gamma|-2}}\int\limits_\alpha^\beta\left((\beta^2-r^2)(r^2-\alpha^2)\right)^{\frac{n+|\gamma|-3}{2}}M_r^\gamma[f(x)]rdr.\tag{8.80}$$

Proof. Let $g(s)$ be an arbitrary continuous finite function of one variable. Consider the integral

$$J=\int\limits_0^\infty \lambda^{n+|\gamma|-1}g(\lambda)I_f^\gamma(x;\lambda,\mu)d\lambda=$$
$$\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_0^\infty \lambda^{n+|\gamma|-1}g(\lambda)d\lambda\int\limits_{S_1^+(n)}\int\limits_{S_1^+(n)}{}^\gamma T_x^{\lambda\zeta}\,{}^\gamma T_x^{\mu\xi}[f(x)]\zeta^\gamma\xi^\gamma dS(\xi)dS(\zeta).$$

Using the property (3.150) of generalized translation and formula (1.104), we can write

$$J=\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}{}^\gamma T_x^{\mu\xi}\left[\int\limits_0^\infty \lambda^{n+|\gamma|-1}g(\lambda)d\lambda\int\limits_{S_1^+(n)}{}^\gamma T_x^{\lambda\zeta}[f(x)]\zeta^\gamma dS(\zeta)\right]\xi^\gamma dS(\xi)=$$
$$\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}{}^\gamma T_x^{\mu\xi}\left[\lim_{R\to+\infty}\int\limits_0^R \lambda^{n+|\gamma|-1}g(\lambda)d\lambda\int\limits_{S_1^+(n)}{}^\gamma T_x^{\lambda\zeta}[f(x)]\zeta^\gamma dS(\zeta)\right]\times$$
$$\xi^\gamma dS(\xi)=$$
$$\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}{}^\gamma T_x^{\mu\xi}\left[\lim_{R\to+\infty}\int\limits_{B_R^+(n)}{}^\gamma T_x^{z}[f(x)]g(|z|)z^\gamma dz\right]\xi^\gamma dS(\xi)=$$
$$\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}{}^\gamma T_x^{\mu\xi}\left[\int\limits_{\mathbb{R}_+^n}{}^\gamma T_x^{z}[f(x)]g(|z|)z^\gamma dz\right]\xi^\gamma dS(\xi).$$

Now, applying the properties of associativity (3.151) and self-adjointness (3.157), we obtain

$$J=\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}\int\limits_{\mathbb{R}_+^n}\left({}^\gamma T_z^{\mu\xi}T_z^x[f(z)]\right)g(|z|)z^\gamma dz\,\xi^\gamma dS(\xi)=$$
$$\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}\int\limits_{\mathbb{R}_+^n}{}^\gamma T_z^x[f(z)]\,{}^\gamma T_z^{\mu\xi}[g(|z|)]z^\gamma dz\,\xi^\gamma dS(\xi)=$$
$$\frac{C(\gamma)}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}\int\limits_{\mathbb{R}_+^n}{}^\gamma T_x^z[f(x)]\int\limits_0^\pi\ldots\int\limits_0^\pi\prod_{i=1}^n\sin^{\gamma_i-1}\alpha_i\times$$

$$
g(\sqrt{\mu^2\xi_1^2+...+\mu^2\xi_n^2+z_1^2+...+z_n^2-2\mu\xi_1 z_1\cos\alpha_1-...-2\mu\xi_n z_n\cos\alpha_n})\times
d\alpha_1...d\alpha_n\, z^\gamma dz\, \xi^\gamma dS(\xi).
$$

Passing to the spherical coordinates $z=r\eta$, $|\eta|=1$, $r\geq 0$ in the integral by z and taking into account that $|\xi|=1$, we obtain

$$
J=\frac{C(\gamma)}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}\int\limits_0^\infty r^{n+|\gamma|-1}dr\int\limits_{S_1^+(n)}{}^\gamma T_x^{r\eta}[f(x)]\times
$$

$$
\int\limits_0^\pi...\int\limits_0^\pi\prod_{i=1}^n\sin^{\gamma_i-1}\alpha_i\, g(\sqrt{\mu^2+r^2-2r\mu\langle\xi,\eta\cos\alpha\rangle})d\alpha_1...d\alpha_n\,\eta^\gamma dS(\eta)\,\xi^\gamma dS(\xi),
$$

where

$$
\langle\xi,\eta\cos\alpha\rangle=\xi_1\eta_1\cos\alpha_1+...+\xi_n\eta_n\cos\alpha_n.
$$

Using the multi-dimensional Poisson operator (3.137) we can write the integral J in the form

$$
J=\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_0^{+\infty} r^{n+|\gamma|-1}dr\int\limits_{S_1^+(n)}{}^\gamma T_x^{r\eta}[f(x)]\eta^\gamma dS(\eta)\times
$$

$$
\int\limits_{S_1^+(n)}\mathcal{P}_\eta^\gamma g(\sqrt{r^2+\mu^2-2r\mu\langle\xi,\eta\rangle})\,\xi^\gamma dS(\xi).
$$

Applying to the integral $\int\limits_{S_1^+(n)}\mathcal{P}_\eta^\gamma g(\sqrt{r^2+\mu^2-2r\mu\langle\xi,\eta\rangle})\,\xi^\gamma dS(\xi)$ the formula (see [247])

$$
\int\limits_{S_1^+(n)}\mathcal{P}_\xi^\gamma g(\langle\xi,x\rangle)x^\gamma dS(x)=C(\gamma)\int\limits_{-1}^1 g(|\xi|p)(1-p^2)^{\frac{n+|\gamma|-3}{2}}\,dp
$$

and noting that $|\eta|=1$, we obtain

$$
J=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_0^\infty r^{n+|\gamma|-1}dr\int\limits_{S_1^+(n)}{}^\gamma T_x^{r\eta}[f(x)]\eta^\gamma dS(\eta)\times
$$

$$
\int\limits_{-1}^1(1-p^2)^{\frac{n+|\gamma|-3}{2}}g(\sqrt{r^2+\mu^2-2r\mu p})\,dp=
$$

$$\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\frac{1}{|S_1^+(n)|_\gamma}\int\limits_0^\infty r^{n+|\gamma|-1}\,M_r^\gamma[f(x)]\,dr\times$$
$$\int\limits_{-1}^{1}(1-p^2)^{\frac{n+|\gamma|-3}{2}}g(\sqrt{r^2+\mu^2-2r\mu p})\,dp.$$

Now, instead of the variable p, we introduce the following formula associating the variable λ associated with p:

$$r^2+\mu^2-2r\mu p=\lambda^2.$$

We get

$$p=\frac{r^2+\mu^2-\lambda^2}{2r\mu},dp=-\frac{\lambda}{r\mu}\,d\lambda,1-p^2=\frac{(\lambda^2-(r-\mu)^2)((r+\mu)^2-\lambda^2)}{(2r\mu)^2},$$

and for $p=-1$, $\lambda=|r+\mu|$, and for $p=1$, $\lambda=|r-\mu|$. Then using (1.107) we obtain

$$J=(2\mu)^{2-n-|\gamma|}\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\times$$
$$\int\limits_0^\infty g(\lambda)\,\lambda\,d\lambda\int\limits_{|\lambda-\mu|}^{|\lambda+\mu|}\Big((\lambda^2-(r-\mu)^2)((r+\mu)^2-\lambda^2)\Big)^{\frac{n+|\gamma|-3}{2}}\,M_r^\gamma[f(x)]rdr.$$

Since $g(\lambda)$ is an arbitrary function, from the equality

$$\int\limits_0^\infty\lambda^{n+|\gamma|-1}g(\lambda)I_f^\gamma(x;\lambda,\mu)d\lambda=(2\mu)^{2-n-|\gamma|}\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\times$$
$$\int\limits_0^\infty g(\lambda)\,\lambda\,d\lambda\int\limits_{\lambda-\mu}^{\lambda+\mu}\Big((\lambda^2-(r-\mu)^2)((r+\mu)^2-\lambda^2)\Big)^{\frac{n+|\gamma|-3}{2}}\,M_r^\gamma[f(x)]rdr$$

it follows that

$$I_f^\gamma(x;\lambda,\mu)=\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\times$$
$$\frac{1}{(2\lambda\mu)^{n+|\gamma|-2}}\int\limits_{\lambda-\mu}^{\lambda+\mu}\Big((\lambda^2-(r-\mu)^2)((r+\mu)^2-\lambda^2)\Big)^{\frac{n+|\gamma|-3}{2}}\,M_r^\gamma[f(x)]rdr. \tag{8.81}$$

We removed the modules within the integration due to the fact that the integrand is odd by r.

Now taking into account (3.154) for $\nu = n+|\gamma|-1$, we get

$$I_f^\gamma(x;\lambda,\mu) = {}^\nu T_\mu^t M_f^\gamma(x;\mu).$$

If in (8.81) we put $\alpha = \lambda-\mu$, $\beta = \lambda+\mu$, $(\beta > \alpha)$, then we obtain

$$I_f^\gamma\left(x;\frac{\beta-\alpha}{2},\frac{\beta+\alpha}{2}\right) =$$
$$\frac{\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\frac{2^{n+|\gamma|-1}}{(\beta^2-\alpha^2)^{n+|\gamma|-2}}\int\limits_\alpha^\beta\Big((\beta^2-r^2)(r^2-\alpha^2)\Big)^{\frac{n+|\gamma|-3}{2}} M_r^\gamma[f(x)]r\,dr.$$

The theorem is proved. $\square$

We give a corollary of the proved theorem expressing the action of iterated weighted averages on Bessel functions.

Corollary 19. *For the function $\mathbf{j}_\gamma(x,\xi)$, the following equality is true:*

$$I^\gamma_{\frac{\beta-\alpha}{2},\frac{\beta+\alpha}{2}}\mathbf{j}_\gamma(x,\xi) = \frac{\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\frac{2^{n+|\gamma|-1}\mathbf{j}_\gamma(x,\xi)}{(\beta^2-\alpha^2)^{n+|\gamma|-2}}\times$$
$$\int\limits_\alpha^\beta\Big((\beta^2-r^2)(r^2-\alpha^2)\Big)^{\frac{n+|\gamma|-3}{2}} j_{\frac{n+|\gamma|-2}{2}}(r)r\,dr. \tag{8.82}$$

Proof. In Theorem 111, let us choose $f(x) = \mathbf{j}_\gamma(x,\xi)$. Using formula (3.190) of the form

$$M_\mu^\gamma\mathbf{j}_\gamma(x,\xi) = \mathbf{j}_\gamma(x,\xi)\; j_{\frac{n+|\gamma|-2}{2}}(\mu),$$

we get

$$I_{\lambda,\mu}^\gamma\mathbf{j}_\gamma(x,\xi) = \mathbf{j}_\gamma(x,\xi)\; j_{\frac{n+|\gamma|-2}{2}}(\mu)\; j_{\frac{n+|\gamma|-2}{2}}(\lambda).$$

Applying (8.80), we obtain (8.82). We can rewrite (8.82) in the form

$$j_{\frac{n+|\gamma|-2}{2}}\left(\frac{\beta-\alpha}{2}\right)\; j_{\frac{n+|\gamma|-2}{2}}\left(\frac{\beta+\alpha}{2}\right) =$$
$$\frac{\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\frac{2^{n+|\gamma|-1}}{(\beta^2-\alpha^2)^{n+|\gamma|-2}}\int\limits_\alpha^\beta\Big((\beta^2-r^2)(r^2-\alpha^2)\Big)^{\frac{n+|\gamma|-3}{2}} j_{\frac{n+|\gamma|-2}{2}}(r)r\,dr.$$

If for $I^{\gamma}_{\lambda,\mu}\mathbf{j}_\gamma(x,\xi)$ we write equality (8.78), we get the known formula (3.152):

$$^{\nu}\mathcal{T}^{\lambda}_{\mu}\, j_{\frac{\nu-1}{2}}(\mu) = j_{\frac{\nu-1}{2}}(\mu)\; j_{\frac{\nu-1}{2}}(\lambda), \qquad \nu = n + |\gamma| - 1. \qquad \square$$

8.4.2 Application of identity for an iterated spherical mean to the task of computed tomography

Consider one application of formula (8.79) from Theorem 111 to computed tomography. In problems of diffraction tomography and backscattering, the Hankel transform is the measured data (see, for example, [168], [302], p. 126, and [3], p. 90). Similar formulas are used to restore function.

Let us prove a formula expressing a function through its Hankel transform and a generalized translation. This is a generalization of the well-known formula for a simpler problem, which uses the representation of a function through its Fourier transform and the usual shift. In this form, similar representations are used to restore functions in the indicated problems of tomography and integral geometry.

In the theory of scattering, the surface of a ball $|x| < 2\lambda$, where λ is the given wavelength and x is the space vector called the Ewald sphere (see [129]). The Ewald sphere can be used to find the maximum resolution available for a given X-ray wavelength and the unit cell dimensions.

Theorem 112. *Let $\widehat{F}$ be a function with support inside part of the ball*

$$B^+_{2\lambda}(n) = \{x \in \mathbb{R}^n_+ : |x| < 2\lambda\}.$$

The equality

$$F(y) = C(n,\gamma) \int\limits_{S_1^+(n)} \int\limits_{S_1^+(n)} {}^{\gamma}T^{\lambda\zeta}_{\lambda\xi} \left[\frac{|\lambda\xi|\widehat{F}(\lambda\xi)\mathbf{j}_\gamma(\lambda\xi, y)}{(4\lambda^2 - |t\xi|^2)^{\frac{n+|\gamma|-3}{2}}} \right] \zeta^\gamma \xi^\gamma dS(\xi) dS(\zeta), \tag{8.83}$$

where

$$C(n,\gamma) = \frac{\sqrt{\pi}\,2^{2n-3}\lambda^{2n+2|\gamma|-4}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}{\Gamma^2\left(\frac{n+|\gamma|}{2}\right) \prod\limits_{j=1}^{n} \Gamma\left(\frac{\gamma_j+1}{2}\right) |S_1^+(n)|^2_\gamma},$$

is true.

Proof. Let us put $\mu = \lambda$ in (8.79). Then

$$I^{\gamma}_f(x;\lambda,\lambda) = \frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)} \frac{1}{(2\lambda^2)^{n+|\gamma|-2}} \int\limits_0^{2\lambda} \left[4\lambda^2 - r^2\right]^{\frac{n+|\gamma|-3}{2}} r^{n+|\gamma|-2} M^{\gamma}_r[f(x)]dr. \tag{8.84}$$

Let us consider the function

$$f_y(x) = \frac{|x|}{(4\lambda^2 - |x|^2)^{\frac{n+|\gamma|-3}{2}}} \widehat{F}(x)\mathbf{j}_\gamma(x, y), \qquad x, y \in \mathbb{R}^n_+.$$

We now find the weighted spherical mean of $f_y(x)$ when $x = 0$:

$$M_r^\gamma f_y(x)|_{x=0} = \frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} [{}^\gamma T_x^{rz} f_y(x)]_{x=0} z^\gamma dS(z) =$$

$$\frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} \left[{}^\gamma T_x^{rz} \frac{|x|}{(4\lambda^2 - |x|^2)^{\frac{n+|\gamma|-3}{2}}} \widehat{F}(x)\mathbf{j}_\gamma(x, y) \right]_{x=0} z^\gamma dS(z) =$$

$$\frac{1}{|S_1^+(n)|_\gamma} \int\limits_{S_1^+(n)} \frac{r|z|}{(4\lambda^2 - r|z|^2)^{\frac{n+|\gamma|-3}{2}}} \widehat{F}(rz)\mathbf{j}_\gamma(rz, y) z^\gamma dS(z) =$$

$$\frac{1}{|S_1^+(n)|_\gamma} \frac{r}{(4\lambda^2 - |r|^2)^{\frac{n+|\gamma|-3}{2}}} \int\limits_{S_1^+(n)} \widehat{F}(rz)\mathbf{j}_\gamma(rz, y) z^\gamma dS(z).$$

Using (8.84) we get

$$I_f^\gamma(0; \lambda, \lambda) =$$

$$\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)} \frac{1}{(2\lambda^2)^{n+|\gamma|-2}} \int\limits_0^{2\lambda} \left[4\lambda^2 - r^2\right]^{\frac{n+|\gamma|-3}{2}} r^{n+|\gamma|-2} M_f^\gamma(0, r) dr =$$

$$\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)} \frac{(2\lambda^2)^{2-n-|\gamma|}}{|S_1^+(n)|_\gamma} \int\limits_0^{2\lambda} \left[4\lambda^2 - r^2\right]^{\frac{n+|\gamma|-3}{2}} r^{n+|\gamma|-2} \frac{r}{(4\lambda^2 - r^2)^{\frac{n+|\gamma|-3}{2}}} dr \times$$

$$\int\limits_{S_1^+(n)} \widehat{F}(rz)\mathbf{j}_\gamma(rz, y) z^\gamma dS(z) =$$

$$\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)} \frac{(2\lambda^2)^{2-n-|\gamma|}}{|S_1^+(n)|_\gamma} \int\limits_0^{2\lambda} r^{n+|\gamma|-1} dr \int\limits_{S_1^+(n)} \widehat{F}(rz)\mathbf{j}_\gamma(rz, y) z^\gamma dS(z).$$

Applying (1.104) we can write

$$I_f^\gamma(0; \lambda, \lambda) = \frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{|\gamma|+n-1}{2}\right)} \frac{(2\lambda^2)^{2-n-|\gamma|}}{|S_1^+(n)|_\gamma} \int\limits_{B_{2\lambda}^+} \widehat{F}(z)\mathbf{j}_\gamma(z, y) z^\gamma dz =$$

$$\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\,\frac{(2\lambda^2)^{2-n-|\gamma|}}{|S_1^+(n)|_\gamma}\int\limits_{\mathbb{R}^n_+}\widehat{F}(z)\mathbf{j}_\gamma(z,y)z^\gamma\,dz=$$

$$\frac{2\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\,\frac{(2\lambda^2)^{2-n-|\gamma|}}{|S_1^+(n)|_\gamma}\,2^{|\gamma|-n}\prod_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)F(y)=$$

$$\frac{\Gamma^2\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}{\sqrt{\pi}2^{2n-3}\lambda^{2n+2|\gamma|-4}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}\,F(y).$$

Therefore,

$$F(y)=\frac{\sqrt{\pi}2^{2n-3}\lambda^{2n+2|\gamma|-4}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}{\Gamma^2\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\,I_f^\gamma(0;\lambda,\lambda). \tag{8.85}$$

From the other side,

$$I_f^\gamma(0;\lambda,\lambda)=\frac{1}{|S_1^+(n)|_\gamma^2}\int\limits_{S_1^+(n)}\int\limits_{S_1^+(n)}{}^\gamma T_{t\xi}^{t\zeta}[f(t\xi)]\zeta^\gamma\xi^\gamma\,dS(\xi)dS(\zeta). \tag{8.86}$$

From (8.85) and (8.86) we get

$$F(y)=\frac{\sqrt{\pi}2^{2n-3}\lambda^{2n+2|\gamma|-4}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)}{\Gamma^2\left(\frac{n+|\gamma|}{2}\right)\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)|S_1^+(n)|_\gamma^2}\times$$

$$\int\limits_{S_1^+(n)}\int\limits_{S_1^+(n)}T_{\lambda\xi}^{\lambda\zeta}\left[\frac{|\lambda\xi|}{(4\lambda^2-|t\xi|^2)^{\frac{n+|\gamma|-3}{2}}}\widehat{F}(\lambda\xi)\mathbf{j}_\gamma(\lambda\xi,y)\right]\zeta^\gamma\xi^\gamma\,dS(\xi)dS(\zeta). \qquad \square$$

Fractional powers of Bessel operators

9

In this chapter we study the fractional powers $(B_\gamma)^\alpha$, $\alpha \in \mathbb{R}$, of the differential Bessel operator in the form

$$B_\gamma = D^2 + \frac{\gamma}{x} D, \qquad \gamma \geq 0, \quad D := \frac{d}{dx}. \tag{9.1}$$

Of course fractional powers of the Bessel operator (9.1) were studied in many papers. But in most of them fractional powers were defined implicitly as a power function multiplication under Hankel transform. This definition via integral transforms leads to many restrictions. Just imagine that for the classical Riemann–Liouville fractional integrals we have to work only with its definitions via Laplace or Mellin transforms and nothing more without explicit integral representations. If it would be true, then 99% of the classical "Bible" [494] and other books on fractional calculus would be empty as they mostly use explicit integral definitions! But for fractional powers of the Bessel operator in most papers implicit definitions via Hankel transform are still used. Such situation is not natural and in some papers different approaches to step closer to explicit formulas were studied. Let us mention that in [367] explicit formulas were derived as compositions of Erdélyi–Kober fractional integrals [494] on distribution spaces; in this monograph results on fractional powers of Bessel and related operators are gathered from McBride's and earlier papers. An important step was made in [555] in which explicit definitions were derived in terms of the Gauss hypergeometric functions with different applications to partial differential equations; we also use basic formulas from [555] in this chapter. The most general study was carried out by I. Dimovski and V. Kiryakova [90–92,252] for the more general class of hyper-Bessel differential operators related to the Obrechkoff integral transform. They constructed explicit integral representations of the fractional powers of these operators by using Meijer G-functions as kernels, and also intensively and successfully used for this the theory of transmutations. Note that in this and other fields of theoretical and applied mathematics, the methods of transmutation theory are very useful and productive and for some problems are even irreplaceable (see, e.g., [89]). In [527,531] simplified representations for fractional powers of the Bessel operator were derived with Legendre functions as kernels, and based on them general definitions were simplified and unified with standard fractional calculus notation as in [494], and also important generalized Taylor formulas were proved which mix integer powers of Bessel operators (instead of derivatives in the classical Taylor formula) with fractional powers of the Bessel operator as integral remainder term (cf. also [268,532]).

In this chapter we study fractional powers of the Bessel differential operator. The fractional powers are defined explicitly in the integral form without the use of integral transforms in the definitions. Some general properties of the fractional powers of the

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Bessel differential operator are proved and others are listed. Among them are different variations of definitions, relations with the Mellin and Hankel transforms, group properties, the generalized Taylor formula with Bessel operators, and an evaluation of the resolvent integral operator in terms of the Wright or generalized Mittag-Leffler functions. At the end, some topics are suggested for further study and possible generalizations.

9.1 Fractional Bessel integrals and derivatives on a segment

In this section we give the definitions of the fractional Bessel integrals and derivatives which correspond of the Riemann–Liouville fractional integrals and fractional derivatives on a finite interval, and we consider their properties. We also consider fractional Bessel derivatives on a finite interval of Gerasimov–Caputo type.

9.1.1 Definitions

Let $[a,b]$ $(0\leq a<b<\infty)$ be a finite interval on the real semiaxis $[0,\infty)$.

Definition 44. *Let $\alpha>0$. The* ***right-sided fractional Bessel integral*** $B^{-\alpha}_{\gamma,b-}$ ***on a segment*** $[a,b]$ *for $f\in L_1(a,b)$, $a,b\in[0,\infty)$, is defined by the formula*

$$(B^{-\alpha}_{\gamma,b-}f)(x)=(IB^{\alpha}_{\gamma,b-}f)(x)=$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_x^b\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)f(y)dy. \tag{9.2}$$

The ***left-sided fractional Bessel integral*** $B^{-\alpha}_{\gamma,a+}$ ***on a segment*** $[a,b]$ *for $f\in L_1(a,b)$, $a,b\in(0,\infty)$, is defined by the formula*

$$(B^{-\alpha}_{\gamma,a+}f)(x)=(IB^{\alpha}_{\gamma,a+}f)(x)=$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_a^x\left(\frac{y}{x}\right)^{\gamma}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)f(y)dy. \tag{9.3}$$

Definition 45. *Let $\alpha>0$, $n=[\alpha]+1$, $f\in L_1(a,b)$, $IB^{n-\alpha}_{\gamma,b-}f$, $IB^{n-\alpha}_{\gamma,a+}f\in C^{2n}_{ev}(a,b)$. The* ***right-sided and left-sided fractional Bessel derivatives on a segment of the Riemann–Liouville type*** *for $\alpha\neq 0,1,1,\ldots$ are defined, respectively, by the equalities*

$$(B^{\alpha}_{\gamma,b-}f)(x)=(DB^{\alpha}_{\gamma,b-}f)(x)=B^n_{\gamma}(IB^{n-\alpha}_{\gamma,b-}f)(x),\qquad n=[\alpha]+1, \tag{9.4}$$

and

$$(B^{\alpha}_{\gamma,a+}f)(x)=(DB^{\alpha}_{\gamma,a+}f)(x)=B^n_{\gamma}(IB^{n-\alpha}_{\gamma,a+}f)(x),\qquad n=[\alpha]+1. \tag{9.5}$$

When $\alpha = n \in \mathbb{N} \cup \{0\}$, *then*

$$(B^0_{\gamma,b-} f)(x) = (B^0_{\gamma,a+} f)(x) = f(x),$$
$$(B^n_{\gamma,b-} f)(x) = (B^n_{\gamma,a+} f)(x) = B^n_\gamma f(x),$$

where B^n_γ *is an iterated Bessel operator* (9.1).

Remark 18. *In some cases for fractional Bessel integrals on a segment* $[a,b]$ *it is convenient to use the notations* $B^{-\alpha}_{\gamma,b-}$ *and* $B^{-\alpha}_{\gamma,a+}$ *and in other cases* $IB^{\alpha}_{\gamma,b-}$ *and* $IB^{\alpha}_{\gamma,a+}$. *Similarly, for fractional Bessel integrals on a segment* $[a,b]$ *sometimes we will use the notations* $B^{\alpha}_{\gamma,b-}$ *and* $B^{\alpha}_{\gamma,a+}$ *and sometimes* $DB^{\alpha}_{\gamma,b-}$ *and* $DB^{\alpha}_{\gamma,a+}$.

Definition 44 is based on integral representations introduced for special cases $a = 1, b = 1$ in [555].

It was noted in [515,527,531] that Definition 44 may be simplified, as the kernels are expressed in a more simple way via Legendre functions (the Legendre functions are a two-parameter family but the Gauss hypergeometric functions are in general a three-parameter family). These simplifications are based on the formula [457]

$$
\begin{aligned}
&{}_2F_1(a,b;2b;z) = \\
&2^{2b-1}\Gamma\left(b+\frac{1}{2}\right) z^{\frac{1}{2}-b}(1-z)^{\frac{1}{2}\left(b-a-\frac{1}{2}\right)} P^{\frac{1}{2}-b}_{a-b-\frac{1}{2}}\left[\left(1-\frac{z}{2}\right)\frac{1}{\sqrt{1-z}}\right],
\end{aligned}
\tag{9.6}
$$

and have forms

$$(B^{-\alpha}_{\gamma,b-} f)(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)} \int\limits_x^b (y^2-x^2)^{\alpha-\frac{1}{2}} \left(\frac{y}{x}\right)^{\frac{\gamma}{2}} P^{\frac{1}{2}-\alpha}_{\frac{\gamma}{2}-1}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right] f(y)dy,$$

$$(B^{-\alpha}_{\gamma,a+} f)(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)} \int\limits_a^x (x^2-y^2)^{\alpha-\frac{1}{2}} \left(\frac{y}{x}\right)^{\frac{\gamma}{2}} P^{\frac{1}{2}-\alpha}_{\frac{\gamma}{2}-1}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right] f(y)dy.$$

Now we would like to have another explicit formula for B^α_γ when $\alpha > 0$. For applications it is better to use the generalization of the Gerasimov–Caputo fractional derivative (2.30).

Definition 46. *Let* $n = [\alpha]+1$, $f \in L[0,\infty)$, $IB^{n-\alpha}_{\gamma,b-} f$, $IB^{n-\alpha}_{\gamma,a+} f \in C^{2n}_{ev}(0,\infty)$.

The ***right-sided fractional Bessel derivatives on a segment*** $[a,b]$ ***of Gerasimov–Caputo type*** *for* $\alpha > 0$, $\alpha \neq 0,1,2,\ldots$, *is defined by the equality*

$$(\mathcal{B}^\alpha_{\gamma,b-} f)(x) = (IB^{n-\alpha}_{\gamma,b-} B^n_\gamma f)(x).$$

The ***left-sided fractional Bessel derivatives on a segment*** $[a,b]$ ***of Gerasimov–Caputo type*** *for* $\alpha > 0$, $\alpha \neq 0,1,2,\ldots$, *is defined by the equality*

$$(\mathcal{B}^\alpha_{\gamma,a+} f)(x) = (IB^{n-\alpha}_{\gamma,a+} B^n_\gamma f)(x).$$

Here $IB^{n-\alpha}_{\gamma,b-}$ is the right-sided fractional Bessel integral (9.2) on a segment $[a,b]$ and $IB^{n-\alpha}_{\gamma,a+}$ is the left-sided fractional Bessel integral (9.3) on a segment $[a,b]$. When $\alpha = n \in \mathbb{N}\cup\{0\}$, then

$$(\mathcal{B}^0_{\gamma,b-}f)(x) = (\mathcal{B}^0_{\gamma,a+}f)(x) = f(x),$$
$$(\mathcal{B}^n_{\gamma,b-}f)(x) = (\mathcal{B}^n_{\gamma,a+}f)(x) = B^n_\gamma f(x),$$

where B^n_γ is an iterated Bessel operator (9.1).

9.1.2 Basic properties of fractional Bessel integrals on a segment

Lemma 29. *For $\gamma = 0$, $f(x)\in L_1(a,b)$, $a \geq 0$, fractional Bessel integrals on a segment $[a,b]$ are*

$$(B^{-\alpha}_{0,b-}f)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_x^b (y-x)^{2\alpha-1}f(y)dy = (I^{2\alpha}_{b-}f)(x)$$

and

$$(B^{-\alpha}_{0,a+}f)(x) = \frac{1}{\Gamma(2\alpha)}\int\limits_a^x (x-y)^{2\alpha-1}f(y)dy = (I^{2\alpha}_{a+}f)(x),$$

where $I^{2\alpha}_{b-}$ and $I^{2\alpha}_{a+}$ are Riemann–Liouville fractional integrals and derivatives on a segment $[a,b]$ defined by (2.11) and (2.12), respectively.

Proof. Indeed, we have

$$(B^{-\alpha}_{0,b-}f)(x) =$$
$$\frac{1}{\Gamma(2\alpha)}\int\limits_x^b \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha-\frac{1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)f(y)dy$$

and

$$(B^{-\alpha}_{0,a+}f)(x) =$$
$$\frac{1}{\Gamma(2\alpha)}\int\limits_a^x \left(\frac{y}{x}\right)^\gamma \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha-\frac{1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)f(y)dy.$$

Using the formula that is obtained from the integral representation of the Gauss hypergeometric function (1.34) of the form

$$
{}_2F_1\left(\alpha-\frac{1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)=\left[\frac{2y}{x+y}\right]^{2\alpha-1},
$$

we obtain provable formulas. □

Consider now the case when $\alpha=1$.

Lemma 30. *The following equalities hold:*

$$
(B^{-1}_{\gamma,b-}f)(x)=\frac{1}{\gamma-1}\int\limits_x^b y\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]f(y)dy,\quad f(x)\in L_1(a,b),\quad a\geq 0,
$$

and

$$
(B^{-1}_{\gamma,a+}f)(x)=\frac{1}{\gamma-1}\int\limits_a^x y\left[1-\left(\frac{y}{x}\right)^{\gamma-1}\right]f(y)dy,\quad f(x)\in L_1(a,b),\quad a\geq 0.
$$

Proof. Applying the formula

$$
{}_2F_1\left(\frac{\gamma+1}{2},1;2;1-\frac{x^2}{y^2}\right)=\frac{2}{1-\gamma}\,\frac{y^2}{x^2-y^2}\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right],
$$

which is valid for the Gauss hypergeometric function, we obtain provable statements. □

In the following lemma we indicate the conditions under which the operators $B^{-1}_{\gamma,b-}$ and $B^{-1}_{\gamma,a+}$ will be left inverse to the Bessel differential operator on the segment.

Lemma 31. *Let $g\in C^2_{ev}(a,b)$, $f(x)=B_\gamma g(x)$, $f(x)\in L_1(a,b)$, $a\geq 0$. The equality*

$$
(B^{-1}_{\gamma,b-}B_\gamma g)(x)=g(x)
$$

is true if

$$
g(b-0)=\lim_{y\to b-0}g(y)=0,\qquad g'(b-0)=\lim_{y\to b-0}g'(y)=0.
$$

The equality

$$
(B^{-1}_{\gamma,a+}B_\gamma g)(x)=g(x)
$$

is true if

$$
g(a+0)=\lim_{x\to a+0}g(x)=0,\qquad g'(a+0)=\lim_{x\to a+0}g'(x)=0.
$$

Proof. Let us consider $B^{-1}_{\gamma,b-}$. Putting $f(x)=B_\gamma g(x)=g''(x)+\frac{\gamma}{x}g'(x)$, we obtain

$$
\begin{gathered}
(B^{-1}_{\gamma,b-}f)(x)=(B^{-1}_{\gamma,b-}B_\gamma g)(x)= \\
\frac{1}{\gamma-1}\int\limits_x^b y\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]\left(g''(y)+\frac{\gamma}{y}g'(y)\right)dy= \\
\frac{1}{\gamma-1}\left[\int\limits_x^b y\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]g''(y)dy+\gamma\int\limits_x^b\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]g'(y)dy\right].
\end{gathered}
\tag{9.7}
$$

Twice integrating by parts the first term in (9.7) leads to

$$
\begin{gathered}
\int\limits_x^b y\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]g''(y)dy= \\
y\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]g'(y)\Bigg|_{y=x}^{y=b}-\int\limits_x^b(\gamma x^{1-\gamma}y^{\gamma-1}-1)g'(y)dy= \\
b\left[\left(\frac{x}{b}\right)^{1-\gamma}-1\right]g'(b-0)-(\gamma x^{1-\gamma}y^{\gamma-1}-1)g(y)\Bigg|_{y=x}^{y=b}+ \\
\gamma(\gamma-1)x^{1-\gamma}\int\limits_x^b y^{\gamma-2}g(y)dy= \\
b\left[\left(\frac{x}{b}\right)^{1-\gamma}-1\right]g'(b-0)-(\gamma x^{1-\gamma}b^{\gamma-1}-1)g(b-0)+ \\
(\gamma-1)g(x)+\gamma(\gamma-1)x^{1-\gamma}\int\limits_x^b y^{\gamma-2}g(y)dy.
\end{gathered}
$$

Integrating in parts the second term in (9.7) we get

$$
\begin{gathered}
\int\limits_x^b\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]g'(y)dy=\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]g(y)\Bigg|_{y=x}^{y=b}-\frac{\gamma-1}{x^{\gamma-1}}\int\limits_x^b y^{\gamma-2}g(y)dy= \\
\left[\left(\frac{x}{b}\right)^{1-\gamma}-1\right]g(b-0)-(\gamma-1)x^{1-\gamma}\int\limits_x^b y^{\gamma-2}g(y)dy.
\end{gathered}
$$

Then

$$
(B^{-1}_{\gamma,b-}B_\gamma g)(x)=
$$

$$\frac{b}{\gamma-1}\left[\left(\frac{x}{b}\right)^{1-\gamma}-1\right]g'(b-0)-\frac{1}{\gamma-1}(\gamma x^{1-\gamma}b^{\gamma-1}-1)g(b-0)+g(x)+$$

$$\gamma x^{1-\gamma}\int\limits_x^b y^{\gamma-2}g(y)dy+$$

$$\frac{\gamma}{\gamma-1}\left[\left(\frac{x}{b}\right)^{1-\gamma}-1\right]g(b-0)-\gamma x^{1-\gamma}\int\limits_x^b y^{\gamma-2}g(y)dy=$$

$$g(x)+\frac{b}{\gamma-1}\left[\left(\frac{x}{b}\right)^{1-\gamma}-1\right]g'(b-0)-g(b-0).$$

From the last equality it is obvious that in order to have $(B^{-1}_{\gamma,b-}B_\gamma g)(x)=g(x)$ it is necessary that $g(b-0)=\lim\limits_{y\to b-0}g(y)=0$, $g'(b-0)=\lim\limits_{y\to b-0}g'(y)=0$.

Similarly, it shows that $(B^{-1}_{\gamma,a+}B_\gamma g)(x)=g(x)$ is true when $\lim\limits_{x\to a+0}g(x)$=0 and $\lim\limits_{x\to a+0}g'(x)$=0. □

9.1.3 Fractional Bessel integrals and derivatives on a segment of elementary and special functions

Statement 21. *For $\mu>-1$ the following formulas hold:*

$$B^{-\alpha}_{\gamma,b-}(b^2-x^2)^\mu=$$
$$\frac{x^{-2\alpha}b^{4\alpha+2\mu}}{2^{2\alpha}\Gamma(2\alpha)}\left(1-\frac{x^2}{b^2}\right)^{2\alpha+\mu}\left(2-\frac{x^2}{b^2}\right)^{\alpha}\times$$
$${}_2F_1\left(\alpha,\alpha+\frac{\gamma-1}{2}+\mu+1;2\alpha+\mu+1;1-\frac{x^2}{b^2}\right)$$

and

$$B^{-\alpha}_{\gamma,a+}(x^2-a^2)^\mu=$$
$$\frac{x^{2\alpha+2\mu}\Gamma(\mu+1)}{2^{2\alpha}\Gamma(2\alpha+\mu+1)}\left(1-\frac{a^2}{x^2}\right)^{2\alpha+\mu}\left(\frac{x}{a}\right)^{1-\gamma}\times$$
$$F_3\left(\frac{1-\gamma}{2},\alpha+\frac{\gamma-1}{2},\mu+1,\alpha,2\alpha+\mu+1;1-\frac{x^2}{a^2};1-\frac{a^2}{x^2}\right).$$

Proof. Let $\mu>-1$. Find the fractional Bessel integral $B^{-\alpha}_{\gamma,b-}$ from $(b^2-x^2)^\mu$:

$$B^{-\alpha}_{\gamma,b-}(b^2-x^2)^\mu=$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_x^b (b^2-y^2)^\mu \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)dy.$$

Replacing variable y by the formula $1-\frac{x^2}{y^2}=t$, we get $y=x(1-t)^{-\frac{1}{2}}$, $dy=\frac{1}{2}x(1-t)^{-\frac{3}{2}}dt$, $y=b$, $t=1-\frac{x^2}{b^2}$, $y=x$, $t=0$, and

$$B^{-\alpha}_{\gamma,b-}(b^2-x^2)^\mu =$$

$$\frac{x^{2\alpha}b^{2\mu}}{2^{2\alpha}\Gamma(2\alpha)}\int\limits_0^{1-\frac{x^2}{b^2}} t^{2\alpha-1}(1-t)^{-\alpha-1-\mu}\left(\left(1-\frac{x^2}{b^2}\right)-t\right)^\mu {}_2F_1\left(\alpha,\alpha+\frac{\gamma-1}{2};2\alpha;t\right)dt.$$

Using formula (2.21.1.21) from [457] of the form

$$\int\limits_0^y x^{c-1}(1-\omega x)^{a-c-\beta}(y-x)^{\beta-1}\,{}_2F_1(a,b;c;\omega x)dx =$$

$$y^{c+\beta-1}(1+\omega y)^\alpha(1-\omega y)^{-c}B(c,\beta)\,{}_2F_1(a,b+\beta;c+\beta;\omega y), \tag{9.8}$$

$$y,\operatorname{Re}c,\operatorname{Re}\beta>0,\ |\arg(1-\omega y)|<\pi,$$

we obtain

$$a=\alpha,\ c=2\alpha,\ \beta=\mu+1,\ a-c-\beta=\alpha-2\alpha-\mu-1=-\alpha-\mu-1$$

and

$$B^{-\alpha}_{\gamma,b-}(b^2-x^2)^\mu =$$

$$\frac{x^{2\alpha}b^{2\mu}}{2^{2\alpha}\Gamma(2\alpha)}\left(1-\frac{x^2}{b^2}\right)^{2\alpha+\mu}\left(2-\frac{x^2}{b^2}\right)^\alpha\left(\frac{x^2}{b^2}\right)^{-2\alpha}\times$$

$${}_2F_1\left(\alpha,\alpha+\frac{\gamma-1}{2}+\mu+1;2\alpha+\mu+1;1-\frac{x^2}{b^2}\right)$$

$$\frac{x^{-2\alpha}b^{4\alpha+2\mu}}{2^{2\alpha}\Gamma(2\alpha)}\left(1-\frac{x^2}{b^2}\right)^{2\alpha+\mu}\left(2-\frac{x^2}{b^2}\right)^\alpha\times$$

$${}_2F_1\left(\alpha,\alpha+\frac{\gamma-1}{2}+\mu+1;2\alpha+\mu+1;1-\frac{x^2}{b^2}\right).$$

Now let us find the fractional Bessel integral $B^{-\alpha}_{\gamma,a+}$ from $(x^2-a^2)^\mu$, $\mu>-1$:

$$B^{-\alpha}_{\gamma,a+}(x^2-a^2)^\mu =$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_a^x \left(\frac{y}{x}\right)^\gamma\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)(y^2-a^2)^\mu dy$$

Replacing the variable y by the formula $1-\frac{y^2}{x^2}=t$, we obtain $y=x(1-t)^{\frac{1}{2}}$, $dy=-\frac{1}{2}x(1-t)^{-\frac{1}{2}}dt$, $y=a$, $t=1-\frac{a^2}{x^2}$, $y=x$, $t=0$, and

$$B^{-\alpha}_{\gamma,a+}(x^2-a^2)^{\mu}=$$
$$\frac{x^{2\alpha+2\mu}}{2^{2\alpha}\Gamma(2\alpha)}\int\limits_0^{1-\frac{a^2}{x^2}} t^{2\alpha-1}(1-t)^{\frac{\gamma-1}{2}}\left(\left(1-\frac{a^2}{x^2}\right)-t\right)^{\mu}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;t\right)dt.$$

Using formula (2.21.1.20) from [457] of the form

$$\int\limits_0^y x^{c-1}(1-zx)^{-\rho}(y-x)^{\beta-1}{}_2F_1(a,b;c;\omega x)dx=$$
$$B(c,\beta)\frac{y^{c+\beta-1}}{(1-yz)^{\rho}}F_3\left(\rho,a,\beta,b,c+\beta;\frac{yz}{yz-1};\omega y\right),$$
$$y,\operatorname{Re}c,\operatorname{Re}\beta>0,\ |\arg(1-\omega y)|,|\arg(1-z)|<\pi,$$

we get

$$B^{-\alpha}_{\gamma,a+}(x^2-a^2)^{\mu}=$$
$$\frac{x^{2\alpha+2\mu}\Gamma(\mu+1)}{2^{2\alpha}\Gamma(2\alpha+\mu+1)}\left(1-\frac{a^2}{x^2}\right)^{2\alpha+\mu}\left(\frac{x}{a}\right)^{1-\gamma}\times$$
$$F_3\left(\frac{1-\gamma}{2},\alpha+\frac{\gamma-1}{2},\mu+1,\alpha,2\alpha+\mu+1;1-\frac{x^2}{a^2};1-\frac{a^2}{x^2}\right).\qquad\square$$

9.1.4 Fractional Bessel derivatives on a segment as inverse to integrals

Theorem 113. *If $f(x)\in L_1(a,b)$. Then $DB^{\alpha}_{\gamma,b-}$ is a left inverse operator to $IB^{\alpha}_{\gamma,b-}$ and $DB^{\alpha}_{\gamma,a+}$ is a left inverse operator to $IB^{\alpha}_{\gamma,a+}$:*

$$(DB^{\alpha}_{\gamma,b-}(IB^{\alpha}_{\gamma,b-}f)(y))(x)=f(x),$$
$$(DB^{\alpha}_{\gamma,a+}(IB^{\alpha}_{\gamma,a+}f)(y))(x)=f(x).$$

Proof. Let us find first $(IB^{n-\alpha}_{\gamma,b-}(IB^{\alpha}_{\gamma,b-}f)(y))(x)$. We obtain

$$(IB^{n-\alpha}_{\gamma,b-}(IB^{\alpha}_{\gamma,b-}f)(y))(x)=$$
$$\frac{1}{\Gamma(2\alpha)}\left(IB^{n-\alpha}_{\gamma,b-}\int\limits_y^b\left(\frac{t^2-y^2}{2t}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{t^2}\right)f(t)dt\right)(x)=$$

$$
\begin{aligned}
&\frac{1}{\Gamma(2\alpha)\Gamma(2n-2\alpha)}\times\\
&\int\limits_x^b \left(\frac{y^2-x^2}{2y}\right)^{2n-2\alpha-1} {}_2F_1\left(n-\alpha+\frac{\gamma-1}{2}, n-\alpha; 2n-2\alpha; 1-\frac{x^2}{y^2}\right)dy\times\\
&\int\limits_y^b \left(\frac{t^2-y^2}{2t}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{t^2}\right)f(t)dt=\\
&\frac{1}{\Gamma(2\alpha)\Gamma(2n-2\alpha)}\int\limits_x^b f(t)dt\int\limits_x^t \left(\frac{y^2-x^2}{2y}\right)^{2n-2\alpha-1}\left(\frac{t^2-y^2}{2t}\right)^{2\alpha-1}\times\\
&{}_2F_1\left(n-\alpha+\frac{\gamma-1}{2}, n-\alpha; 2n-2\alpha; 1-\frac{x^2}{y^2}\right){}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{t^2}\right)dy=\\
&\frac{1}{2^{2n-2}\Gamma(2\alpha)\Gamma(2n-2\alpha)}\int\limits_x^b t^{1-2\alpha}f(t)dt\int\limits_x^t y^{2\alpha-2n+1}(y^2-x^2)^{2n-2\alpha-1}(t^2-y^2)^{2\alpha-1}\times\\
&{}_2F_1\left(n-\alpha+\frac{\gamma-1}{2}, n-\alpha; 2n-2\alpha; 1-\frac{x^2}{y^2}\right){}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{t^2}\right)dy.
\end{aligned}
\tag{9.9}
$$

Let us denote the internal integral in (9.9) by I. Replacing variables by the formulas $y^2=\eta, x^2=\xi, t^2=\tau$ in the internal integral in (9.9) we get

$$
\begin{aligned}
&I=\frac{1}{2}\int\limits_\xi^\tau \eta^{\alpha-n}(\eta-\xi)^{2n-2\alpha-1}(\tau-\eta)^{2\alpha-1}\times\\
&{}_2F_1\left(n-\alpha+\frac{\gamma-1}{2}, n-\alpha; 2n-2\alpha; 1-\frac{\xi}{\eta}\right){}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{\eta}{\tau}\right)d\eta.
\end{aligned}
$$

Now, introducing a new variable substitution $\eta=\tau-w(\tau-\xi)$, we can write

$$
\begin{aligned}
&\eta=\tau-w(\tau-\xi), \quad \eta=\tau\Rightarrow w=0, \quad \eta=\xi\Rightarrow w=1, \quad d\eta=-(\tau-\xi)dw,\\
&\eta^{\alpha-n}=(\tau-w(\tau-\xi))^{\alpha-n}=\tau^{\alpha-n}\left(1-\left(1-\frac{\xi}{\tau}\right)w\right)^{\alpha-n},\\
&(\eta-\xi)^{2n-2\alpha-1}=(\tau-\xi)^{2n-2\alpha-1}(1-w)^{2n-2\alpha-1},\\
&(\tau-\eta)^{2\alpha-1}=(\tau-\xi)^{2\alpha-1}w^{2\alpha-1},\\
&1-\frac{\xi}{\eta}=\frac{\left(1-\frac{\xi}{\tau}\right)(1-w)}{1-\left(1-\frac{\xi}{\tau}\right)w}, \quad 1-\frac{\eta}{\tau}=\left(1-\frac{\xi}{\tau}\right)w,
\end{aligned}
$$

and

$$I=\frac{1}{2}\tau^{\alpha-n}(\tau-\xi)^{2n-1}\int_0^1 w^{2\alpha-1}(1-w)^{2n-2\alpha-1}\left(1-\left(1-\frac{\xi}{\tau}\right)w\right)^{\alpha-n}\times$$
$$ {}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;\left(1-\frac{\xi}{\tau}\right)w\right)\times$$
$$ {}_2F_1\left(n-\alpha+\frac{\gamma-1}{2},n-\alpha;2n-2\alpha;\frac{\left(1-\frac{\xi}{\tau}\right)(1-w)}{1-\left(1-\frac{\xi}{\tau}\right)w}\right)dw.$$

For the product of two Gauss hypergeometric functions (see [19]), we have

$$ {}_2F_1(a,b;c;x)\,{}_2F_1(a',b';c';y)=\sum_{m,k=0}^{\infty}\frac{(a)_m(a')_k(b)_m(b')_k}{(c)_m(c')_k}\frac{x^m}{m!}\frac{y^k}{k!}.$$

Therefore,

$$I=\frac{1}{2}\sum_{m,k=0}^{\infty}\tau^{\alpha-n-m-k}(\tau-\xi)^{2n-1+k+m}\times$$
$$\frac{(\alpha+\frac{\gamma-1}{2})_m(n-\alpha+\frac{\gamma-1}{2})_k(\alpha)_m(n-\alpha)_k}{(2\alpha)_m(2n-2\alpha)_k}\frac{1}{m!}\frac{1}{k!}\times$$
$$\int_0^1 w^{2\alpha-1+m}(1-w)^{2n-2\alpha+k-1}\left(1-\left(1-\frac{\xi}{\tau}\right)w\right)^{\alpha-n-k}dw=$$
$$\frac{1}{2}\sum_{m,k=0}^{\infty}\tau^{\alpha-n-m-k}(\tau-\xi)^{2n-1+k+m}\times$$
$$\frac{(\alpha+\frac{\gamma-1}{2})_m(n-\alpha+\frac{\gamma-1}{2})_k(\alpha)_m(n-\alpha)_k}{(2\alpha)_m(2n-2\alpha)_k}\frac{1}{m!}\frac{1}{k!}\times$$
$$\frac{\Gamma(2\alpha+m)\Gamma(2n+k-2\alpha)}{\Gamma(2n+k+m)}\,{}_2F_1\left(n+k-\alpha,2\alpha+m;2n+k+m;1-\frac{\xi}{\tau}\right)=$$
$$\frac{1}{2}\sum_{m,k=0}^{\infty}\tau^{n+\alpha-1}\left(1-\frac{\xi}{\tau}\right)^{2n-1+k+m}\times$$
$$\frac{(\alpha+\frac{\gamma-1}{2})_m(n-\alpha+\frac{\gamma-1}{2})_k(\alpha)_m(n-\alpha)_k}{(2\alpha)_m(2n-2\alpha)_k}\frac{1}{m!}\frac{1}{k!}\times$$
$$\frac{\Gamma(2\alpha+m)\Gamma(2n+k-2\alpha)}{\Gamma(2n+k+m)}\,{}_2F_1\left(n+k-\alpha,2\alpha+m;2n+k+m;1-\frac{\xi}{\tau}\right).$$

Consider the expression

$$S=\sum_{k=0}^{\infty}\frac{(n-\alpha+\frac{\gamma-1}{2})_k(n-\alpha)_k\Gamma(2n+k-2\alpha)}{(2n-2\alpha)_k\Gamma(2n+k+m)}\frac{1}{k!}\left(1-\frac{\xi}{\tau}\right)^{2n-1+k+m}\times$$
$${}_2F_1\left(n+k-\alpha,2\alpha+m;2n+k+m;1-\frac{\xi}{\tau}\right).$$

Noting that

$$\frac{\Gamma(2n+k-2\alpha)}{(2n-2\alpha)_k}=\frac{\Gamma(2n+k-2\alpha)\Gamma(2n-2\alpha)}{\Gamma(2n-2\alpha+k)}=\Gamma(2n-2\alpha),$$
$$\Gamma(2n+k+m)=(2n+m)_k\Gamma(2n+m),$$

we obtain

$$S=\frac{\Gamma(2n-2\alpha)}{\Gamma(2n+m)}\left(1-\frac{\xi}{\tau}\right)^{2n-1+m}\sum_{k=0}^{\infty}\frac{(n-\alpha)_k(n-\alpha+\frac{\gamma-1}{2})_k}{(2n+m)_k}\frac{1}{k!}\left(1-\frac{\xi}{\tau}\right)^{k}\times$$
$${}_2F_1\left(n-\alpha+k,2\alpha+m;2n+m+k;1-\frac{\xi}{\tau}\right).$$

Using formula (6.7.1.7) from [457] of the form

$$\sum_{k=0}\frac{(a)_k(b')_k}{k!(c)_k}x^k\,{}_2F_1(a+k,b;c+k;x)={}_2F_1(a,b+b';c;x),$$

we obtain

$$S=\frac{\Gamma(2n-2\alpha)}{\Gamma(2n+m)}\left(1-\frac{\xi}{\tau}\right)^{2n-1+m}\times$$
$${}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m;1-\frac{\xi}{\tau}\right).$$

Returning to variables x and t by the formulas $\xi=x^2$, $\tau=t^2$, we can write

$$I=\frac{\Gamma(2n-2\alpha)}{2}\tau^{n+\alpha-1}\sum_{m=0}^{\infty}\frac{\Gamma(2\alpha+m)(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{(2\alpha)_m\Gamma(2n+m)}\frac{1}{m!}\times$$
$$\left(1-\frac{\xi}{\tau}\right)^{2n-1+m}{}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m;1-\frac{\xi}{\tau}\right)=$$
$$\frac{\Gamma(2n-2\alpha)}{2}t^{2(n+\alpha-1)}\sum_{m=0}^{\infty}\frac{\Gamma(2\alpha+m)(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{(2\alpha)_m\Gamma(2n+m)}\frac{1}{m!}\times$$
$$\left(1-\frac{x^2}{t^2}\right)^{2n-1+m}{}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m;1-\frac{x^2}{t^2}\right).$$

Taking into account the form of I, let us write

$$(IB^{n-\alpha}_{\gamma,b-}(IB^{\alpha}_{\gamma,b-}f)(y))(x)=$$

$$\frac{1}{2^{2n-2}\Gamma(2\alpha)\Gamma(2n-2\alpha)}\frac{\Gamma(2n-2\alpha)}{2}\sum_{m=0}^{\infty}\frac{\Gamma(2\alpha+m)(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{(2\alpha)_m\Gamma(2n+m)}\frac{1}{m!}\times$$

$$\int\limits_x^b t^{1-2\alpha}t^{2(n+\alpha-1)}\left(1-\frac{x^2}{t^2}\right)^{2n-1+m}\times$$

$${}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m;1-\frac{x^2}{t^2}\right)f(t)dt=$$

$$\frac{1}{2^{2n-1}\Gamma(2\alpha)}\sum_{m=0}^{\infty}\frac{\Gamma(2\alpha+m)(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{(2\alpha)_m\Gamma(2n+m)}\frac{1}{m!}\int\limits_x^b t^{2n-1}\left(1-\frac{x^2}{t^2}\right)^{2n-1+m}\times$$

$${}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m;1-\frac{x^2}{t^2}\right)f(t)dt.$$

Since

$$\frac{\Gamma(2\alpha+m)}{(2\alpha)_m}=\Gamma(2\alpha),$$

we have

$$(IB^{n-\alpha}_{\gamma,b-}(IB^{\alpha}_{\gamma,b-}f)(y))(x)=\frac{1}{2^{2n-1}}\sum_{m=0}^{\infty}\frac{(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{\Gamma(2n+m)}\frac{1}{m!}\int\limits_x^b\left(1-\frac{x^2}{t^2}\right)^{2n-1+m}\times$$

$$t^{2n-1}\,{}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m;1-\frac{x^2}{t^2}\right)f(t)dt.$$

Now we show that

$$(B^n_{\gamma})_x(IB^{n-\alpha}_{\gamma,b-}(IB^{\alpha}_{\gamma,b-}f)(y))(x)=f(x).$$

Let

$$M_n(x,t)=\left(1-\frac{x^2}{t^2}\right)^{2n-1+m}\,{}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m;1-\frac{x^2}{t^2}\right).$$

It is obvious that

$$M_n(x,x)=0,\qquad n=1,2,\dots.$$

Let us find $(B_\gamma)_x M_n$ applying formula (15.2.4) from [2] of the form

$$\frac{d}{dz}[z^{c-1}\,{}_2F_1(a,b;c;z)]=(c-1)z^{c-2}\,{}_2F_1(a,b;c-1;z).$$

We obtain

$$(B_\gamma)_x M_n = \frac{1}{x^\gamma}\frac{\partial}{\partial x}x^\gamma\frac{\partial}{\partial x}\left[\left(1-\frac{x^2}{t^2}\right)^{2n-1+m}\right.$$
$$\left.{}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m;1-\frac{x^2}{t^2}\right)\right]=$$
$$-(2n+m-1)\frac{2}{t^2}\frac{1}{x^\gamma}\frac{\partial}{\partial x}x^{\gamma+1}\left(1-\frac{x^2}{t^2}\right)^{2n+m-2}\times$$
$${}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m-1;1-\frac{x^2}{t^2}\right)=$$
$$-(2n+m-1)\frac{2t^{\gamma+1}}{t^2}\frac{1}{x^\gamma}\frac{\partial}{\partial x}\left(\frac{x^2}{t^2}\right)^{\frac{\gamma+1}{2}}\left(1-\frac{x^2}{t^2}\right)^{2n+m-2}\times$$
$${}_2F_1\left(n-\alpha,\alpha+m+n+\frac{\gamma-1}{2};2n+m-1;1-\frac{x^2}{t^2}\right).$$

Using formula (15.2.9) from [2] of the form

$$\frac{d}{dz}[z^{c-1}(1-z)^{a+b-c}\,{}_2F_1(a,b;c;z)]=$$
$$(c-1)\,z^{c-2}(1-z)^{a+b-c-1}\,{}_2F_1(a-1,b-1;c-1;z),$$

we can find

$$(B_\gamma)_x M_n =$$
$$(2n+m-1)(2n+m-2)\frac{2t^{\gamma+1}}{t^2}\frac{1}{x^\gamma}\frac{2x}{t^2}\left(1-\frac{x^2}{t^2}\right)^{2n+m-3}\left(\frac{x^2}{t^2}\right)^{\frac{\gamma-1}{2}}\times$$
$${}_2F_1\left(n-1-\alpha,\alpha+m+n-1+\frac{\gamma-1}{2};2n+m-2;1-\frac{x^2}{t^2}\right)=$$
$$(2n+m-1)(2n+m-2)\left(\frac{2}{t}\right)^2\left(1-\frac{x^2}{t^2}\right)^{2n+m-3}\times$$
$${}_2F_1\left(n-1-\alpha,\alpha+m+n-1+\frac{\gamma-1}{2};2n+m-2;1-\frac{x^2}{t^2}\right)=$$
$$(2n+m-1)(2n+m-2)\left(\frac{2}{t}\right)^2 M_{n-1}.$$

That gives

$$(B_\gamma)_x M_n = (2n+m-1)(2n+m-2)\left(\frac{2}{t}\right)^2 M_{n-1}. \tag{9.10}$$

Applying (9.10) $(n-1)$ times, we obtain

$$(B_\gamma)_x^{n-1} M_n = (2n+m-1)(2n+m-2)...(2+m)\left(\frac{2}{t}\right)^{2(n-1)} M_1 =$$

$$(2n+m-1)(2n+m-2)...(2+m)\left(\frac{2}{t}\right)^{2(n-1)} \times$$

$$\left(1-\frac{x^2}{t^2}\right)^{1+m} {}_2F_1\left(1-\alpha, \alpha+m+\frac{\gamma+1}{2}; 2+m; 1-\frac{x^2}{t^2}\right) =$$

$$\frac{\Gamma(2n+m)}{\Gamma(m+2)}\left(\frac{2}{t}\right)^{2(n-1)} \times$$

$$\left(1-\frac{x^2}{t^2}\right)^{1+m} {}_2F_1\left(1-\alpha, \alpha+m+\frac{\gamma+1}{2}; 2+m; 1-\frac{x^2}{t^2}\right).$$

So

$$(B_\gamma^n)_x (IB_{\gamma,b-}^{n-\alpha}(IB_{\gamma,b-}^{\alpha} f)(y))(x) = \frac{1}{2^{2n-1}} \sum_{m=0}^{\infty} \frac{(\alpha+\frac{\gamma-1}{2})_m (\alpha)_m}{\Gamma(2n+m)} \frac{1}{m!} \frac{\Gamma(2n+m)}{\Gamma(m+2)} \times$$

$$(B_\gamma)_x \int\limits_x^b \left(\frac{2}{t}\right)^{2(n-1)} t^{2n-1}\left(1-\frac{x^2}{t^2}\right)^{1+m} \times$$

$${}_2F_1\left(1-\alpha, \alpha+m+\frac{\gamma+1}{2}; 2+m; 1-\frac{x^2}{t^2}\right) f(t)dt =$$

$$\frac{1}{2}\sum_{m=0}^{\infty} \frac{(\alpha+\frac{\gamma-1}{2})_m (\alpha)_m}{\Gamma(2+m)} \frac{1}{m!} \times$$

$$(B_\gamma)_x \int\limits_x^b t\left(1-\frac{x^2}{t^2}\right)^{1+m} {}_2F_1\left(1-\alpha, \alpha+m+\frac{\gamma+1}{2}; 2+m; 1-\frac{x^2}{t^2}\right) f(t)dt.$$

Applying formula (15.3.3) from [2] of the form

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z),$$

we obtain

$${}_2F_1\left(1-\alpha, \alpha+m+\frac{\gamma+1}{2}; 2+m; 1-\frac{x^2}{t^2}\right) =$$

$$\left(\frac{x^2}{t^2}\right)^{\frac{1-\gamma}{2}} {}_2F_1\left(\alpha+m+1, 1-\alpha+\frac{1-\gamma}{2}; 2+m; 1-\frac{x^2}{t^2}\right).$$

Now

$$\frac{\partial}{\partial x}\left(1-\frac{x^2}{t^2}\right)^{m+1}{}_2F_1\left(1-\alpha,\alpha+m+\frac{\gamma+1}{2};2+m;1-\frac{x^2}{t^2}\right)=$$
$$\frac{\partial}{\partial x}\left(\frac{x^2}{t^2}\right)^{\frac{1-\gamma}{2}}\left(1-\frac{x^2}{t^2}\right)^{m+1}{}_2F_1\left(\alpha+m+1,1-\alpha+\frac{1-\gamma}{2};2+m;1-\frac{x^2}{t^2}\right).$$

Using formula (15.2.9) from [2] of the form

$$\frac{d}{dz}[z^{c-1}(1-z)^{a+b-c}\,{}_2F_1(a,b;c;z)]=$$
$$(c-1)\,z^{c-2}(1-z)^{a+b-c-1}\,{}_2F_1(a-1,b-1;c-1;z),$$

we obtain

$$\frac{\partial}{\partial x}\left(1-\frac{x^2}{t^2}\right)^{m+1}{}_2F_1\left(1-\alpha,\alpha+m+\frac{\gamma+1}{2};2+m;1-\frac{x^2}{t^2}\right)=$$
$$\frac{\partial}{\partial x}\left(\frac{x^2}{t^2}\right)^{\frac{1-\gamma}{2}}\left(1-\frac{x^2}{t^2}\right)^{m+1}\times$$
$${}_2F_1\left(\alpha+m+1,1-\alpha+\frac{1-\gamma}{2};2+m;1-\frac{x^2}{t^2}\right)=$$
$$(m+1)\left(-\frac{2x}{t^2}\right)\left(\frac{x^2}{t^2}\right)^{\frac{1-\gamma}{2}-1}\left(1-\frac{x^2}{t^2}\right)^{m}\times$$
$${}_2F_1\left(\alpha+m,-\alpha+\frac{1-\gamma}{2};m+1;1-\frac{x^2}{t^2}\right).$$

Returning to the series, we write

$$(B_\gamma^n)_x(IB_{\gamma,b-}^{n-\alpha}(IB_{\gamma,b-}^{\alpha}f)(y))(x)=$$
$$\frac{1}{2}\sum_{m=0}^{\infty}\frac{(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{\Gamma(2+m)}\frac{1}{m!}\times$$
$$(B_\gamma)_x\int\limits_x^b t\left(1-\frac{x^2}{t^2}\right)^{1+m}{}_2F_1\left(1-\alpha,\alpha+m+\frac{\gamma+1}{2};2+m;1-\frac{x^2}{t^2}\right)f(t)dt=$$
$$\frac{1}{2}\sum_{m=0}^{\infty}\frac{(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{\Gamma(2+m)}\frac{1}{m!}\frac{1}{x^\gamma}\frac{\partial}{\partial x}x^\gamma\frac{\partial}{\partial x}\int\limits_x^b t\left(1-\frac{x^2}{t^2}\right)^{1+m}\times$$
$${}_2F_1\left(1-\alpha,\alpha+m+\frac{\gamma+1}{2};2+m;1-\frac{x^2}{t^2}\right)f(t)dt=$$

$$\frac{1}{2}\sum_{m=0}^{\infty}\frac{(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{(m+1)!}\frac{1}{m!}\times$$

$$\frac{1}{x^\gamma}\frac{\partial}{\partial x}x^\gamma\int\limits_x^b t\,(m+1)\left(-\frac{2x}{t^2}\right)\left(\frac{x^2}{t^2}\right)^{\frac{1-\gamma}{2}-1}\left(1-\frac{x^2}{t^2}\right)^m\times$$

$${}_2F_1\left(\alpha+m,-\alpha+\frac{1-\gamma}{2};m+1;1-\frac{x^2}{t^2}\right)f(t)dt=$$

$$-\sum_{m=0}^{\infty}\frac{(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{m!}\frac{1}{m!}\times$$

$$\frac{1}{x^\gamma}\frac{\partial}{\partial x}\int\limits_x^b t^\gamma\left(1-\frac{x^2}{t^2}\right)^m{}_2F_1\left(\alpha+m,-\alpha+\frac{1-\gamma}{2};m+1;1-\frac{x^2}{t^2}\right)f(t)dt.$$

Applying formula (6.7.1.7) from [457] of the form

$$\sum_{k=0}\frac{(a)_k(b')_k}{k!(c)_k}x^k{}_2F_1(a+k,b;c+k;x)={}_2F_1(a,b+b';c;x),$$

we sum up the row:

$$\sum_{m=0}^{\infty}\frac{(\alpha+\frac{\gamma-1}{2})_m(\alpha)_m}{m!}\frac{1}{m!}\left(1-\frac{x^2}{t^2}\right)^m{}_2F_1\left(\alpha+m,-\alpha+\frac{1-\gamma}{2};m+1;1-\frac{x^2}{t^2}\right)=$$

$${}_2F_1\left(\alpha,0;1;1-\frac{x^2}{t^2}\right)=1.$$

So

$$(B_\gamma^n)_x(IB_{\gamma,b-}^{n-\alpha}(IB_{\gamma,b-}^{\alpha}f)(y))(x)=-\frac{1}{x^\gamma}\frac{\partial}{\partial x}\int\limits_x^b t^\gamma f(t)dt=$$

$$\frac{1}{x^\gamma}x^\gamma f(x)=f(x).$$

It is similarly proved that $(DB_{\gamma,a+}^{\alpha}(IB_{\gamma,a+}^{\alpha}f)(y))(x)=f(x)$. □

9.2 Fractional Bessel integral and derivatives on a semiaxis

The fractional powers $(B_\gamma)^\alpha,\alpha\in\mathbb{R}$, in the case of a semiaxis were not studied in [555] as they require more delicate considerations and estimates when applied. But

they seem to be very important as in most applications boundary conditions for differential equations are given exactly at zero or infinity. So we introduce fractional Bessel integrals and derivatives for these special values.

9.2.1 Definitions

Definition 47. *Let* $\alpha > 0$. *The* ***right-sided fractional Bessel integral on a semiaxis*** $B_{\gamma,-}^{-\alpha}$ *for* $f(x)\in C^{[2\alpha]+1}(0,+\infty)$ *is defined by the formula*

$$(B_{\gamma,-}^{-\alpha} f)(x) = (IB_{\gamma,-}^{\alpha} f)(x) =$$
$$\frac{1}{\Gamma(2\alpha)}\int\limits_x^\infty \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right) f(y)dy. \tag{9.11}$$

The ***left-sided fractional Bessel integral on a semiaxis*** $B_{\gamma,0+}^{-\alpha}$ *for* $f(x)\in C^{[2\alpha]+1}[0,+\infty)$ *is defined by the formula*

$$(B_{\gamma,0+}^{-\alpha} f)(x) = (IB_{\gamma,0+}^{\alpha} f)(x) =$$
$$\frac{1}{\Gamma(2\alpha)}\int\limits_0^x \left(\frac{y}{x}\right)^{\gamma}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right) f(y)dy. \tag{9.12}$$

Using (9.6), we can write

$$(B_{\gamma,-}^{-\alpha} f)(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_x^\infty (y^2-x^2)^{\alpha-\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{\gamma}{2}} P_{\frac{\gamma}{2}-1}^{\frac{1}{2}-\alpha}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right] f(y)dy$$

and

$$(B_{\gamma,0+}^{-\alpha} f)(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^x (x^2-y^2)^{\alpha-\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{\gamma}{2}} P_{\frac{\gamma}{2}-1}^{\frac{1}{2}-\alpha}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right] f(y)dy.$$

The expression of the fractional Bessel integrals through the Legendre functions is useful and is a simplification of the original definition, since the Gauss hypergeometric function depends on three parameters, and the Legendre function depends only on two parameters.

Definition 48. *Let* $\alpha > 0$, $n = [\alpha]+1$, $f\in L_1(a,b)$, $IB_{\gamma,-}^{n-\alpha} f$, $IB_{\gamma,0+}^{n-\alpha} f\in C_{ev}^{2n}(0,\infty)$. *The* ***right-sided and left-sided fractional Bessel derivatives on a semiaxis of the Riemann–Liouville type*** *for* $\alpha\neq 0,1,1,...$ *are defined, respectively, by the equalities*

$$(B_{\gamma,-}^{\alpha} f)(x) = (DB_{\gamma,-}^{\alpha} f)(x) = B_\gamma^n (IB_{\gamma,-}^{n-\alpha} f)(x), \qquad n=[\alpha]+1, \tag{9.13}$$

and

$$(B^{\alpha}_{\gamma,0+}f)(x)=(DB^{\alpha}_{\gamma,0+}f)(x)=B^{n}_{\gamma}(IB^{n-\alpha}_{\gamma,0+}f)(x),\qquad n=[\alpha]+1. \tag{9.14}$$

When $\alpha=n\in\mathbb{N}\cup\{0\}$, *then*

$$(B^{0}_{\gamma,-}f)(x)=(B^{0}_{\gamma,0+}f)(x)=f(x),$$
$$(B^{n}_{\gamma,-}f)(x)=(B^{n}_{\gamma,0+}f)(x)=B^{n}_{\gamma}f(x),$$

where B^n_γ *is an iterated Bessel operator* (9.1).

In [367] spaces adapted to work with operators of the form $B^{\alpha}_{\gamma,0+}$ and $B^{\alpha}_{\gamma,-}$, $\alpha\in\mathbb{R}$, were introduced:

$$F_p=\left\{\varphi\in C^{\infty}(0,\infty):x^k\frac{d^k\varphi}{dx^k}\in L^p(0,\infty)\,\text{for}\,k=0,1,2,...\right\},\qquad 1\le p<\infty,$$
$$F_{\infty}=\left\{\varphi\in C^{\infty}(0,\infty):x^k\frac{d^k\varphi}{dx^k}\to 0\,\text{as}\,x\to 0+\ \text{and as}\,x\to\infty\,\text{for}\,k=0,1,2,...\right\},$$

and

$$F_{p,\mu}=\left\{\varphi:x^{-\mu}\varphi(x)\in F_p\right\},\qquad 1\le p\le\infty,\qquad \mu\in\mathbb{C}.$$

We present here two theorems that are special cases of theorems from [367].

Theorem 114. *Let* $\alpha\in\mathbb{R}$. *For all* p,μ, *and* $\gamma>0$ *such that* $\mu\neq\frac{1}{p}-2m$, $\gamma\neq\frac{1}{p}-\mu-2m+1$, $m=1,2...$, *the operator* $B^{\alpha}_{\gamma,0+}$ *is a continuous linear mapping from* F_p,μ *into* $F_{p,\mu-2\alpha}$. *If also* $2\alpha\neq\mu-\frac{1}{p}+2m$ *and* $\gamma-2\alpha\neq\frac{1}{p}-\mu-2m+1$, $m=1,2...$, *then* $B^{\alpha}_{\gamma,0+}$ *is a homeomorphism from* F_p,μ *onto* $F_{p,\mu-2\alpha}$ *with inverse* $B^{-\alpha}_{\gamma,0+}$.

Theorem 115. *Let* $\alpha\in\mathbb{R}$. *For all* p,μ, *and* $\gamma>0$ *such that* $\mu\neq\frac{1}{p}-2m+1$, $\gamma\neq\frac{1}{p}-\mu-2m$, $m=1,2...$, *the operator* $B^{\alpha}_{\gamma,-}$ *is a continuous linear mapping from* $F_{q,-\mu+2\alpha}$ *into* $F_{q,\mu}$, *where* $\frac{1}{q}=1-\frac{1}{p}$. *If also* $2\alpha\neq\mu-\frac{1}{p}+2m-1$ *and* $\gamma+2\alpha\neq\mu-\frac{1}{p}+2m$, $m=1,2...$, *then* $B^{\alpha}_{\gamma,-}$ *is a homeomorphism from* $F_{q,-\mu+2\alpha}$ *onto* $F_{q,-\mu}$ *with inverse* $B^{-\alpha}_{\gamma,-}$.

Definition 49. *Let* $n=[\alpha]+1$, $f\in L[0,\infty)$, $IB^{n-\alpha}_{\gamma,-}f$, $IB^{n-\alpha}_{\gamma,0+}f\in C^{2n}_{ev}(0,\infty)$.

The ***right-sided fractional Bessel derivatives on a semiaxis of Gerasimov–Caputo type*** *for* $\alpha>0$, $\alpha\neq 0,1,2,...$ *is defined by the equality*

$$(\mathcal{B}^{\alpha}_{\gamma,-}f)(x)=(IB^{n-\alpha}_{\gamma,-}B^{n}_{\gamma}f)(x).$$

The ***left-sided fractional Bessel derivatives on a semiaxis of Gerasimov–Caputo type*** *for* $\alpha>0$, $\alpha\neq 0,1,2,...$ *is defined by the equality*

$$(\mathcal{B}^{\alpha}_{\gamma,0+}f)(x)=(IB^{n-\alpha}_{\gamma,0+}B^{n}_{\gamma}f)(x).$$

Here $IB^{n-\alpha}_{\gamma,-}$ *is the right-sided fractional Bessel integral* (9.11) *on a semiaxis, and* $IB^{n-\alpha}_{\gamma,0+}$ *is the left-sided fractional Bessel integral* (9.12) *on a semiaxis. When* $\alpha = n \in \mathbb{N} \cup \{0\}$*, then*

$$(\mathcal{B}^0_{\gamma,-} f)(x) = (\mathcal{B}^0_{\gamma,0+} f)(x) = f(x),$$
$$(\mathcal{B}^n_{\gamma,-} f)(x) = (\mathcal{B}^n_{\gamma,0+} f)(x) = B^n_\gamma f(x),$$

where B^n_γ *is an iterated Bessel operator* (9.1)*.*

9.2.2 Basic properties of fractional Bessel integrals on a semiaxis

Lemma 32. *For* $\gamma = 0$*,* $f \in L_1(0,\infty)$ *the following formulas are valid:*

$$(B^{-\alpha}_{0,-} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_x^\infty (y-x)^{2\alpha-1} f(y)dy = (I^{2\alpha}_{-} f)(x)$$

and

$$(B^{-\alpha}_{0,0+} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_0^x (x-y)^{2\alpha-1} f(y)dy = (I^{2\alpha}_{0+} f)(x),$$

where $I^{2\alpha}_{-}$ *and* $I^{2\alpha}_{0+}$ *are right-sided* (2.25) *and left-sided* (2.26) *fractional Bessel integrals on a semiaxis, respectively.*

Proof. Indeed, we have

$$(B^{-\alpha}_{0,-} f)(x) =$$
$$\frac{1}{\Gamma(2\alpha)} \int\limits_x^\infty \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha - \frac{1}{2}, \alpha; 2\alpha; 1 - \frac{x^2}{y^2}\right) f(y)dy$$

and

$$(B^{-\alpha}_{0,0+} f)(x) =$$
$$\frac{1}{\Gamma(2\alpha)} \int\limits_0^x \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha - \frac{1}{2}, \alpha; 2\alpha; 1 - \frac{y^2}{x^2}\right) f(y)dy.$$

Using the formula that is obtained from the integral representation of the Gauss hypergeometric function (1.34)

$${}_2F_1\left(\alpha - \frac{1}{2}, \alpha; 2\alpha; 1 - \frac{x^2}{y^2}\right) = \left[\frac{2y}{x+y}\right]^{2\alpha-1},$$

we obtain provable formulas. □

Let us now consider the connection of fractional Bessel integrals $B_{\gamma,-}^{-\alpha}$ and $B_{\gamma,0+}^{-\alpha}$ with corresponding fractional Saigo integrals (2.41) and (2.42).

Lemma 33. *Let $f\in L_1(0,\infty)$. The following equalities hold:*

$$(B_{\gamma,-}^{-\alpha}f)(x)=\frac{1}{2^{2\alpha}}J_{x^2}^{2\alpha,\frac{\gamma-1}{2}-\alpha,-\alpha}\left(x^{\frac{\gamma-1}{2}}f(\sqrt{x})\right), \tag{9.15}$$

$$(B_{\gamma,0+}^{-\alpha}f)(x)=\frac{x^{\frac{1-\gamma}{2}-\alpha}}{2^{2\alpha}}I_{x^2}^{2\alpha,\frac{\gamma-1}{2}-\alpha,-\alpha}\left(x^{\frac{\gamma-1}{2}}f(\sqrt{x})\right), \tag{9.16}$$

where $J_x{}^{\gamma,\beta,\eta}$ is the fractional Saigo integral (2.41) and $I_x^{\gamma,\beta,\eta}$ is the fractional Saigo integral (2.42). Here $\gamma>0$, β,θ are real numbers.

Proof. Replacing the variable $y^2=t$ in the Bessel fractional integral on the semiaxis (9.11), we get

$$(B_{\gamma,-}^{-\alpha}f)(x)=$$
$$\frac{1}{2^{2\alpha}\Gamma(2\alpha)}\int\limits_{x^2}^{\infty}(t-x^2)^{2\alpha-1}t^{-\alpha}\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{t}\right)f(\sqrt{t})dt.$$

Comparing the resulting expression with (2.41), we obtain

$$\gamma=2\alpha,\qquad \beta=\frac{\gamma-1}{2}-\alpha,\qquad -\gamma-\beta=-\alpha-\frac{\gamma-1}{2},\qquad \eta=-\alpha,$$

Which gives (9.15). Similarly, we obtain (9.16). □

We consider now the case when $\alpha=1$.

Lemma 34. *The following equalities hold:*

$$(B_{\gamma,-}^{-1}f)(x)=\frac{1}{\gamma-1}\int\limits_x^{\infty}y\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]f(y)dy,\qquad f(x)\in L_1(0,\infty),$$

$$(B_{\gamma,0+}^{-1}f)(x)=\frac{1}{\gamma-1}\int\limits_0^{x}y\left[1-\left(\frac{y}{x}\right)^{\gamma-1}\right]f(y)dy,\qquad f(x)\in L_1(0,\infty).$$

Proof. Applying the formula

$${}_2F_1\left(\frac{\gamma+1}{2},1;2;1-\frac{x^2}{y^2}\right)=\frac{2}{1-\gamma}\,\frac{y^2}{x^2-y^2}\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right],$$

which is valid for the Gauss hypergeometric function, we obtain provable statements. □

Lemma 35. *Let* $g \in C^2_{ev}(0,\infty)$, $f(x) = B_\gamma g(x)$, $f(x) \in L_1(0,\infty)$. *When*

$$\lim_{x\to+\infty} g(x) = 0, \qquad \lim_{x\to+\infty} g'(x) = 0,$$

we have

$$(B^{-1}_{\gamma,-} B_\gamma g)(x) = g(x).$$

When

$$\lim_{x\to 0+} g(x)=0, \qquad \lim_{x\to 0+} g'(x)=0,$$

we have

$$(B^{-1}_{\gamma,0+} B_\gamma g)(x) = g(x).$$

Proof. Let $f(x) = B_\gamma g(x) = g''(x) + \frac{\gamma}{x} g'(x)$. Then

$$(B^{-1}_{\gamma,-} f)(x) = (B^{-1}_{\gamma,-} B_\gamma g)(x) =$$

$$\frac{1}{\gamma-1}\int\limits_x^\infty y\left[\left(\frac{x}{y}\right)^{1-\gamma} - 1\right]\left(g''(y) + \frac{\gamma}{y} g'(y)\right) dy =$$

$$\frac{1}{\gamma-1}\left(\int\limits_x^\infty y\left[\left(\frac{x}{y}\right)^{1-\gamma} -1\right] g''(y)dy + \gamma\int\limits_x^\infty \left[\left(\frac{x}{y}\right)^{1-\gamma} -1\right] g'(y)dy\right). \tag{9.17}$$

Twice integrating by parts the first term in (9.17), we obtain

$$\int\limits_x^\infty y\left[\left(\frac{x}{y}\right)^{1-\gamma} -1\right] g''(y)dy=$$

$$y\left[\left(\frac{x}{y}\right)^{1-\gamma} -1\right] g'(y)\Bigg|_{y=x}^{y=\infty} - \int\limits_x^\infty (\gamma x^{1-\gamma} y^{\gamma-1} - 1) g'(y)dy =$$

$$y\left[\left(\frac{x}{y}\right)^{1-\gamma} -1\right] g'(y)\Bigg|_{y=x}^{y=\infty} - (\gamma x^{1-\gamma} y^{\gamma-1} - 1) g(y)\Bigg|_{y=x}^{y=\infty} +$$

$$\gamma(\gamma-1)x^{1-\gamma}\int\limits_x^\infty y^{\gamma-2} g(y)dy.$$

Integrating by parts the second term in (9.17), we obtain

$$\int\limits_x^\infty \left[\left(\frac{x}{y}\right)^{1-\gamma} - 1\right] g'(y)dy =$$

$$\left[\left(\frac{x}{y}\right)^{1-\gamma}-1\right]g(y)\Big|_{y=x}^{y=\infty}-\frac{\gamma-1}{x^{\gamma-1}}\int\limits_{x}^{\infty}y^{\gamma-2}g(y)dy.$$

Then, obviously, when $\lim\limits_{x\to+\infty}g(x)=0$, $\lim\limits_{x\to+\infty}g'(x)=0$ leads to $(B_{\gamma,-}^{-1}B_\gamma g)(x)=g(x)$. Similarly, when $\lim\limits_{x\to 0+}g(x)=0$ and $\lim\limits_{x\to 0+}g'(x)=0$, we have $(B_{\gamma,0+}^{-1}B_\gamma g)(x)=g(x)$. □

Lemma 36. *Let $f(x)\in L_1(0,\infty)$. Fractional Bessel integrals on the semiaxis are related by the equality*

$$\int\limits_0^\infty f(x)(B_{\gamma,0+}^{-\alpha}g)(x)x^\gamma dx=\int\limits_0^\infty g(x)(B_{\gamma,-}^{-\alpha}f)(x)x^\gamma dx. \tag{9.18}$$

Proof. Let us consider $(B_{\gamma,0+}^{-\alpha}f)(x)$ using its kernel representation as a Legendre function:

$$\int\limits_0^\infty f(x)(B_{\gamma,0+}^{-\alpha}g)(x)x^\gamma dx=$$

$$\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)}\int\limits_0^\infty f(x)x^\gamma dx\int\limits_0^x(x^2-y^2)^{\alpha-\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{\gamma}{2}}P_{\frac{\gamma}{2}-1}^{\frac{1}{2}-\alpha}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right]g(y)dy=$$

$$\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)}\int\limits_0^\infty g(y)dy\int\limits_y^\infty(x^2-y^2)^{\alpha-\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{\gamma}{2}}P_{\frac{\gamma}{2}-1}^{\frac{1}{2}-\alpha}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right]f(x)x^\gamma dx=$$

$$\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)}\int\limits_0^\infty g(y)y^\gamma dy\int\limits_y^\infty(x^2-y^2)^{\alpha-\frac{1}{2}}\left(\frac{x}{y}\right)^{\frac{\gamma}{2}}P_{\frac{\gamma}{2}-1}^{\frac{1}{2}-\alpha}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right]f(x)dx=$$

$$\frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha)}\int\limits_0^\infty g(x)x^\gamma dx\int\limits_x^\infty(y^2-x^2)^{\alpha-\frac{1}{2}}\left(\frac{y}{x}\right)^{\frac{\gamma}{2}}P_{\frac{\gamma}{2}-1}^{\frac{1}{2}-\alpha}\left[\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}\right)\right]f(y)dy=$$

$$\int\limits_0^\infty g(x)(B_{\gamma,-}^{-\alpha}f)(x)x^\gamma dx.$$

This proves the lemma. □

9.2.3 Factorization

Following [555] and [367] we present the following results.

Let $\operatorname{Re}(2\eta+\mu)+2>1/p$ and $f\in F_{p,\mu}$. For $\operatorname{Re}\alpha>0$, we define $I_2^{\eta,\alpha}f$ by the formula

$$I_2^{\eta,\alpha}f(x)=\frac{2}{\Gamma(\alpha)}x^{-2\eta-2\alpha}\int\limits_0^x(x^2-u^2)^{\alpha-1}u^{2\eta+1}f(u)du. \tag{9.19}$$

Let $\operatorname{Re}(2\eta-\mu)>-1/p$ and $f\in F_{p,\mu}$. For $\operatorname{Re}\alpha>0$, we define $K_2^{\eta,\alpha}f$ by the formula

$$K_2^{\eta,\alpha}f(x)=\frac{2}{\Gamma(\alpha)}x^{2\eta}\int\limits_x^\infty(u^2-x^2)^{\alpha-1}u^{1-2(\eta+\alpha)}f(u)du. \tag{9.20}$$

The definitions are extended to $\operatorname{Re}\alpha\le 0$ by means of the formulas

$$I_2^{\eta,\alpha}f=(\eta+\alpha+1)I_2^{\eta,\alpha+1}f+\frac{1}{2}I_2^{\eta,\alpha+1}x\frac{df}{dx} \tag{9.21}$$

and

$$K_2^{\eta,\alpha}f=(\eta+\alpha)K_2^{\eta,\alpha+1}f-\frac{1}{2}K_2^{\eta,\alpha+1}x\frac{df}{dx}. \tag{9.22}$$

Theorem 116. *The following factorizations of* (9.11) *and* (9.12) *are valid:*

$$(B_{\gamma,-}^{-\alpha}f)(x)=2^{-2\alpha}K_2^{\frac{1-\gamma}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}f(x) \tag{9.23}$$

and

$$(B_{\gamma,0+}^{-\alpha}f)(x)=\left(\frac{x}{2}\right)^{2\alpha}I_2^{\frac{\gamma-1}{2},\alpha}I_2^{0,\alpha}f(x), \tag{9.24}$$

where

$$K_2^{0,\alpha}f(x)=\frac{2}{\Gamma(\alpha)}\int\limits_x^\infty(u^2-x^2)^{\alpha-1}u^{1-2\alpha}f(u)du,$$

$$K_2^{\frac{1-\gamma}{2},\alpha}f(x)=\frac{2}{\Gamma(\alpha)}x^{1-\gamma}\int\limits_x^\infty(u^2-x^2)^{\alpha-1}u^{\gamma-2\alpha}f(u)du,$$

$$I_2^{0,\alpha}f(x)=\frac{2}{\Gamma(\alpha)}x^{-2\alpha}\int\limits_0^x(x^2-u^2)^{\alpha-1}uf(u)du,$$

$$I_2^{\frac{\gamma-1}{2},\alpha}f(x)=\frac{2}{\Gamma(\alpha)}x^{1-\gamma-2\alpha}\int\limits_0^x(x^2-u^2)^{\alpha-1}u^{\gamma}f(u)du.$$

Proof. We have

$$B_{\gamma,-}^{-\alpha} f = 2^{-2\alpha} K_2^{\frac{1-\gamma}{2},\alpha} K_2^{0,\alpha} x^{2\alpha} f =$$

$$\frac{2^{1-2\alpha}}{\Gamma(\alpha)} K_2^{\frac{1-\gamma}{2},\alpha} \int\limits_y^\infty (u^2-y^2)^{\alpha-1} u f(u) du =$$

$$\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)} x^{1-\gamma} \int\limits_x^\infty (y^2-x^2)^{\alpha-1} y^{\gamma-2\alpha} dy \int\limits_y^\infty (u^2-y^2)^{\alpha-1} u f(u) du =$$

$$\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)} x^{1-\gamma} \int\limits_x^\infty u f(u) du \int\limits_x^u (u^2-y^2)^{\alpha-1} (y^2-x^2)^{\alpha-1} y^{\gamma-2\alpha} dy.$$

For the inner integral we have

$$\int\limits_x^u (y^2-x^2)^{\alpha-1} (u^2-y^2)^{\alpha-1} y^{\gamma-2\alpha} dy =$$

$$\frac{1}{2} \frac{2^{1-2\alpha}\sqrt{\pi}\,\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)} \left(u^2-x^2\right)^{2\alpha-1} x^{\gamma-1} u^{-2\alpha} {}_2F_1\left(\alpha, \alpha+\frac{\gamma-1}{2}; 2\alpha; 1-\frac{x^2}{u^2}\right)$$

and

$$B_{\gamma,-}^{-\alpha} f = \frac{2^{1-2\alpha}}{\Gamma^2(\alpha)} \frac{2^{1-2\alpha}\sqrt{\pi}\,\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)} x^{1-\gamma} \times$$

$$\int\limits_x^\infty \left(u^2-x^2\right)^{2\alpha-1} x^{\gamma-1} u^{-2\alpha} {}_2F_1\left(\alpha, \alpha+\frac{\gamma-1}{2}; 2\alpha; 1-\frac{x^2}{u^2}\right) u f(u) du =$$

$$\frac{2^{1-2\alpha}}{\Gamma(2\alpha)} \int\limits_x^\infty \left(u^2-x^2\right)^{2\alpha-1} u^{1-2\alpha} {}_2F_1\left(\alpha, \alpha+\frac{\gamma-1}{2}; 2\alpha; 1-\frac{x^2}{u^2}\right) f(u) du =$$

$$\frac{1}{\Gamma(2\alpha)} \int\limits_x^\infty \left(\frac{u^2-x^2}{2u}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{u^2}\right) f(u) du.$$

This coincides with formula (9.23).

Now we proof (9.24). We have

$$(B_{\gamma,0+}^{-\alpha} f)(x) =$$

$$\frac{1}{\Gamma(2\alpha)} \int\limits_0^x \left(\frac{u}{x}\right)^\gamma \left(\frac{x^2-u^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{u^2}{x^2}\right) f(u) du =$$

$$2^{-2\alpha}x^{2\alpha}I_2^{\frac{\gamma-1}{2},\alpha}I_2^{0,\alpha}f=$$
$$\frac{2^{1-2\alpha}x^{2\alpha}}{\Gamma(\alpha)}I_2^{\frac{\gamma-1}{2},\alpha}y^{-2\alpha}\int\limits_0^y(y^2-u^2)^{\alpha-1}uf(u)du=$$
$$\frac{2^{2-2\alpha}x^{2\alpha}}{\Gamma^2(\alpha)}x^{-\gamma+1-2\alpha}\int\limits_0^x(x^2-y^2)^{\alpha-1}y^{\gamma-2\alpha}dy\int\limits_0^y(y^2-u^2)^{\alpha-1}uf(u)du=$$
$$\frac{2^{2-2\alpha}}{\Gamma^2(\alpha)}x^{1-\gamma}\int\limits_0^x uf(u)du\int\limits_u^x(y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\gamma-2\alpha}dy.$$

Let us find

$$\int\limits_u^x(y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\gamma-2\alpha}dy=\{y^2=t\}=$$
$$\frac{1}{2}\int\limits_{u^2}^{x^2}(t-u^2)^{\alpha-1}(x^2-t)^{\alpha-1}t^{\frac{\gamma-1}{2}-\alpha}dt=$$
$$\frac{\sqrt{\pi}\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(x^2-u^2\right)^{2\alpha-1}u^{-2\alpha+\gamma-1}\,{}_2F_1\left(\alpha+\frac{1-\gamma}{2},\alpha;2\alpha;1-\frac{x^2}{u^2}\right).$$

Using the formula

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right),$$

we obtain

$${}_2F_1\left(\alpha+\frac{1-\gamma}{2},\alpha;2\alpha;1-\frac{x^2}{u^2}\right)={}_2F_1\left(\alpha,\alpha+\frac{1-\gamma}{2};2\alpha;1-\frac{x^2}{u^2}\right)=$$
$$\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\gamma-1}{2};2\alpha;1-\frac{u^2}{x^2}\right)=$$
$$\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)$$

and

$$\int\limits_u^x(y^2-u^2)^{\alpha-1}(x^2-y^2)^{\alpha-1}y^{\gamma-2\alpha}dy=\frac{\sqrt{\pi}\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\times$$
$$\left(x^2-u^2\right)^{2\alpha-1}u^{-2\alpha+\gamma-1}\left(\frac{x^2}{u^2}\right)^{-\alpha}{}_2F_1\left(\alpha,\alpha+\frac{\gamma-1}{2};2\alpha;1-\frac{u^2}{x^2}\right)=$$

$$\frac{\sqrt{\pi}\Gamma(\alpha)}{2^{2\alpha}\Gamma\left(\alpha+\frac{1}{2}\right)}\left(x^2-u^2\right)^{2\alpha-1}u^{\gamma-1}x^{-2\alpha}\,{}_2F_1\left(\alpha,\alpha+\frac{\gamma-1}{2};2\alpha;1-\frac{u^2}{x^2}\right).$$

Finally,

$$(B_{\gamma,0+}^{-\alpha}f)(x)=\frac{2^{2(1-2\alpha)}\sqrt{\pi}}{\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)}x^{1-\gamma-2\alpha}\times$$

$$\int\limits_0^x\left(x^2-u^2\right)^{2\alpha-1}u^{\gamma}\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)f(u)du.$$

Applying the duplication formula (1.7), we obtain

$$(B_{\gamma,0+}^{-\alpha}f)(x)=\frac{2^{1-2\alpha}}{\Gamma(2\alpha)}x^{1-\gamma-2\alpha}\times$$

$$\int\limits_0^x\left(x^2-u^2\right)^{2\alpha-1}u^{\gamma}\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)f(u)du=$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_0^x\left(\frac{x^2-u^2}{2x}\right)^{2\alpha-1}\left(\frac{u}{x}\right)^{\gamma}\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{u^2}{x^2}\right)f(u)du,$$

which gives (9.24). The proof is complete. □

9.2.4 Fractional Bessel integrals on semiaxes of elementary and special functions

Statement 22. *Let $f(x)=x^m$, $x>0$, $m\in\mathbb{R}$. Then the integrals $B_{\gamma,-}^{-\alpha}$ and $B_{\gamma,0+}^{-\alpha}$ of the power function are defined by the formulas*

$$B_{\gamma,-}^{-\alpha}x^m=x^{2\alpha+m}2^{-2\alpha}\Gamma\left[\begin{array}{cc}-\alpha-\frac{m}{2}, & -\frac{\gamma-1}{2}-\alpha-\frac{m}{2}\\ \frac{1-\gamma-m}{2}, & -\frac{m}{2}\end{array}\right],\quad m+2\alpha+\gamma<1,$$

$$B_{\gamma,0+}^{-\alpha}x^m=x^{2\alpha+m}2^{-2\alpha}\Gamma\left[\begin{array}{cc}\frac{m+\gamma+1}{2}, & \frac{m}{2}+1\\ \alpha+\frac{m}{2}+1, & \alpha+\frac{m+\gamma+1}{2}\end{array}\right].$$

Proof. We have

$$B_{\gamma,-}^{-\alpha}x^m=\frac{1}{\Gamma(2\alpha)}\int\limits_x^{\infty}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)y^m dy=$$

$$\left\{\frac{x^2}{y^2}=t,\,y=xt^{-\frac{1}{2}},\,dy=-\frac{1}{2}xt^{-\frac{3}{2}}dt,\,y=x,\,t=1,\,y=+\infty,\,t=0\right\}=$$

$$\frac{1}{2}\frac{1}{\Gamma(2\alpha)}\int\limits_0^1\left(\frac{x^2t^{-1}-x^2}{2xt^{-\frac{1}{2}}}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-t\right)(xt^{-\frac{1}{2}})^m xt^{-\frac{3}{2}}dt=$$
$$\frac{x^{2\alpha+m}}{2^{2\alpha}\Gamma(2\alpha)}\int\limits_0^1 t^{-\alpha-\frac{m}{2}-1}(1-t)^{2\alpha-1}\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-t\right)dt.$$

Using the following formula (2.21.1.11) from [457], p. 265,

$$\int\limits_0^z x^{\mu-1}(z-x)^{c-1}\,{}_2F_1\left(a,b;c;1-\frac{x}{z}\right)dx=$$
$$z^{c+\mu-1}\Gamma\left[\begin{array}{ccc} c, & \mu, & c-a-b+\mu \\ c-a+\mu, & & c-b+\mu\end{array}\right],$$
$$z>0,\ \mathrm{Re}\,c>0,\ \mathrm{Re}\,(c-a-b+\mu)>0,$$

we have

$$z=1,\mu=-\alpha-\frac{m}{2},a=\alpha+\frac{\gamma-1}{2},b=\alpha,c=2\alpha\Rightarrow$$
$$c-a-b+\mu=-\frac{\gamma-1}{2}-\alpha-\frac{m}{2}>0$$

and

$$B_{\gamma,-}^{-\alpha}\,x^m=\frac{x^{2\alpha+m}}{2^{2\alpha}\Gamma(2\alpha)}\Gamma\left[\begin{array}{ccc} 2\alpha, & -\alpha-\frac{m}{2}, & -\frac{\gamma-1}{2}-\alpha-\frac{m}{2} \\ \frac{1-\gamma-m}{2}, & & -\frac{m}{2}\end{array}\right]=$$
$$x^{2\alpha+m}\,2^{-2\alpha}\,\Gamma\left[\begin{array}{cc} -\alpha-\frac{m}{2}, & -\frac{\gamma-1}{2}-\alpha-\frac{m}{2} \\ \frac{1-\gamma-m}{2}, & -\frac{m}{2}\end{array}\right].$$

Now let us consider $B_{\gamma,0+}^{-\alpha}\,x^m$:

$$B_{\gamma,0+}^{-\alpha}\,x^m=$$
$$\frac{1}{\Gamma(2\alpha)}\int\limits_0^x\left(\frac{y}{x}\right)^\gamma\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)y^m dy=$$
$$\{\frac{y^2}{x^2}=t,\ y=xt^{\frac{1}{2}},\ dy=\frac{1}{2}xt^{-\frac{1}{2}}dt,\ y=0,t=0,\ y=x,t=1\}=$$
$$\frac{x}{2\Gamma(2\alpha)}\int\limits_0^1 t^{\frac{\gamma-1}{2}}\left(\frac{x(1-t)}{2}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-t\right)x^m t^{\frac{m}{2}}dt=$$

$$\frac{x^{2\alpha+m}}{2^{2\alpha}\Gamma(2\alpha)}\int\limits_0^1 t^{\frac{m+\gamma+1}{2}-1}(1-t)^{2\alpha-1}\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-t\right)dt.$$

Using the following formula (2.21.1.11) from [457], p. 265,

$$\int\limits_0^z x^{\mu-1}(z-x)^{c-1}\,{}_2F_1\left(a,b;c;1-\frac{x}{z}\right)dx =$$
$$= z^{c+\mu-1}\Gamma\left[\begin{array}{ccc} c, & \mu, & c-a-b+\mu \\ c-a+\mu, & & c-b+\mu \end{array}\right],$$
$$z>0,\ \operatorname{Re} c>0,\ \operatorname{Re}(c-a-b+\mu)>0,$$

we have

$$z=1,\mu=\frac{m+\gamma+1}{2}, a=\alpha+\frac{\gamma-1}{2}, b=\alpha,$$
$$c=2\alpha \Rightarrow c-a-b+\mu=\frac{m}{2}+1>0,$$
$$B_{\gamma,0+}^{-\alpha}\,x^m=\frac{x^{2\alpha+m}}{2^{2\alpha}\Gamma(2\alpha)}\Gamma\left[\begin{array}{ccc} 2\alpha, & \frac{m+\gamma+1}{2}, & \frac{m}{2}+1 \\ \alpha+\frac{m}{2}+1, & & \alpha+\frac{m+\gamma+1}{2} \end{array}\right]=$$
$$x^{2\alpha+m}\,2^{-2\alpha}\,\Gamma\left[\begin{array}{cc} \frac{m+\gamma+1}{2}, & \frac{m}{2}+1 \\ \alpha+\frac{m}{2}+1, & \alpha+\frac{m+\gamma+1}{2} \end{array}\right]. \qquad \square$$

Corollary 20. *The operator $\frac{1}{x^{2\alpha}}B_{\gamma,-}^{-\alpha}$ is of the so-called Dzhrbashyan–Gelfond–Leontiev type (see [252,494]) when $m+2\alpha+\gamma<1$. This means that it acts on power series by the rule*

$$\frac{1}{z^{2\alpha}}B_{\gamma,-}^{-\alpha}\left(\sum_{k=0}^{\infty}a_k z^k\right)=\left(\sum_{k=0}^{\infty}c(\alpha,k)a_k z^k\right),$$
$$c(\alpha,k)=2^{-2\alpha}\,\Gamma\left[\begin{array}{cc} -\alpha-\frac{k}{2}, & -\frac{\gamma-1}{2}-\alpha-\frac{k}{2} \\ \frac{1-\gamma-k}{2}, & -\frac{k}{2} \end{array}\right], \quad a_k\in\mathbb{R}.$$

Corollary 21. *The operator $\frac{1}{x^{2\alpha}}B_{\gamma,0+}^{-\alpha}$ is also of Dzhrbashyan–Gelfond–Leontiev type:*

$$\frac{1}{z^{2\alpha}}B_{\gamma,0+}^{-\alpha}\left(\sum_{k=0}^{\infty}a_k z^k\right)=\left(\sum_{k=0}^{\infty}d(\alpha,k)a_k z^k\right),$$
$$d(\alpha,k)=2^{-2\alpha}\,\Gamma\left[\begin{array}{cc} \frac{m+\gamma+1}{2}, & \frac{m}{2}+1 \\ \alpha+\frac{m}{2}+1, & \alpha+\frac{m+\gamma+1}{2} \end{array}\right], \quad a_k\in\mathbb{R}.$$

Statement 23. *Let* $f(x)=k_{\frac{\gamma-1}{2}}(x\xi))(x)$*, defined by* (1.21)*,* $x>0$*,* $\gamma>0$*. Then the integral* $(B_{\gamma,-}^{-\alpha}f)(x)$ *is defined by the formula*

$$(B_{\gamma,-}^{-\alpha}k_{\frac{\gamma-1}{2}}(x\xi))(x)=\xi^{-2\alpha}k_{\frac{\gamma-1}{2}}(x\xi).$$

Proof. Using the factorization (9.23), we get

$$(B_{\gamma,-}^{-\alpha}f)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_x^\infty\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)f(y)dy=$$

$$2^{-2\alpha}K_2^{\frac{1-\gamma}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}f=$$

$$\frac{2^{2(2-\alpha)}}{\Gamma^2(\alpha)}x^{1-\gamma}\int\limits_x^\infty(u^2-x^2)^{\alpha-1}u^{\gamma-2\alpha}du\int\limits_u^\infty(t^2-u^2)^{\alpha-1}tf(t)dt.$$

Applying formula (2.16.3.7) from [456] of the form

$$\int\limits_a^\infty x^{1\pm\rho}(x^2-a^2)^{\beta-1}K_\rho(cx)dx=2^{\beta-1}a^{\beta\pm\rho}c^{-\beta}\Gamma(\beta)K_{\rho\pm\beta}(ac),$$

where $a,c,\beta>0$ and K_ρ is the modified Bessel function of the second kind (1.17), we obtain

$$(B_{\gamma,-}^{-\alpha}k_{\frac{\gamma-1}{2}}(x\xi))(x)=$$

$$\frac{2^{2(1-\alpha)}}{\Gamma^2(\alpha)}x^{1-\gamma}\int\limits_x^\infty(u^2-x^2)^{\alpha-1}u^{\gamma-2\alpha}du\int\limits_u^\infty(t^2-u^2)^{\alpha-1}\,t\,k_{\frac{\gamma-1}{2}}(t\xi)dt=$$

$$\frac{2^{2(1-\alpha)}}{\Gamma(\alpha)}x^{1-\gamma}\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{\xi^{\frac{\gamma-1}{2}}}2^{\alpha-1}\xi^{-\alpha}\int\limits_x^\infty(u^2-x^2)^{\alpha-1}u^{1+\left(\frac{\gamma-1}{2}-\alpha\right)}K_{\frac{\gamma-1}{2}-\alpha}(u\xi)du=$$

$$2^{2(1-\alpha)}x^{1-\gamma}\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{\xi^{\frac{\gamma-1}{2}}}2^{2(\alpha-1)}\xi^{-2\alpha}x^{\frac{\gamma-1}{2}}K_{\frac{\gamma-1}{2}}(x\xi)=$$

$$\xi^{-2\alpha}\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{(x\xi)^{\frac{\gamma-1}{2}}}K_{\frac{\gamma-1}{2}}(x\xi)=\xi^{-2\alpha}k_{\frac{\gamma-1}{2}}(x\xi).$$

□

9.3 Integral transforms of fractional powers of Bessel operators

An integral transform maps the original space into or onto the image space. Usually difficult operations in the original space are converted into simple operations in the image space. For example, the Fourier transform converts a derivative of order n into multiplication by the power n of the variable with some constant. This is the reason why the Fourier transform is beneficial to use for solution to differential equations. Since the Hankel transform applied to a Bessel operator of order n gives multiplication of a Hankel image of a function by the power $2n$ of the variable with some constant, this transform is used instead of the Fourier transform when a differential equation with Bessel operator is solved. The action of the Hankel transform on fractional Bessel derivatives of order α on semiaxes has the next property: it does not involve multiplication by some power in the dual variable under the Hankel transform (see Theorem 119). In this section we collect some integral transforms which can be used to solve differential equations with fractional Bessel derivatives on semiaxes.

9.3.1 The Mellin transform

Using formula (2.21.1.11) from [457], p. 265, of the form

$$
\begin{aligned}
&\int\limits_0^z x^{\alpha-1}(z-x)^{c-1}\,{}_2F_1\left(a,b;c;1-\frac{x}{z}\right)dx = \\
&z^{c+\alpha-1}\Gamma\left[\begin{array}{cc} c, & \alpha,\ \ c-a-b+\alpha \\ c-a+\alpha, & c-b+\alpha \end{array}\right], \\
&z>0,\ \operatorname{Re}c>0,\ \operatorname{Re}(c-a-b+\alpha)>0,
\end{aligned} \tag{9.25}
$$

we prove the following theorems.

Theorem 117. *Let $\alpha>0$. Mellin transforms of $IB^{\alpha}_{\gamma,-}$ and $IB^{\alpha}_{\gamma,0+}$ are*

$$
\mathcal{M}IB^{\alpha}_{\gamma,-}f(s)=\frac{1}{2^{2\alpha}}\,\Gamma\left[\begin{array}{cc} \frac{s}{2}, & \frac{s}{2}-\frac{\gamma-1}{2} \\ \alpha+\frac{s}{2}-\frac{\gamma-1}{2}, & \alpha+\frac{s}{2} \end{array}\right]f^*(2\alpha+s), \tag{9.26}
$$

where $s>\gamma-1$, $IB^{\alpha}_{\gamma,-}f\in P^b_a$,

$$
\mathcal{M}IB^{\alpha}_{\gamma,0+}f(s)=\frac{1}{2^{2\alpha}}\Gamma\left[\begin{array}{cc} \frac{\gamma-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha \\ 1-\frac{s}{2}, & \frac{\gamma-s+1}{2} \end{array}\right]f^*(2\alpha+s), \tag{9.27}
$$

where $IB^{\alpha}_{\gamma,0+}f\in P^b_a$, $2\alpha+s<2$.

Proof. Let us start from the definitions

$$((IB^{\alpha}_{\gamma,-}f)(x))^*(s)=\int\limits_0^{\infty}x^{s-1}(IB^{\alpha}_{\gamma,-}f)(x)dx=$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty}x^{s-1}dx\int\limits_x^{+\infty}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)f(y)dy$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty}f(y)(2y)^{1-2\alpha}dy\int\limits_0^{y}(y^2-x^2)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)x^{s-1}dx.$$

Using (9.25), let us find the inner integral for $s>\gamma-1$:

$$\int\limits_0^{y}(y^2-x^2)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{y^2}\right)x^{s-1}dx=$$

$$\frac{y^{4\alpha+s-2}}{2}\,\Gamma\left[\begin{array}{ccc}2\alpha, & \frac{s}{2}, & \frac{s}{2}-\frac{\gamma-1}{2}\\ \alpha+\frac{s}{2}-\frac{\gamma-1}{2}, & & \alpha+\frac{s}{2}\end{array}\right].$$

We obtain

$$((IB^{\alpha}_{\gamma,-}f)(x))^*(s)=\frac{1}{2^{2\alpha}}\Gamma\left[\begin{array}{cc}\frac{s}{2}, & \frac{s}{2}-\frac{\gamma-1}{2}\\ \alpha+\frac{s}{2}-\frac{\gamma-1}{2}, & \alpha+\frac{s}{2}\end{array}\right]\int\limits_0^{\infty}f(y)y^{2\alpha+s-1}dy=$$

$$\frac{1}{2^{2\alpha}}\,\Gamma\left[\begin{array}{cc}\frac{s}{2}, & \frac{s}{2}-\frac{\gamma-1}{2}\\ \alpha+\frac{s}{2}-\frac{\gamma-1}{2}, & \alpha+\frac{s}{2}\end{array}\right]f^*(2\alpha+s).$$

Similarly, we have

$$((IB^{\alpha}_{\gamma,0+}f)(x))^*(s)=\int\limits_0^{\infty}x^{s-1}(B^{-\alpha}_{\gamma,0+}f)(x)dx=$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty}x^{s-1}dx\int\limits_0^{x}\left(\frac{y}{x}\right)^{\gamma}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}\times$$

$${}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)f(y)dy=$$

$$\frac{1}{\Gamma(2\alpha)}\int\limits_0^{\infty}f(y)y^{\gamma}dy\int\limits_y^{\infty}\left(\frac{1}{x}\right)^{\gamma}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1}\times$$

$${}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)x^{s-1}dx.$$

Let us find the inner integral:

$$\int\limits_y^\infty \left(\frac{1}{x}\right)^\gamma \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{x^2}\right) x^{s-1}dx =$$

$$2^{1-2\alpha}\int\limits_y^\infty \left(\frac{1}{x}\right)^{2\alpha-s+\gamma} \left(x^2-y^2\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{x^2}\right) dx =$$

$$\left\{\frac{1}{x}=t\right\} =$$

$$2^{1-2\alpha}\int\limits_0^{1/y} t^{\gamma-2\alpha-s}\left(1-t^2y^2\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-t^2y^2\right)dt = \{ty=z\} =$$

$$2^{1-2\alpha}y^{2\alpha+s-\gamma-1}\int\limits_0^1 z^{\gamma-2\alpha-s}\left(1-z^2\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-z^2\right)dz =$$

$$\{z^2=s\} =$$

$$\frac{1}{2^{2\alpha}}y^{2\alpha+s-\gamma-1}\int\limits_0^1 s^{\frac{\gamma-s-1}{2}-\alpha}(1-s)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-s\right)ds.$$

Using (9.25), for $2\alpha+s<2$ we get

$$\int\limits_0^1 s^{\frac{\gamma-s-1}{2}-\alpha}(1-s)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-s\right)ds =$$

$$\frac{1}{2^{2\alpha}}y^{2\alpha+s-\gamma-1}\Gamma\left[\begin{array}{ccc} 2\alpha, & \frac{\gamma-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha \\ 1-\frac{s}{2}, & & \frac{\gamma-s+1}{2} \end{array}\right]$$

and

$$((B_{\gamma,0+}^{-\alpha}f)(x))^*(s) = \frac{1}{2^{2\alpha}}\Gamma\left[\begin{array}{cc} \frac{\gamma-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha \\ 1-\frac{s}{2}, & \frac{\gamma-s+1}{2} \end{array}\right]\int\limits_0^\infty f(y)y^{2\alpha+s-1}dy =$$

$$\frac{1}{2^{2\alpha}}\Gamma\left[\begin{array}{cc} \frac{\gamma-s+1}{2}-\alpha, & 1-\frac{s}{2}-\alpha \\ 1-\frac{s}{2}, & \frac{\gamma-s+1}{2} \end{array}\right]f^*(2\alpha+s).$$

This completes the proof. □

In order to obtain formulas for Mellin transform of fractional Bessel derivatives on semiaxes we should proof the next statement.

Lemma 37. *Let $B_\gamma^n f \in P_a^b$. Then for $n \in \mathbb{N}$*

$$\mathcal{M}B_\gamma^n f(s) = 2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\gamma}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\gamma}{2} \end{array}\right] f^*(s-2n). \tag{9.28}$$

Proof. Using formulas for Mellin transform from [94], we obtain

$$\mathcal{M}f'(s) = (1-s)\mathcal{M}f(s-1), \qquad \mathcal{M}\frac{1}{x}f(s) = \mathcal{M}f(s-1),$$
$$\mathcal{M}\frac{1}{x}f'(s) = (\mathcal{M}f'(t-1))(s) = (2-s)\mathcal{M}f(s-2),$$
$$\mathcal{M}f''(s) = (2-s)(1-s)\mathcal{M}f(s-2),$$
$$\mathcal{M}B_\gamma f(s) = (2-s)(1-s)f^*(s-2) + \gamma(2-s)f^*(s-2) =$$
$$(2-s)(1-s+\gamma)f^*(s-2).$$

So

$$\mathcal{M}B_\gamma f)(s) = (2-s)(1-s+\gamma)f^*(s-2). \tag{9.29}$$

Applying formula (9.29) n times, we obtain

$$\mathcal{M}B_\gamma^n f(s) = (2-s)(4-s)...(2n-s)(1-s+\gamma)(3-s+\gamma)...\times$$
$$(2n-1-s+\gamma)f^*(s-2n).$$

Since

$$(2-s)(4-s)...(2n-s) = 2^n\left(1-\frac{s}{2}\right)\left(2-\frac{s}{2}\right)...\left(n-\frac{s}{2}\right) =$$
$$2^n\left(1-\frac{s}{2}\right)_n = \frac{2^n\Gamma\left(n+1-\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}$$

and

$$(1-s+\gamma)(3-s+\gamma)...(2n-1-s+\gamma) =$$
$$2^n\left(\frac{1-s+\gamma}{2}\right)\left(\frac{1-s+\gamma}{2}+1\right)...\left(\frac{1-s+\gamma}{2}+n-1\right) =$$
$$2^n\left(\frac{1-s+\gamma}{2}\right)_n = \frac{2^n\Gamma\left(\frac{1-s+\gamma}{2}+n\right)}{\Gamma\left(\frac{1-s+\gamma}{2}\right)},$$

we have

$$\mathcal{M}B_\gamma^n f(s) = 2^{2n}\frac{\Gamma\left(n+1-\frac{s}{2}\right)\Gamma\left(\frac{1-s+\gamma}{2}+n\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{1-s+\gamma}{2}\right)}f^*(s-2n) =$$

$$2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\gamma}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\gamma}{2} \end{array}\right] f^*(s-2n).$$

This completes the proof. □

Theorem 118. *Let $\alpha > 0$, $n = [\alpha]+1$. Mellin transforms of $DB^{\alpha}_{\gamma,-}$ and $DB^{\alpha}_{\gamma,0+}$ are*

$$\mathcal{M}DB^{\alpha}_{\gamma,-}f(s) = 2^{2\alpha}\,\Gamma\left[\begin{array}{cc} \frac{s}{2}, & \frac{s}{2}-\frac{\gamma-1}{2} \\ \frac{s}{2}-\alpha-\frac{\gamma-1}{2}, & \frac{s}{2}-\alpha \end{array}\right] f^*(s-2\alpha), \tag{9.30}$$

where $s-2n > \gamma-1$, $IB^{n-\alpha}_{\gamma,-}f \in P^b_a$, and

$$\mathcal{M}DB^{\alpha}_{\gamma,0+}f(s) = 2^{2\alpha}\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\gamma-s+1}{2}+\alpha \\ 1-\frac{s}{2}, & \frac{\gamma-s+1}{2} \end{array}\right] f^*(s-2\alpha), \tag{9.31}$$

where $2\alpha-2n+s < 2$, $IB^{n-\alpha}_{\gamma,0+}f \in P^b_a$.

Proof. Applying (9.26) and (9.28), we obtain

$$\begin{aligned} &((DB^{\alpha}_{\gamma,-}f)(x))^*(s) = ((B^n_\gamma(IB^{n-\alpha}_{\gamma,-}f(x))^*(s) = \\ &2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\gamma}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\gamma}{2} \end{array}\right]((IB^{n-\alpha}_{\gamma,-}f(x))^*(s-2n) = \\ &2^{2\alpha}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\gamma}{2}+n \\ 1-\frac{s}{2} & \frac{1-s+\gamma}{2} \end{array}\right]\Gamma\left[\begin{array}{cc} \frac{s}{2}-n, & \frac{s}{2}-n-\frac{\gamma-1}{2} \\ \frac{s}{2}-\alpha-\frac{\gamma-1}{2}, & \frac{s}{2}-\alpha \end{array}\right] f^*(s-2\alpha). \end{aligned} \tag{9.32}$$

Using the formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \qquad z \notin \mathbb{Z},$$

in the numerator, we obtain

$$\begin{aligned} &\Gamma\left(1+n-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}-n\right) = \frac{\pi}{\sin(\frac{s}{2}-n)\pi} = \frac{(-1)^n\pi}{\sin(\frac{s}{2})\pi}, \\ &\Gamma\left(\frac{1-s+\gamma}{2}+n\right)\Gamma\left(\frac{s-\gamma+1}{2}-n\right) = \\ &\Gamma\left(1-\frac{1-s+\gamma}{2}-n\right)\Gamma\left(\frac{1-s+\gamma}{2}+n\right) = \\ &\frac{\pi}{\sin(\frac{1-s+\gamma}{2}+n)\pi} = \frac{(-1)^n\pi}{\sin(\frac{1-s+\gamma}{2})\pi}. \end{aligned}$$

So

$$\frac{(-1)^n\pi}{\Gamma\left(\frac{1-s+\gamma}{2}\right)\sin(\frac{1-s+\gamma}{2})\pi}=(-1)^n\Gamma\left(\frac{1+s-\gamma}{2}\right),$$
$$\frac{(-1)^n\pi}{\Gamma\left(1-\frac{s}{2}\right)\sin(\frac{s}{2})\pi}=(-1)^n\Gamma\left(\frac{s}{2}\right).$$

Substituting the obtained expressions in (9.32), we obtain (9.30).

Similarly, using (9.27) and (9.28) we have

$$((DB^{\alpha}_{\gamma,0+}f)(x))^*(s)=((B^n_\gamma(IB^{n-\alpha}_{\gamma,0+}f(x))^*(s)=$$
$$2^{2n}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\gamma}{2}+n\\ 1-\frac{s}{2} & \frac{1-s+\gamma}{2}\end{array}\right]((IB^{n-\alpha}_{\gamma,0+}f(x))^*(s-2n)=$$
$$2^{2\alpha}\Gamma\left[\begin{array}{cc} n+1-\frac{s}{2} & \frac{1-s+\gamma}{2}+n\\ 1-\frac{s}{2} & \frac{1-s+\gamma}{2}\end{array}\right]\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\gamma-s+1}{2}+\alpha\\ 1-\frac{s}{2}+n, & \frac{\gamma-s+1}{2}+n\end{array}\right]f^*(s-2\alpha)=$$
$$2^{2\alpha}\Gamma\left[\begin{array}{cc} 1-\frac{s}{2}+\alpha, & \frac{\gamma-s}{2}+\alpha\\ 1-\frac{s}{2}, & \frac{\gamma-s+1}{2}\end{array}\right]f^*(s-2\alpha).\qquad\square$$

9.3.2 The Hankel transform

Theorem 119. *Let $B^{-\alpha}_{\gamma,0+}f$, $B^{-\alpha}_{\gamma,-}f\in L^\gamma_1(\mathbb{R}_+)$. Then*

$$F_\gamma[(B^{-\alpha}_{\gamma,0+}f)(x)](\xi)=\xi^{-2\alpha}\int\limits_0^\infty f(t)\Big[\cos(\alpha\pi)j_{\frac{\gamma-1}{2}}(\xi t)-\sin(\alpha\pi)y_{\frac{\gamma-1}{2}}(\xi t)\Big]t^\gamma dt,\tag{9.33}$$

where $4\alpha-2<\gamma<4-2\alpha$ *and*

$$F_\gamma[(B^{-\alpha}_{\gamma,-}f)](\xi)=\xi^{-2\alpha}\int\limits_0^\infty j^1_{\frac{\gamma-1}{2},\alpha}(t\xi)f(t)t^\gamma dt,\tag{9.34}$$

where

$$j^1_{\frac{\gamma-1}{2},\alpha}(t\xi)=\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{+1}{2}\right)}{(t\xi)^{\frac{\gamma-1}{2}}}J^1_{\frac{\gamma-1}{2},\alpha}(t\xi),$$
$$J^1_{\frac{\gamma-1}{2},\alpha}(t\xi)=\sum_{n=0}^\infty\frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\frac{\gamma+1}{2}+\alpha+n\right)}\left(\frac{t\xi}{2}\right)^{2n+\frac{\gamma-1}{2}+2\alpha}.$$

Proof. Using the factorization formula (9.24) and denoting $g(x)=I_2^{0,\alpha}f(x)$, we obtain

$$F_\gamma[(B_{\gamma,0+}^{-\alpha}f)(x)](\xi)=\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\,(B_{\gamma,0+}^{-\alpha}f)(x)x^\gamma\,dx=$$

$$\frac{1}{2^{2\alpha}}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\;I_2^{\frac{\gamma-1}{2},\alpha}I_2^{0,\alpha}f(x)x^{2\alpha+\gamma}\,dx=$$

$$\frac{1}{2^{2\alpha}}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\;I_2^{\frac{\gamma-1}{2},\alpha}g(x)x^{2\alpha+\gamma}\,dx=$$

$$\frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\,x\,dx\int\limits_0^x(x^2-u^2)^{\alpha-1}u^\gamma g(u)du=$$

$$\frac{1}{2^{2\alpha-1}\Gamma(\alpha)}\int\limits_0^\infty u^\gamma g(u)du\int\limits_u^\infty(x^2-u^2)^{\alpha-1}j_{\frac{\gamma-1}{2}}(x\xi)\,x\,dx.$$

Let us consider the inner integral:

$$\int\limits_u^\infty(x^2-u^2)^{\alpha-1}j_{\frac{\gamma-1}{2}}(x\xi)\,x\,dx=$$

$$\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{\xi^{\frac{\gamma-1}{2}}}\int\limits_u^\infty(x^2-u^2)^{\alpha-1}J_{\frac{\gamma-1}{2}}(x\xi)\,x^{1-\frac{\gamma-1}{2}}\,dx.$$

Using formula (2.12.4.17) from [456] of the form

$$\int\limits_a^\infty x^{1-\rho}(x^2-a^2)^{\beta-1}J_\rho(cx)dx=2^{\beta-1}a^{\beta-\rho}c^{-\beta}\Gamma(\beta)J_{\rho-\beta}(ac),$$
$$a,c,\beta>0,\qquad(2\beta-\rho)<3/2,$$

we obtain for $4\alpha-\gamma<2$

$$\int\limits_u^\infty(x^2-u^2)^{\alpha-1}J_{\frac{\gamma-1}{2}}(x\xi)\,x^{1-\frac{\gamma-1}{2}}\,dx=2^{\alpha-1}u^{\alpha-\frac{\gamma-1}{2}}\xi^{-\alpha}\Gamma(\alpha)J_{\frac{\gamma-1}{2}-\alpha}(u\xi)$$

and

$$F_\gamma[(B_{\gamma,0+}^{-\alpha}f)(x)](\xi)=\frac{2^{\frac{\gamma-1}{2}-\alpha}\Gamma\left(\frac{\gamma+1}{2}\right)}{\xi^{\frac{\gamma-1}{2}+\alpha}}\int\limits_0^\infty u^{\alpha+\frac{\gamma+1}{2}}J_{\frac{\gamma-1}{2}-\alpha}(u\xi)g(u)du=$$

$$\frac{2^{\frac{\gamma+1}{2}-\alpha}\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\gamma-1}{2}+\alpha}}\int\limits_0^\infty u^{\frac{\gamma+1}{2}-\alpha}J_{\frac{\gamma-1}{2}-\alpha}(u\xi)du\int\limits_0^u(u^2-y^2)^{\alpha-1}yf(y)dy=$$

$$\frac{2^{\frac{\gamma+1}{2}-\alpha}\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma(\alpha)\xi^{\frac{\gamma-1}{2}+\alpha}}\int\limits_0^\infty yf(y)dy\int\limits_y^\infty(u^2-y^2)^{\alpha-1}u^{\frac{\gamma+1}{2}-\alpha}J_{\frac{\gamma-1}{2}-\alpha}(u\xi)du.$$

Let us calculate the inner integral using formula (2.12.4.17) from [456] of the form

$$\int\limits_a^\infty x^{1+\rho}(x^2-a^2)^{\beta-1}J_\rho(cx)dx=$$
$$2^{\beta-1}a^{\beta+\rho}c^{-\beta}\Gamma(\beta)[\cos(\beta\pi)J_{\rho+\beta}(ac)-\sin(\beta\pi)Y_{\rho+\beta}(ac)],$$
$$a,c,\beta>0,\qquad(2\beta+\rho)<3/2.$$

We obtain

$$\int\limits_y^\infty(u^2-y^2)^{\alpha-1}u^{\frac{\gamma+1}{2}-\alpha}J_{\frac{\gamma-1}{2}-\alpha}(u\xi)du=$$
$$2^{\alpha-1}y^{\frac{\gamma-1}{2}}\xi^{-\alpha}\Gamma(\alpha)[\cos(\alpha\pi)J_{\frac{\gamma-1}{2}}(\xi y)-\sin(\alpha\pi)Y_{\frac{\gamma-1}{2}}(\xi y)]$$

for $2\alpha+\gamma<4$ and

$$F_\gamma[(B_{\gamma,0+}^{-\alpha}f)(x)](\xi)=$$
$$\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{\xi^{\frac{\gamma-1}{2}+2\alpha}}\int\limits_0^\infty y^{\frac{\gamma+1}{2}}f(y)[\cos(\alpha\pi)J_{\frac{\gamma-1}{2}}(\xi y)-\sin(\alpha\pi)Y_{\frac{\gamma-1}{2}}(\xi y)]dy=$$
$$\xi^{-2\alpha}\int\limits_0^\infty f(t)\left[\cos(\alpha\pi)j_{\frac{\gamma-1}{2}}(\xi t)-\sin(\alpha\pi)y_{\frac{\gamma-1}{2}}(\xi t)\right]t^\gamma dt.$$

So (9.33) is proved.

Now let us consider (9.34). Let $g(x)=K_2^{0,\alpha}x^{2\alpha}f(x)$. Using the factorization (9.23), we obtain

$$F_\gamma[(B_{\gamma,-}^{-\alpha}f)](\xi)=2^{-2\alpha}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)x^\gamma K_2^{\frac{1-\gamma}{2},\alpha}K_2^{0,\alpha}x^{2\alpha}f(x)dx=$$

$$2^{-2\alpha}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\,x^\gamma\,K_2^{\frac{1-\gamma}{2},\alpha}g(x)dx=$$

$$\frac{2^{1-2\alpha}}{\Gamma(\alpha)}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\,xdx\int\limits_x^\infty (u^2-x^2)^{\alpha-1}u^{\gamma-2\alpha}g(u)du=$$

$$\frac{2^{1-2\alpha}}{\Gamma(\alpha)}\int\limits_0^\infty g(u)u^{\gamma-2\alpha}du\int\limits_0^u j_{\frac{\gamma-1}{2}}(x\xi)(u^2-x^2)^{\alpha-1}xdx.$$

Using formula (2.12.4.7) from [456] of the form

$$\int\limits_0^a x^{1-\rho}(a^2-x^2)^{\beta-1}J_\rho(cx)dx=\frac{2^{1-\rho}a^{\beta-\rho}}{c^\beta\Gamma(\rho)}s_{\rho+\beta-1,\beta-\rho}(ac),$$
$$a>0,\qquad \operatorname{Re}\beta>0,$$

we obtain for the inner integral

$$\int\limits_0^u (u^2-x^2)^{\alpha-1}j_{\frac{\gamma-1}{2}}(x\xi)\,x\,dx=$$

$$\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{\xi^{\frac{\gamma-1}{2}}}\int\limits_0^u (u^2-x^2)^{\alpha-1}J_{\frac{\gamma-1}{2}}(x\xi)\,x^{1-\frac{\gamma-1}{2}}\,dx=$$

$$\frac{\Gamma(\alpha)}{2\Gamma(\alpha+1)}u^{2\alpha}\,{}_1F_2\left(1;\alpha+1,\frac{\gamma+1}{2};-\frac{u^2\xi^2}{4}\right).$$

So

$$F_\gamma[(B_{\gamma,-}^{-\alpha}f)](\xi)=\frac{1}{2^{2\alpha}\Gamma(\alpha+1)}\int\limits_0^\infty {}_1F_2\left(1;\alpha+1,\frac{\gamma+1}{2};-\frac{u^2\xi^2}{4}\right)g(u)u^\gamma du=$$

$$\frac{1}{2^{2\alpha}\Gamma(\alpha+1)}\int\limits_0^\infty {}_1F_2\left(1;\alpha+1,\frac{\gamma+1}{2};-\frac{u^2\xi^2}{4}\right)u^\gamma K_2^{0,\alpha}u^{2\alpha}f(u)du=$$

$$\frac{2^{1-2\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int\limits_0^\infty {}_1F_2\left(1;\alpha+1,\frac{\gamma+1}{2};-\frac{u^2\xi^2}{4}\right)u^\gamma du\int\limits_u^\infty (t^2-u^2)^{\alpha-1}tf(t)dt=$$

$$\frac{2^{1-2\alpha}}{\Gamma(\alpha)\Gamma(\alpha+1)}\int\limits_0^\infty tf(t)dt\int\limits_0^t (t^2-u^2)^{\alpha-1}\,{}_1F_2\left(1;\alpha+1,\frac{\gamma+1}{2};-\frac{u^2\xi^2}{4}\right)u^\gamma du.$$

Using Wolfram Mathematica, we obtain

$$\int\limits_0^t (t^2-u^2)^{\alpha-1}\,{}_1F_2\left(1;\alpha+1,\frac{\gamma+1}{2};-\frac{u^2\xi^2}{4}\right)u^\gamma\,du=$$

$$\frac{\Gamma(\alpha)\Gamma\left(\frac{\gamma+1}{2}\right)}{2\Gamma\left(\alpha+\frac{\gamma+1}{2}\right)}t^{2\alpha+\gamma-1}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{\gamma+1}{2};-\frac{t^2\xi^2}{4}\right)$$

and

$$F_\gamma[(B_{\gamma,-}^{-\alpha}f)](\xi)=$$

$$\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{2^{2\alpha}\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\gamma+1}{2}\right)}\int\limits_0^\infty f(t)\,t^{2\alpha+\gamma}\,{}_1F_2\left(1;\alpha+1,\alpha+\frac{\gamma+1}{2};-\frac{t^2\xi^2}{4}\right)dt.$$

Since

$${}_1F_2\left(1;\alpha+1,\alpha+\frac{\gamma+1}{2};-\frac{t^2\xi^2}{4}\right)=$$

$$\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{\gamma+1}{2}\right)\sum_{n=0}^\infty\frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\alpha+\frac{\gamma+1}{2}+n\right)}\left(\frac{t\xi}{2}\right)^{2n}$$

and the Wright function through which the Hankel transform of $B_{\gamma,-}^{-\alpha}f$ is expressed in [515] is given by

$$J^1_{\frac{\gamma-1}{2},\alpha}(t\xi)=\sum_{n=0}^\infty\frac{(-1)^n}{\Gamma(\alpha+n+1)\Gamma\left(\frac{\gamma+1}{2}+\alpha+n\right)}\left(\frac{t\xi}{2}\right)^{2n+\frac{\gamma-1}{2}+2\alpha},$$

we obtain

$$F_\gamma[(B_{\gamma,-}^{-\alpha}f)](\xi)=\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{\xi^{\frac{\gamma-1}{2}}}\xi^{-2\alpha}\int\limits_0^\infty f(t)\,t^{\frac{\gamma+1}{2}}\,J^1_{\frac{\gamma-1}{2},\alpha}(t\xi)dt=$$

$$\xi^{-2\alpha}\int\limits_0^\infty j^1_{\frac{\gamma-1}{2},\alpha}(t\xi)f(t)t^\gamma\,dt.$$

Thus, (9.34) is proved.

$\square$

Since $F_\gamma[(B_\gamma^n f)](\xi)=(-1)^n\xi^{2n}F_\gamma[f](\xi)$, we obtain for $B_{\gamma,0+}^\alpha f$, $B_{\gamma,-}^\alpha f\in L_1^\gamma(\mathbb{R}_+)$

$$F_\gamma[(B_{\gamma,0+}^\alpha f)(x)](\xi)=F_\gamma[(B_\gamma^n B_{\gamma,0+}^{-(n-\alpha)}f)(x)](\xi)=$$

$$(-1)^n \xi^{2n} F_\gamma [B_{\gamma,0+}^{-(n-\alpha)} f(x)](\xi) =$$

$$(-1)^n \xi^{2\alpha} \int\limits_0^\infty f(t) \left[\cos((n-\alpha)\pi) j_{\frac{\gamma-1}{2}}(\xi t) - \sin((n-\alpha)\pi) y_{\frac{\gamma-1}{2}}(\xi t) \right] t^\gamma dt,$$

$$n = [\alpha] + 1, \qquad 4(n-\alpha) - 2 < \gamma < 4 - 2(n-\alpha)$$

and

$$F_\gamma [(B_{\gamma,-}^\alpha f)(x)](\xi) = F_\gamma [(B_\gamma^n B_{\gamma,-}^{-(n-\alpha)} f)(x)](\xi) =$$

$$(-1)^n \xi^{2n} F_\gamma [B_{\gamma,-}^{-(n-\alpha)} f(x)](\xi) =$$

$$(-1)^n \xi^{2\alpha} \int\limits_0^\infty j^1_{\frac{\gamma-1}{2}, n-\alpha}(t\xi) f(t) t^\gamma dt, \qquad n = [\alpha] + 1.$$

9.3.3 The Meijer transform

The integral Meijer transform (1.58) plays the same role for the left-sided Bessel fractional derivative at a semiaxis as the Laplace transform (1.54) plays for the left-sided Riemann–Liouville fractional derivative at a semiaxis (compare (2.43) and (9.35)).

Theorem 120. *Let $\alpha > 0$. The Meijer transform of $B_{\gamma,0+}^{-\alpha}$ for proper functions is*

$$\mathcal{K}_\gamma [(B_{\gamma,0+}^{-\alpha} f)(x)](\xi) = \xi^{-2\alpha} \mathcal{K}_\gamma f(\xi). \tag{9.35}$$

Proof. Let $g(x) = I_2^{0,\alpha} f(x)$. Then using the factorization (9.24), we obtain

$$\mathcal{K}_\gamma [(B_{\gamma,0+}^{-\alpha} f)(x)](\xi) = \int\limits_0^\infty k_{\frac{\gamma-1}{2}}(x\xi) (B_{\gamma,0+}^{-\alpha} f)(x) x^\gamma \, dx =$$

$$\frac{1}{2^{2\alpha}} \int\limits_0^\infty k_{\frac{\gamma-1}{2}}(x\xi) \; I_2^{\frac{\gamma-1}{2},\alpha} I_2^{0,\alpha} f(x) x^{2\alpha+\gamma} \, dx =$$

$$\frac{1}{2^{2\alpha}} \int\limits_0^\infty k_{\frac{\gamma-1}{2}}(x\xi) \; I_2^{\frac{\gamma-1}{2},\alpha} g(x) x^{2\alpha+\gamma} \, dx =$$

$$\frac{1}{2^{2\alpha-1}\Gamma(\alpha)} \int\limits_0^\infty k_{\frac{\gamma-1}{2}}(x\xi) \, x \, dx \int\limits_0^x (x^2-u^2)^{\alpha-1} u^\gamma g(u) du =$$

$$\frac{1}{2^{2\alpha-1}\Gamma(\alpha)} \int\limits_0^\infty u^\gamma g(u) du \int\limits_u^\infty (x^2-u^2)^{\alpha-1} k_{\frac{\gamma-1}{2}}(x\xi) \, x \, dx.$$

Let us consider the inner integral. Using formula (2.16.3.7) from [456] of the form

$$\int\limits_a^\infty x^{1\pm\rho}(x^2-a^2)^{\beta-1}K_\rho(cx)dx=2^{\beta-1}a^{\beta\pm\rho}c^{-\beta}\Gamma(\beta)K_{\rho\pm\beta}(ac),\qquad a,c,\beta>0, \tag{9.36}$$

we get

$$\int\limits_u^\infty (x^2-u^2)^{\alpha-1}k_{\frac{\gamma-1}{2}}(x\xi)\,x\,dx=$$
$$\frac{2^{\frac{1-\gamma}{2}}}{\Gamma\left(\frac{\gamma+1}{2}\right)\xi^{\frac{\gamma-1}{2}}}\int\limits_u^\infty (x^2-u^2)^{\alpha-1}K_{\frac{\gamma-1}{2}}(x\xi)\,x^{1-\frac{\gamma-1}{2}}\,dx=$$
$$\frac{2^{\frac{1-\gamma}{2}}}{\Gamma\left(\frac{\gamma+1}{2}\right)\xi^{\frac{\gamma-1}{2}}}\cdot 2^{\alpha-1}u^{\alpha-\frac{\gamma-1}{2}}\xi^{-\alpha}\Gamma(\alpha)K_{\frac{\gamma-1}{2}-\alpha}(u\xi)$$

and

$$\mathcal{K}_\gamma[(B^{-\alpha}_{\gamma,0+}f)(x)](\xi)=\frac{2^{\frac{1-\gamma}{2}-\alpha}}{\Gamma\left(\frac{\gamma+1}{2}\right)\xi^{\frac{\gamma-1}{2}+\alpha}}\int\limits_0^\infty u^{\alpha+\frac{\gamma+1}{2}}K_{\frac{\gamma-1}{2}-\alpha}(u\xi)g(u)du=$$
$$\frac{2^{\frac{3-\gamma}{2}-\alpha}}{\Gamma(\alpha)\Gamma\left(\frac{\gamma+1}{2}\right)\xi^{\frac{\gamma-1}{2}+\alpha}}\int\limits_0^\infty u^{\frac{\gamma+1}{2}-\alpha}K_{\frac{\gamma-1}{2}-\alpha}(u\xi)du\int\limits_0^u (u^2-t^2)^{\alpha-1}tf(t)dt=$$
$$\frac{2^{\frac{3-\gamma}{2}-\alpha}}{\Gamma(\alpha)\Gamma\left(\frac{\gamma+1}{2}\right)\xi^{\frac{\gamma-1}{2}+\alpha}}\int\limits_0^\infty tf(t)dt\int\limits_t^\infty (u^2-t^2)^{\alpha-1}u^{\frac{\gamma+1}{2}-\alpha}K_{\frac{\gamma-1}{2}-\alpha}(u\xi)du.$$

Using again (9.36), we can write

$$\int\limits_t^\infty (u^2-t^2)^{\alpha-1}u^{\frac{\gamma+1}{2}-\alpha}K_{\frac{\gamma-1}{2}-\alpha}(u\xi)du=2^{\alpha-1}t^{\frac{\gamma-1}{2}}\xi^{-\alpha}\Gamma(\alpha)K_{\frac{\gamma-1}{2}}(t\xi)$$

and

$$\mathcal{K}_\gamma[(B^{-\alpha}_{\gamma,0+}f)(x)](\xi)=$$
$$\frac{2^{\frac{3-\gamma}{2}-\alpha}}{\Gamma(\alpha)\Gamma\left(\frac{\gamma+1}{2}\right)\xi^{\frac{\gamma-1}{2}+\alpha}}\cdot 2^{\alpha-1}\xi^{-\alpha}\Gamma(\alpha)\int\limits_0^\infty f(t)K_{\frac{\gamma-1}{2}}(t\xi)t^{\frac{\gamma+1}{2}}dt=$$

$$\xi^{-2\alpha}\int\limits_0^\infty f(t)k_{\frac{\gamma-1}{2}}(t\xi)t^\gamma\,dt=\xi^{-2\alpha}\mathcal{K}_\gamma f. \qquad \square$$

Lemma 38. *Let $n\in\mathbb{N}$ and let the Meijer transform of $B_\gamma^n f$ exist. Then for $0\leq\gamma<1$*

$$\mathcal{K}_\gamma[B_\gamma^n f](\xi)=\xi^{2n}\mathcal{K}_\gamma[f](\xi)-$$
$$\sum_{k=1}^n\xi^{2k-1-\gamma}B_\gamma^{n-k}f(0+)-\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to 0+}\sum_{k=1}^n\xi^{2k-2}x^\gamma\frac{d}{dx}[B_\gamma^{n-k}f(x)], \tag{9.37}$$

for $\gamma=1$

$$\mathcal{K}_\gamma[B_\gamma^n f](\xi)=\xi^{2n}\mathcal{K}_\gamma[f](\xi)-\sum_{k=1}^n\xi^{2k-1-\gamma}B_\gamma^{n-k}f(0+)+$$
$$\lim_{x\to 0+}\sum_{k=1}^n\xi^{2k-2}\ln x\xi\frac{d}{dx}[B_\gamma^{n-k}f(x)], \tag{9.38}$$

and for $1<\gamma$

$$\mathcal{K}_\gamma[B_\gamma^n f](\xi)=\xi^{2n}\mathcal{K}_\gamma[f](\xi)-$$
$$\sum_{k=1}^n\xi^{2k-1-\gamma}B_\gamma^{n-k}f(0+)-\frac{1}{\gamma-1}\lim_{x\to 0+}\sum_{k=1}^n\xi^{2k-1-\gamma}x\frac{d}{dx}[B_\gamma^{n-k}f(x)], \tag{9.39}$$

where

$$B_\gamma^{n-k}f(0+)=\lim_{x\to+0}B_\gamma^{n-k}f(x).$$

Proof. Let us find $\mathcal{K}_\gamma[B_\gamma^n f](\xi)$:

$$\mathcal{K}_\gamma[B_\gamma^n f](\xi)=\int\limits_0^\infty k_{\frac{\gamma-1}{2}}(x\xi)\,[B_\gamma^n f(x)]\,x^\gamma\,dx=$$
$$\int\limits_0^\infty k_{\frac{\gamma-1}{2}}(x\xi)\,\frac{d}{dx}x^\gamma\frac{d}{dx}[B_\gamma^{n-1}f(x)]\,dx=$$
$$k_{\frac{\gamma-1}{2}}(x\xi)\,x^\gamma\frac{d}{dx}[B_\gamma^{n-1}f(x)]\bigg|_{x=0}^\infty-\int\limits_0^\infty x^\gamma\frac{d}{dx}k_{\frac{\gamma-1}{2}}(x\xi)\,\frac{d}{dx}[B_\gamma^{n-1}f(x)]\,dx=$$
$$-k_{\frac{\gamma-1}{2}}(x\xi)\,x^\gamma\frac{d}{dx}[B_\gamma^{n-1}f(x)]\bigg|_{x=0}+\left(x^\gamma\frac{d}{dx}k_{\frac{\gamma-1}{2}}(x\xi)\right)[B_\gamma^{n-1}f(x)]\bigg|_{x=0}+$$

$$\int_0^\infty [B_\gamma k_{\frac{\gamma-1}{2}}(x\xi)][B_\gamma^{n-1}f(x)]x^\gamma\,dx = -k_{\frac{\gamma-1}{2}}(x\xi)\,x^\gamma\frac{d}{dx}[B_\gamma^{n-1}f(x)]\Big|_{x=0}+$$

$$\left(x^\gamma\frac{d}{dx}k_{\frac{\gamma-1}{2}}(x\xi)\right)[B_\gamma^{n-1}f(x)]\Big|_{x=0}+\xi^2\int_0^\infty k_{\frac{\gamma-1}{2}}(x\xi)\,[B_\gamma^{n-1}f(x)]x^\gamma\,dx = ...$$

$$... = \xi^{2n}\int_0^\infty k_{\frac{\gamma-1}{2}}(x\xi)\,f(x)x^\gamma\,dx+$$

$$\sum_{k=0}^{n-1}\xi^{2k}\left(\left(x^\gamma\frac{d}{dx}k_{\frac{\gamma-1}{2}}(x\xi)\right)[B_\gamma^{n-1-k}f(x)]-k_{\frac{\gamma-1}{2}}(x\xi)\,x^\gamma\frac{d}{dx}[B_\gamma^{n-1-k}f(x)]\right)\Big|_{x=0}.$$

Let $0\leq\gamma<1$. Then using (1.27), we obtain

$$\lim_{x\to 0+}k_{\frac{\gamma-1}{2}}(x\xi)\,x^\gamma\frac{d}{dx}[B_\gamma^{n-1-k}f(x)] = \frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to 0+}x^\gamma\frac{d}{dx}[B_\gamma^{n-1-k}f(x)].$$

For $\gamma=1$ using (1.28) we obtain

$$\lim_{x\to 0+}k_0(x\xi)\,\frac{d}{dx}[B_\gamma^{n-1-k}f(x)] = -\lim_{x\to 0+}\ln x\xi\,\frac{d}{dx}[B_\gamma^{n-1-k}f(x)].$$

When $1<\gamma$ using (1.26) we obtain

$$\lim_{x\to 0+}k_{\frac{\gamma-1}{2}}(x\xi)\,x^\gamma\frac{d}{dx}[B_\gamma^{n-1-k}f(x)] = \frac{1}{\gamma-1}\lim_{x\to 0+}x\xi^{1-\gamma}\frac{d}{dx}[B_\gamma^{n-1-k}f(x)].$$

Next we have

$$\frac{d}{dx}k_{\frac{\gamma-1}{2}}(x\xi) = -\frac{2^{\frac{1-\gamma}{2}}\xi^{\frac{3-\gamma}{2}}x^{\frac{1-\gamma}{2}}}{\Gamma\left(\frac{\gamma+1}{2}\right)}K_{\frac{\gamma+1}{2}}(x\xi),$$

and using (1.18) for small x

$$x^\gamma\frac{d}{dx}k_{\frac{\gamma-1}{2}}(x\xi) = -\frac{2^{\frac{1-\gamma}{2}}}{\Gamma\left(\frac{\gamma+1}{2}\right)}x^{\frac{\gamma+1}{2}}\xi^{\frac{3-\gamma}{2}}K_{\frac{\gamma+1}{2}}(x\xi)\sim$$

$$-\frac{2^{\frac{1-\gamma}{2}}}{\Gamma\left(\frac{\gamma+1}{2}\right)}x^{\frac{\gamma+1}{2}}\xi^{\frac{3-\gamma}{2}}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{2^{1-\frac{\gamma+1}{2}}}(\xi x)^{-\frac{\gamma+1}{2}} = -\xi^{1-\gamma},\qquad x\to 0+.$$

Therefore,

$$\lim_{x\to 0+}\left(x^\gamma\frac{d}{dx}k_{\frac{\gamma-1}{2}}(x\xi)\right)[B_\gamma^{n-1-k}f(x)] = -\xi^{1-\gamma}B_\gamma^{n-1-k}f(0+),$$

and for $0 \le \gamma < 1$

$$\mathcal{K}_\gamma[B_\gamma^n f](\xi) = \xi^{2n}\mathcal{K}_\gamma[f](\xi) - \sum_{k=0}^{n-1} \xi^{2k+1-\gamma} B_\gamma^{n-1-k} f(0+) -$$

$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma \Gamma\left(\frac{\gamma+1}{2}\right)} \sum_{k=0}^{n-1} \xi^{2k} \lim_{x\to 0+} x^\gamma \frac{d}{dx}[B_\gamma^{n-1-k} f(x)] =$$

$$\xi^{2n}\mathcal{K}_\gamma[f](\xi) - \sum_{k=1}^{n} \xi^{2k-1-\gamma} B_\gamma^{n-k} f(0+) -$$

$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma \Gamma\left(\frac{\gamma+1}{2}\right)} \lim_{x\to 0+} \sum_{k=1}^{n} \xi^{2k-2} x^\gamma \frac{d}{dx}[B_\gamma^{n-k} f(x)],$$

for $\gamma = 1$

$$\mathcal{K}_\gamma[B_\gamma^n f](\xi) = \xi^{2n}\mathcal{K}_\gamma[f](\xi) - \sum_{k=1}^{n} \xi^{2k-1-\gamma} B_\gamma^{n-k} f(0+) +$$

$$\lim_{x\to 0+} \sum_{k=1}^{n} \xi^{2k-2} \ln x\xi \frac{d}{dx}[B_\gamma^{n-k} f(x)],$$

and for $1 < \gamma$

$$\mathcal{K}_\gamma[B_\gamma^n f](\xi) =$$

$$\xi^{2n}\mathcal{K}_\gamma[f](\xi) - \sum_{k=1}^{n} \xi^{2k-1-\gamma} B_\gamma^{n-k} f(0+) -$$

$$\frac{1}{\gamma-1} \lim_{x\to 0+} \sum_{k=1}^{n} \xi^{2k-1-\gamma} x \frac{d}{dx}[B_\gamma^{n-k} f(x)]. \qquad \square$$

Remark 19. *Let $n \in \mathbb{N}$, let $\frac{d}{dx}[B_\gamma^{n-k} f(x)]$ be bounded, let the Meijer transform of $B_\gamma^n f$ exist, and let $\gamma \neq 1$. Then*

$$\mathcal{K}_\gamma[B_\gamma^n f](\xi) = \xi^{2n}\mathcal{K}_\gamma[f](\xi) - \sum_{k=1}^{n} \xi^{2k-1-\gamma} B_\gamma^{n-k} f(0+). \tag{9.40}$$

If $\frac{d}{dx}[B_\gamma^{n-k} f(x)] \sim x^\beta$, $\beta > 0$, when $x \to 0+$, then (9.40) holds for $\gamma = 1$.

Remark 20. *Since $k_{-\frac{1}{2}}(x) = e^{-x}$. we have*

$$\mathcal{K}_0[f](\xi) = \mathcal{L}[f](\xi),$$

where $\mathcal{L}[f]$ is a Laplace transform of f. It is well known that

$$\mathcal{L}[f''](\xi)=\xi^2\mathcal{L}[f](\xi)-\xi f(0)-f'(0).$$

From the other side,

$$\left.\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\right|_{\gamma=0}=1,\qquad \left.\sum_{k=1}^{n}x^\gamma\frac{d}{dx}[B_\gamma^{n-k}f(x)]\right|_{\gamma=0,n=1}=f'(x),$$

and

$$\mathcal{K}_0[B_0f](\xi)=Lf''(\xi)=\xi^2\mathcal{K}_0[f](\xi)-\xi f(0)-f'(0)=\mathcal{L}[f''](\xi).$$

The same situation holds for $\mathcal{K}_0[B_0^nf](\xi)$.

Theorem 121. *Let $n=[\alpha]+1$ for fractional α and $n=\alpha$ for $\alpha\in\mathbb{N}$ and let the Meijer transform of the left-sided fractional Bessel derivatives on a semiaxis of the Riemann–Liouville type $B_{\gamma,0+}^\alpha f$ exist. Then*

for $0\leq\gamma<1$

$$\mathcal{K}_\gamma[B_{\gamma,0+}^\alpha f](\xi)=\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=1}^{n}\xi^{2k-1-\gamma}B_{\gamma,0+}^{\alpha-k}f(0+)-$$
$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to0+}\sum_{k=1}^{n}x^\gamma\frac{d}{dx}B_{\gamma,0+}^{\alpha-k}f(x),\tag{9.41}$$

for $\gamma=1$

$$\mathcal{K}_\gamma[B_{\gamma,0+}^\alpha f](\xi)=$$
$$\xi^{2n}\mathcal{K}_\gamma[f](\xi)-\sum_{k=1}^{n}\xi^{2k-1-\gamma}B_{\gamma,0+}^{\alpha-k}f(0+)+\lim_{x\to0+}\sum_{k=1}^{n}\ln x\frac{d}{dx}B_{\gamma,0+}^{\alpha-k}f(x),\tag{9.42}$$

and for $1<\gamma$

$$\mathcal{K}_\gamma[B_{\gamma,0+}^\alpha f](\xi)=$$
$$\xi^{2n}\mathcal{K}_\gamma[f](\xi)-\sum_{k=1}^{n}\xi^{2k-1-\gamma}B_{\gamma,0+}^{\alpha-k}f(0+)-\frac{1}{\gamma-1}\lim_{x\to0+}\sum_{k=1}^{n}x\frac{d}{dx}B_{\gamma,0+}^{\alpha-k}f(x),\tag{9.43}$$

where

$$B_{\gamma,0+}^{\alpha-k}f(0+)=\lim_{x\to+0}B_{\gamma,0+}^{\alpha-k}f(x).$$

Proof. Using (9.40) and (9.35) for $0 \leq \gamma < 1$, we obtain

$$
\begin{aligned}
&\mathcal{K}_\gamma[B^\alpha_{\gamma,0+}f](\xi) = \mathcal{K}_\gamma[B^n_\gamma(IB^{n-\alpha}_{\gamma,0+}f)(x)](\xi) = \\
&\xi^{2n}\mathcal{K}_\gamma[(IB^{n-\alpha}_{\gamma,0+}f)(x)](\xi) - \sum_{k=1}^{n}\xi^{2k-1-\gamma}B^{n-k}_\gamma(IB^{n-\alpha}_{\gamma,0+}f)(x)|_{x=0} - \\
&\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to 0+}\sum_{k=1}^{n}x^\gamma\frac{d}{dx}[B^{n-k}_\gamma(IB^{n-\alpha}_{\gamma,0+}f)(x)] = \\
&\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi) - \sum_{k=1}^{n}\xi^{2k-1-\gamma}B^{\alpha-k}_{\gamma,0+}f(0+) - \\
&\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to 0}\sum_{k=1}^{n}x^\gamma\frac{d}{dx}(B^{\alpha-k}_{\gamma,0+}f)(x),
\end{aligned}
$$

where we put

$$
\lim_{x\to+0}B^{\alpha-k}_{\gamma,0+}f(x) = B^{\alpha-k}_{\gamma,0+}f(0+).
$$

Similarly, we get (9.45) and (9.43). □

Theorem 122. *Let $n = [\alpha]+1$ for fractional α and $n = \alpha$ for $\alpha \in \mathbb{N}$ and let the Meijer transform of the left-sided fractional Bessel derivatives on a semiaxis of Gerasimov–Caputo type $\mathcal{B}^\alpha_{\gamma,0+}f$ exist. Then*

for $0 \leq \gamma < 1$

$$
\begin{aligned}
&\mathcal{K}_\gamma[\mathcal{B}^\alpha_{\gamma,0+}f](\xi) = \xi^{2\alpha}\mathcal{K}_\gamma[f](\xi) - \sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}B^k_\gamma f(0+) - \\
&\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to 0+}\sum_{k=0}^{n-1}\xi^{2\alpha-2k-2}x^\gamma\frac{d}{dx}[B^k_\gamma f(x)],
\end{aligned}
\tag{9.44}
$$

for $\gamma = 1$

$$
\begin{aligned}
&\mathcal{K}_\gamma[\mathcal{B}^\alpha_{\gamma,0+}f](\xi) = \xi^{2\alpha}\mathcal{K}_\gamma[f](\xi) - \sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}B^k_\gamma f(0+) + \\
&\lim_{x\to 0+}\sum_{k=0}^{n-1}\xi^{2\alpha-2k-2}\ln x\xi\frac{d}{dx}[B^k_\gamma f(x)],
\end{aligned}
\tag{9.45}
$$

and for $1<\gamma$

$$\mathcal{K}_\gamma[\mathcal{B}^\alpha_{\gamma,0+}f](\xi)=\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}\,B^k_\gamma f(0+)-$$

$$\frac{1}{\gamma-1}\lim_{x\to0+}\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}\,x\frac{d}{dx}[B^k_\gamma f(x)], \tag{9.46}$$

where

$$B^{\alpha-k}_{\gamma,0+}f(0+)=\lim_{x\to+0}B^{\alpha-k}_{\gamma,0+}f(x).$$

Proof. Using (9.35) and (9.40) for $0\le\gamma<1$, we obtain

$$\mathcal{K}_\gamma[\mathcal{B}^\alpha_{\gamma,0+}f](\xi)=\mathcal{K}_\gamma[(IB^{n-\alpha}_{\gamma,0+}B^n_\gamma f)(x)](\xi)=\xi^{2\alpha-2n}\mathcal{K}_\gamma[B^n_\gamma f](\xi)=$$

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=1}^{n}\xi^{2\alpha-2n+2k-1-\gamma}\,B^{n-k}_\gamma f(0+)-$$

$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to0+}\sum_{k=1}^{n}\xi^{2\alpha-2n+2k-2}x^\gamma\frac{d}{dx}[B^{n-k}_\gamma f(x)]=$$

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}\,B^k_\gamma f(0+)-$$

$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to0+}\sum_{k=0}^{n-1}\xi^{2\alpha-2k-2}x^\gamma\frac{d}{dx}[B^k_\gamma f(x)],$$

where we put

$$\lim_{x\to+0}B^k_{\gamma,0+}f(x)=B^k_{\gamma,0+}f(0+).$$

Similarly, for $\gamma=1$ we have

$$\mathcal{K}_\gamma[\mathcal{B}^\alpha_{\gamma,0+}f](\xi)=\mathcal{K}_\gamma[(IB^{n-\alpha}_{\gamma,0+}B^n_\gamma f)(x)](\xi)=\xi^{2\alpha-2n}\mathcal{K}_\gamma[B^n_\gamma f](\xi)=$$

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=1}^{n}\xi^{2\alpha-2n+2k-1-\gamma}\,B^{n-k}_\gamma f(0+)+$$

$$\lim_{x\to0+}\sum_{k=1}^{n}\xi^{2\alpha-2n+2k-2}\ln x\xi\frac{d}{dx}[B^{n-k}_\gamma f(x)]=$$

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}\,B^k_\gamma f(0+)+$$

$$\lim_{x\to 0+}\sum_{k=0}^{n-1}\xi^{2\alpha-2k-2}\ln x\xi\,\frac{d}{dx}[B_\gamma^k f(x)]$$

and for $\gamma > 1$

$$\mathcal{K}_\gamma[\mathcal{B}^\alpha_{\gamma,0+}f](\xi)=\mathcal{K}_\gamma[(IB^{n-\alpha}_{\gamma,0+}B^n_\gamma f)(x)](\xi)=\xi^{2\alpha-2n}\mathcal{K}_\gamma[B^n_\gamma f](\xi)=$$

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}B^k_\gamma f(0+)-$$

$$\frac{1}{\gamma-1}\lim_{x\to 0+}\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}x\frac{d}{dx}[B^k_\gamma f(x)]. \qquad \square$$

Remark 21. *Let $k\in\mathbb{N}$, let $\frac{d}{dx}[B^k_\gamma f(x)]$ be bounded, let the Meijer transform of $\mathcal{B}^\alpha_{\gamma,0+}f$ exist, and let $\gamma\neq 1$. Then*

$$\mathcal{K}_\gamma[\mathcal{B}^\alpha_{\gamma,0+}f](\xi)=\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}B^k_\gamma f(0+). \tag{9.47}$$

If $\frac{d}{dx}[B^k_\gamma f(x)]\sim x^\beta$, $\beta>0$, when $x\to 0+$, then (9.47) holds for $\gamma=1$.

9.3.4 Generalized Whittaker transform

Theorem 123. *The generalized Whittaker transform of $B^{-\alpha}_{\gamma,0+}$ for proper functions is*

$$\left(W^{\frac{\gamma-1}{2}}_{\rho,\frac{\gamma-1}{4}}B^{-\alpha}_{\gamma,0+}f\right)(x)=C(\gamma,\alpha,\rho)x^{-2\alpha}\left(W^{\frac{\gamma-1}{2}}_{\rho+\alpha,\frac{\gamma-1}{4}}f\right)(x),$$

where

$$C(\gamma,\alpha,\rho)=\frac{\Gamma\left(\frac{\gamma+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\alpha-\rho\right)}{2^{2\alpha}\Gamma\left(\frac{\gamma+1}{4}-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\rho\right)}.$$

Proof. We have

$$\left(W^{\frac{\gamma-1}{2}}_{\rho,\frac{\gamma-1}{4}}B^{-\alpha}_{\gamma,0+}f\right)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_0^\infty (xt)^{\frac{\gamma-1}{2}}e^{\frac{x^2t^2}{2}}W_{\rho,\frac{\gamma-1}{4}}(x^2t^2)dt\times$$

$$\int\limits_0^t\left(\frac{y}{t}\right)^\gamma\left(\frac{t^2-y^2}{2x}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{t^2}\right)f(y)dy=$$

$$\frac{x^{\frac{\gamma-1}{2}}}{2^{2\alpha-1}\Gamma(2\alpha)}\int\limits_0^\infty f(y)y^\gamma dy\int\limits_y^\infty t^{\frac{\gamma-1}{2}-\gamma-2\alpha+1}e^{\frac{x^2t^2}{2}}(t^2-y^2)^{2\alpha-1}\times$$

$$W_{\rho,\frac{\gamma-1}{4}}(x^2t^2)\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{t^2}\right)dt.$$

Using the formula

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right),$$

we obtain

$${}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{y^2}{t^2}\right)=\left(\frac{y}{t}\right)^{1-\gamma-2\alpha}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{t^2}{y^2}\right)$$

and

$$\left(W^{\frac{\gamma-1}{2}}_{\rho,\frac{\gamma-1}{4}}B^{-\alpha}_{\gamma,0+}f\right)(x)=$$
$$\frac{x^{\frac{\gamma-1}{2}}}{2^{2\alpha-1}\Gamma(2\alpha)}\int\limits_0^\infty f(y)y^{1-2\alpha}dy\int\limits_y^\infty t^{\frac{\gamma-1}{2}}e^{\frac{x^2t^2}{2}}(t^2-y^2)^{2\alpha-1}\times$$
$$W_{\rho,\frac{\gamma-1}{4}}(x^2t^2)\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{t^2}{y^2}\right)dt.$$

Let us consider the inner integral. We have

$$\int\limits_y^\infty t^{\frac{\gamma-1}{2}}e^{\frac{x^2t^2}{2}}(t^2-y^2)^{2\alpha-1}W_{\rho,\frac{\gamma-1}{4}}(x^2t^2)\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{t^2}{y^2}\right)dt=$$
$$\{t^2\to t,\ y^2=p\}=$$
$$\frac{1}{2}\int\limits_p^\infty t^{\frac{\gamma-1}{4}-\frac{1}{2}}e^{\frac{x^2t}{2}}(t-p)^{2\alpha-1}W_{\rho,\frac{\gamma-1}{4}}(x^2t)\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{t}{p}\right)dt.$$

Using formula (2.21.8.2) from [457] of the form

$$\int\limits_p^\infty t^{\frac{a+b-c-1}{2}}(t-p)^{c-1}e^{\frac{\sigma t}{2}}W_{\rho,\frac{a+b-c}{2}}(\sigma t)\,{}_2F_1\left(a,b;c;1-\frac{t}{p}\right)dt=$$
$$\frac{p^{\frac{a+b-1}{2}}}{\sigma^{\frac{c}{2}}}\frac{\Gamma(c)\,\Gamma\left(\frac{a-b-c+1}{2}-\rho\right)\Gamma\left(\frac{b-a-c+1}{2}-\rho\right)}{\Gamma\left(\frac{a+b-c+1}{2}-\rho\right)\Gamma\left(\frac{c-a-b+1}{2}-\rho\right)}e^{\frac{\sigma p}{2}}W_{\rho+\frac{c}{2},\frac{a-b}{2}}(\sigma p),$$
$$p,\operatorname{Re}c>0,\ \operatorname{Re}(c+2\rho)<1-|\operatorname{Re}(a-b)|;\ |\arg\sigma|<\frac{3\pi}{2},$$

we obtain

$$a=\alpha+\frac{\gamma-1}{2},\ b=\alpha,\ c=2\alpha,\ \sigma=x^2,\ 2\alpha+2\rho<1-\left|\frac{\gamma-1}{2}\right|$$

and

$$\frac{1}{2}\int\limits_p^\infty t^{\frac{\gamma-1}{4}-\frac{1}{2}}e^{\frac{x^2t}{2}}(t-p)^{2\alpha-1}W_{\rho,\frac{\gamma-1}{4}}(x^2t)\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{t}{p}\right)dt=$$

$$\frac{1}{2}\frac{p^{\alpha+\frac{\gamma-3}{4}}}{x^{2\alpha}}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\gamma+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\gamma+1}{4}-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\rho\right)}e^{\frac{x^2p}{2}}W_{\rho+\alpha,\frac{\gamma-1}{4}}(x^2p)=$$

$$\frac{1}{2}\frac{y^{2\alpha+\frac{\gamma-3}{2}}}{x^{2\alpha}}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\gamma+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\gamma+1}{4}-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\rho\right)}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\gamma-1}{4}}(x^2y^2)=$$

$$A(\gamma,\alpha,\rho)x^{-2\alpha}y^{2\alpha+\frac{\gamma-3}{2}}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\gamma-1}{4}}(x^2y^2),$$

where

$$A(\gamma,\alpha,\rho)=\frac{1}{2}\frac{\Gamma(2\alpha)\,\Gamma\left(\frac{\gamma+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\alpha-\rho\right)}{\Gamma\left(\frac{\gamma+1}{4}-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\rho\right)}.$$

Then

$$\left(W^{\frac{\gamma-1}{2}}_{\rho,\frac{\gamma-1}{4}}B^{-\alpha}_{\gamma,0+}f\right)(x)=$$

$$A(\gamma,\alpha,\rho)\frac{x^{\frac{\gamma-1}{2}-2\alpha}}{2^{2\alpha-1}\Gamma(2\alpha)}\int\limits_0^\infty f(y)y^{\frac{\gamma-1}{2}}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\gamma-1}{4}}(x^2y^2)dy=$$

$$C(\gamma,\alpha,\rho)x^{-2\alpha}\int\limits_0^\infty f(y)(xy)^{\frac{\gamma-1}{2}}e^{\frac{x^2y^2}{2}}W_{\rho+\alpha,\frac{\gamma-1}{4}}(x^2y^2)dy=$$

$$C(\gamma,\alpha,\rho)x^{-2\alpha}\left(W^{\frac{\gamma-1}{2}}_{\rho+\alpha,\frac{\gamma-1}{4}}f\right)(x),$$

where

$$C(\gamma,\alpha,\rho)=\frac{\Gamma\left(\frac{\gamma+1}{4}-\alpha-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\alpha-\rho\right)}{2^{2\alpha}\Gamma\left(\frac{\gamma+1}{4}-\rho\right)\Gamma\left(\frac{3-\gamma}{4}-\rho\right)}.\qquad\square$$

9.4 Further properties of fractional powers of Bessel operators

9.4.1 Resolvent for the right-sided fractional Bessel integral on a semiaxis

We consider resolvents for integral operators at the standard setting (cf. [266]). For any linear operator A on some Banach space Φ, let us consider the equation

$$(A-\lambda I)\, g = f, \qquad \lambda \in C, \qquad f, g \in \Phi, \tag{9.48}$$

and its solution as resolvent operator due to the well-known formula from [266]

$$g = R_\lambda f = (A-\lambda I)^{-1} f = -(\lambda I - A)^{-1} f = -\frac{1}{\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1} f =$$
$$-\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{1}{\lambda}A\right)^k f = -\frac{1}{\lambda} f - \frac{1}{\lambda}\left(\sum_{k=1}^{\infty}\frac{A^k}{\lambda^k} f\right). \tag{9.49}$$

Note that if integral representations are known for all powers A^k, then an integral representation for the resolvent readily follows from (9.48), of course if the series are convergent. In this way it is possible to get resolvent operators for the Riemann–Liouville fractional integrals, known as the Hille–Tamarkin formula [494] (in fact first proved by M. M. Dzhrbashyan in [98]), and also for the Erdélyi–Kober fractional integrals, but we omit it here.

Theorem 124. *For a resolvent operator of $(B_{\gamma,-}^{-\alpha})$, the following formula is valid:*

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda^2}\int\limits_x^{+\infty} f(y)\left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} dy \int\limits_0^1 t^{\alpha-1}(1-t)^{\alpha-1}\times$$
$$\left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha-\frac{\gamma-1}{2}} E_{(\alpha,\alpha),(\alpha,\alpha)}\left(\frac{1}{\lambda}\left(\frac{1}{4}\frac{t(1-t)(y^2-x^2)^2}{y^2-(y^2-x^2)t}\right)^{\alpha}\right) dt,$$

with the Wright or generalized (multi-index) Mittag-Leffler function

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+k/\rho_1)\ldots\Gamma(\mu_m+k/\rho_m)} \tag{9.50}$$

(cf. [252]).

Proof. Let us consider

$$(B_{\gamma,-}^{-\alpha} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_{x}^{+\infty} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} \times$$

$${}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy.$$

Using the group property or index law, we have

$$(B_{\gamma,-}^{-\alpha} f)^k = B_{\gamma,-}^{-\alpha k} f.$$

Then from (9.49) we obtain

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda}\left(\sum_{k=1}^{\infty} \frac{1}{\lambda^k} B_{\gamma,-}^{-\alpha k} f\right) = -\frac{1}{\lambda} f - -\frac{1}{\lambda}\left(\sum_{k=1}^{\infty} \frac{1}{\lambda^k \Gamma(2\alpha k)} \times\right.$$

$$\left.\int\limits_{x}^{+\infty} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1} {}_2F_1\left(\alpha k+\frac{\gamma-1}{2}, \alpha k; 2\alpha k; 1-\frac{x^2}{y^2}\right) f(y)dy\right) =$$

$$-\frac{1}{\lambda} f - \frac{1}{\lambda}\left(\int\limits_{x}^{+\infty} f(y)dy \sum_{k=1}^{\infty}\left[\frac{1}{\lambda^k \Gamma(2\alpha k)}\left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1} \times\right.\right.$$

$$\left.\left.{}_2F_1\left(\alpha k+\frac{\gamma-1}{2}, \alpha k; 2\alpha k; 1-\frac{x^2}{y^2}\right)\right]\right).$$

Using the integral representation for the hypergeometric function for $c-a-b>0$,

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_{0}^{1} t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt,$$

we obtain

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda} \int\limits_{x}^{+\infty} f(y)dy \int\limits_{0}^{1} \sum_{k=1}^{\infty} \frac{1}{\lambda^k \Gamma^2(\alpha k)} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha k-1} \times$$

$$t^{\alpha k-1}(1-t)^{\alpha k-1}\left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha k-\frac{\gamma-1}{2}} dt = \{k=p+1\} =$$

$$-\frac{1}{\lambda} f - \frac{1}{\lambda} \int\limits_{x}^{+\infty} f(y)dy \int\limits_{0}^{1} \sum_{p=0}^{\infty} \frac{1}{\lambda^{p+1} \Gamma^2(\alpha(p+1))} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha(p+1)-1} \times$$

$$t^{\alpha(p+1)-1}(1-t)^{\alpha(p+1)-1}\left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha(p+1)-\frac{\gamma-1}{2}} dt = -\frac{1}{\lambda} f - \frac{1}{\lambda} \times$$

$$\int_x^{+\infty} f(y) \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} dy \int_0^1 t^{\alpha-1}(1-t)^{\alpha-1} \left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha-\frac{\gamma-1}{2}} \times$$

$$\sum_{p=0}^{\infty} \frac{1}{\lambda^{p+1}\Gamma^2(\alpha(p+1))} \left(\frac{y^2-x^2}{2y}\right)^{2\alpha p} t^{\alpha p}(1-t)^{\alpha p} \left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha p} dt =$$

$$-\frac{1}{\lambda} f - \frac{1}{\lambda^2} \int_x^{+\infty} f(y) \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} dy \int_0^1 \left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha-\frac{\gamma-1}{2}} \times$$

$$t^{\alpha-1}(1-t)^{\alpha-1} \sum_{p=0}^{\infty} \frac{1}{\Gamma^2(\alpha+\alpha p)} \left[\frac{1}{\lambda}\left(\frac{1}{4}\frac{t(1-t)(y^2-x^2)^2}{y^2-(y^2-x^2)t}\right)^{\alpha}\right]^p dt. \qquad (9.51)$$

The function in (9.51) is a special case of the Wright generalized hypergeometric function defined above as (9.50). So it follows that

$$\sum_{p=0}^{\infty} \frac{1}{\Gamma^2(\alpha+\alpha p)} \left[\frac{1}{\lambda}\left(\frac{1}{4}\frac{t(1-t)(y^2-x^2)^2}{y^2-(y^2-x^2)t}\right)^{\alpha}\right]^p =$$

$$E_{(\alpha,\alpha),(\alpha,\alpha)} \left(\frac{1}{\lambda}\left(\frac{1}{4}\frac{t(1-t)(y^2-x^2)^2}{y^2-(y^2-x^2)t}\right)^{\alpha}\right),$$

and we finally derive

$$R_\lambda f = -\frac{1}{\lambda} f - \frac{1}{\lambda^2} \int_x^{+\infty} f(y) \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} dy \int_0^1 t^{\alpha-1}(1-t)^{\alpha-1} \times$$

$$\left(1-\left(1-\frac{x^2}{y^2}\right)t\right)^{-\alpha-\frac{\gamma-1}{2}} E_{(\alpha,\alpha),(\alpha,\alpha)} \left(\frac{1}{\lambda}\left(\frac{1}{4}\frac{t(1-t)(y^2-x^2)^2}{y^2-(y^2-x^2)t}\right)^{\alpha}\right) dt. \qquad \square$$

9.4.2 The generalized Taylor formula with powers of Bessel operators

Many applications of the Riemann–Liouville fractional integrals are based on the fact that they are remainder terms in the Taylor formula. Such formulas exist also with powers of Bessel operators – they are the so-called Taylor–Delsarte series (cf. [83, 317] and especially [139]). But in the Taylor–Delsarte series not a function itself is expanded but its generalized translation; these series are in fact just operator versions of Bessel function series. But for application to numerical partial differential equation solutions, we need the classical form of the Taylor formula $f(x+t) = f(x)+\ldots$, as only with this formula we may calculate partial differential equation solutions layer by layer. Such formulas are much harder to prove. With the abovementioned motivation as a tool for solving singular partial differential equations numerically, a first

attempt to construct the generalized Taylor formula with Bessel operators was made in [225,227]. But these results were rather vague as neither coefficients nor the integral remainder term were found explicitly: For coefficients the recurrent system of equations was found and the remainder term was evaluated as a multi-term composition of simple integral operators. The solution to the problem of finding the generalized Taylor formula with Bessel operators in the explicit form was found in [268] (cf. also [234,527,531]). Of course it is based on explicit forms for fractional powers of Bessel operators.

Theorem 125. *The following generalized Taylor formula is valid for proper functions:*

$$
\begin{aligned}
f(x) = \sum_{i=1}^{k} \Bigg\{ & \frac{1}{\Gamma(2i-1)} \left(\frac{b^2-x^2}{2b}\right)^{2i-2} {}_2F_1\left(i+\frac{\gamma-1}{2}, i-1; 2i-1; 1-\frac{x^2}{b^2}\right) \times \\
& (B^{i-1} f)|_b - \frac{1}{\Gamma(2i)} \left(\frac{b^2-x^2}{2b}\right)^{2i-1} {}_2F_1\left(i+\frac{\gamma-1}{2}, i; 2i; 1-\frac{x^2}{b^2}\right) \times \\
& (DB^{i-1} f)|_b \Bigg\} + B_{\gamma,b-}^{-k}(B^k f).
\end{aligned}
$$

Theorem 126. *The following generalized Taylor formula is valid for proper functions:*

$$
\begin{aligned}
f(x) = \sum_{i=1}^{k} \Bigg\{ & \frac{1}{\Gamma(2i-1)} \left(\frac{x^2-a^2}{2x}\right)^{2i-2} \left(\frac{a}{x}\right) {}_2F_1\left(i+\frac{\gamma-1}{2}, i; 2i-1; 1-\frac{a^2}{x^2}\right) \times \\
& (C_\gamma^{i-1} f)|_a + \frac{1}{\Gamma(2i)} \left(\frac{x^2-a^2}{2x}\right)^{2i-1} \times \\
& {}_2F_1\left(i+\frac{\gamma-1}{2}, i; 2i; 1-\frac{a^2}{x^2}\right) a^\gamma \, (Dx^{-\gamma} C_\gamma^{i-1} f)|_a \Bigg\} + B_{\gamma,a+}^{-k}(C_\gamma^k f).
\end{aligned}
$$

B-potentials theory

10

10.1 Definitions of hyperbolic B-potentials, absolute convergence, and boundedness

The theory of fractional powers of elliptic operators with Bessel operator

$$B_\nu = D^2 + \frac{\nu}{x} D, \qquad D = \frac{d}{dx}$$

acting instead of all or some second derivatives in Δ is well developed (see [174,206,207,343–347,351–353,501–504,506,507,511]).

Fractional powers of hyperbolic operators, with Bessel operators instead of all or some second derivatives, are much less studied. Such operators have wide areas of application, such as singular differential equations, differential geometry, and random walks.

In this chapter we study real powers of

$$\Box_\gamma = B_{\gamma_1} - B_{\gamma_2} - \ldots - B_{\gamma_n}, \qquad B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \qquad i = 1, \ldots, n.$$

The composition method (see [229,230,234,523]) was used for the construction of $(\Box_\gamma)^{-\frac{\alpha}{2}}$, $\alpha > 0$.

10.1.1 Negative fractional powers of the hyperbolic expression with Bessel operators

We consider fractional powers of the hyperbolic expression with Bessel operators

$$\Box_\gamma = B_{\gamma_1} - B_{\gamma_2} - \ldots - B_{\gamma_n}, \qquad B_{\gamma_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \qquad i = 1, \ldots, n,$$

in S_{ev} and L_p^γ. We will call negative real powers of $\Box_\gamma$ **hyperbolic B-potentials**.

Definition 50. *Hyperbolic B-potentials $I^\alpha_{P\pm i0,\gamma}$ for $\alpha > n + |\gamma| - 2$ are defined by formulas*

$$(I^\alpha_{P\pm i0,\gamma} f)(x) = \frac{e^{\pm\frac{n-1+|\gamma'|}{2} i\pi}}{H_{n,\gamma}(\alpha)} \int\limits_{\mathbb{R}^n_+} (P \pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}} ({}^\gamma \mathbf{T}^y_x f)(x) y^\gamma dy, \qquad y^\gamma = \prod_{i=1}^n y_i^{\gamma_i}, \tag{10.1}$$

where $\gamma' = (\gamma_2, \ldots, \gamma_n)$, $|\gamma'| = \gamma_2 + \ldots + \gamma_n$,

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$$H_{n,\gamma}(\alpha)=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{2^{n-\alpha}\Gamma\left(\frac{n+|\gamma|-\alpha}{2}\right)}.$$

For $0\leq\alpha\leq n+|\gamma|-2$ *hyperbolic B-potentials* $I^{\alpha}_{P\pm i0,\gamma}$ *are defined as*

$$(I^{\alpha}_{P\pm i0,\gamma}f)(x)=(\Box_\gamma)^k(I^{\alpha+2k}_{P\pm i0,\gamma}f)(x)=$$
$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2k)}(\Box_\gamma)^k\int\limits_{\mathbb{R}^n_+}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma\,dy, \tag{10.2}$$

where $k=\left[\frac{n+|\gamma|-\alpha}{2}\right]$.

It is well known (see for example [242]) that generalized convolution of weighted generalized functions and a regular function is a regular function.

Using the property of weighted generalized functions $(P\pm i0)^{\lambda}_{\gamma}$ (see [505]) we can rewrite (10.1) as

$$(I^{\alpha}_{P\pm i0,\gamma}f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\left[\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma\,dy+\right.$$
$$\left.e^{\pm\frac{\alpha-n-|\gamma|}{2}\pi i}\int\limits_{K^-}|r(y)|^{\alpha-n-|\gamma|}({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma\,dy\right], \tag{10.3}$$

where

$$K^+=\{x:x\in\mathbb{R}^n_+:P(x)\geq 0\},\qquad K^-=\{x:x\in\mathbb{R}^n_+:P(x)\leq 0\},$$
$$r(y)=\sqrt{P(y)}=\sqrt{y_1^2-y_2^2-...-y_n^2}.$$

Function $r(y)$ is a Lorentz distance and K^+ is a part of a light cone.

Introducing the notations

$$(I^{\alpha}_{P_+,\gamma}f)(x)=\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma\,dy, \tag{10.4}$$
$$(I^{\alpha}_{P_-,\gamma}f)(x)=\int\limits_{K^-}|r(y)|^{\alpha-n-|\gamma|}({}^\gamma\mathbf{T}^y_x f)(x)y^\gamma\,dy, \tag{10.5}$$

we can write

$$(I^{\alpha}_{P\pm i0,\gamma}f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\left[(I^{\alpha}_{P_+,\gamma}f)(x)+e^{\pm\frac{\alpha-n-|\gamma|}{2}\pi i}(I^{\alpha}_{P_-,\gamma}f)(x)\right]. \tag{10.6}$$

Remark 22. *Let* $y'=(y_2,...,y_n)$, $|y'|=\sqrt{y_2^2+...+y_n^2}$, $(y')^{\gamma'}=y_2^{\gamma_2}...y_n^{\gamma_n}$. *For* $n\geq 3$, *we have*

$$(I_{P_+,\gamma}^{\alpha}f)(x)=\int\limits_0^{\infty}y_1^{\gamma_1}dy_1\int\limits_{\{|y'|<y_1\}^+}(y_1^2-|y'|^2)^{\frac{\alpha-n-|\gamma|}{2}}({}^{\gamma}\mathbf{T}_x^y f)(x)(y')^{\gamma'}dy', \tag{10.7}$$

$$(I_{P_-,\gamma}^{\alpha}f)(x)=\int\limits_0^{\infty}y_1^{\gamma_1}dy_1\int\limits_{\{|y'|>y_1\}^+}(|y'|^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}}({}^{\gamma}\mathbf{T}_x^y f)(x)(y')^{\gamma'}dy', \tag{10.8}$$

where $\{|y'|<y_1\}^+=\{y\in\mathbb{R}_+^n:|y'|<y_1\}$, $\{|y'|>y_1\}^+=\{y\in\mathbb{R}_+^n:|y'|>y_1\}$.

For $n=2$, *we have*

$$(I_{P_+,\gamma}^{\alpha}f)(x)=\int\limits_0^{\infty}y_1^{\gamma_1}dy_1\int\limits_0^{y_1}(y_1^2-y_2^2)^{\frac{\alpha-2-|\gamma|}{2}}({}^{\gamma}\mathbf{T}_x^y f)(x)y_2^{\gamma_2}dy_2,$$

$$(I_{P_-,\gamma}^{\alpha}f)(x)=\int\limits_0^{\infty}y_1^{\gamma_1}dy_1\int\limits_{y_1}^{\infty}(y_2^2-y_1^2)^{\frac{\alpha-2-|\gamma|}{2}}({}^{\gamma}\mathbf{T}_x^y f)(x)y_2^{\gamma_2}dy_2.$$

Passing to the spherical coordinates $y'=\rho\sigma$ in (10.7) and (10.8), we obtain

$$(I_{P_+,\gamma}^{\alpha}f)(x)=|S_1^+(n-1)|_{\gamma}\times$$
$$\int\limits_0^{\infty}y_1^{\gamma_1}dy_1\int\limits_0^{y_1}(y_1^2-\rho^2)^{\frac{\alpha-n-|\gamma|}{2}}\rho^{n+|\gamma'|-2}({}^{\gamma_1}T_{x_1}^{y_1})(M_{\rho}^{\gamma'})_{x'}[f(x_1,x')]d\rho, \tag{10.9}$$
$$(I_{P_-,\gamma}^{\alpha}f)(x)=|S_1^+(n-1)|_{\gamma}\times$$
$$\int\limits_0^{\infty}y_1^{\gamma_1}dy_1\int\limits_{y_1}^{\infty}(\rho^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}}\rho^{n+|\gamma'|-2}({}^{\gamma_1}T_{x_1}^{y_1})(M_{\rho}^{\gamma'})_{x'}[f(x_1,x')]d\rho, \tag{10.10}$$

where

$$(M_{\rho}^{\gamma'})_{x'}[f(x_1,x')]=\frac{1}{|S_1^+(n-1)|_{\gamma}}\int\limits_{S_1^+(n-1)}{}^{\gamma'}\mathbf{T}_{x'}^{\rho\sigma}f(x_1,x')\sigma^{\gamma'}dS$$

is the weighted spherical mean (3.183).

If $f(x)=\varphi(x_1)G(x')$, then (10.9) and (10.10) have forms

$$(I^{\alpha}_{P_+,\gamma}f)(x){=}|S_1^+(n{-}1)|_{\gamma}\times$$
$$\int\limits_0^{\infty}({}^{\gamma_1}T^{y_1}_{x_1})[\varphi(x_1)]y_1^{\gamma_1}dy_1\int\limits_0^{y_1}(M^{\gamma'}_{\rho})_{x'}[G(x')](y_1^2-\rho^2)^{\frac{\alpha-n-|\gamma|}{2}}\rho^{n+|\gamma'|-2}d\rho, \tag{10.11}$$

$$(I^{\alpha}_{P_-,\gamma}f)(x){=}|S_1^+(n{-}1)|_{\gamma}\times$$
$$\int\limits_0^{\infty}({}^{\gamma_1}T^{y_1}_{x_1})[\varphi(x_1)]y_1^{\gamma_1}dy_1\int\limits_{y_1}^{\infty}(M^{\gamma'}_{\rho})_{x'}[G(x')](\rho^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}}\rho^{n+|\gamma'|-2}d\rho. \tag{10.12}$$

10.1.2 Absolute convergence and boundedness

Theorem 127. *Let $f\in S_{ev}$ and $\alpha>n+|\gamma|-2$. Then integrals $(I^{\alpha}_{P\pm i0,\gamma}f)(x)$ converge absolutely for $x\in\mathbb{R}^n_+$.*

Proof. Let us prove absolute convergence of each term in (10.3). Passing in (10.3) to spherical coordinates $y{=}\rho\sigma$, $\rho{=}|y|$, $\sigma'{=}(\sigma_2,...,\sigma_n)$, we obtain

$$\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)({}^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy=$$
$$\int\limits_0^{\infty}\rho^{\alpha-1}d\rho\int\limits_{\{S_1^+(n),|\sigma'|<\sigma_1\}}(\sigma_1^2-|\sigma'|^2)^{\frac{\alpha-n-|\gamma|}{2}}({}^{\gamma}\mathbf{T}^{\rho\sigma}f)(x)\sigma^{\gamma}dS,$$

where

$$\{S_1^+(n),|\sigma'|<\sigma_1\}=\{\sigma'\in\mathbb{R}^{n-1}_+:\sigma_1^2+|\sigma'|^2=1,\ |\sigma'|<\sigma_1\}.$$

Using the formula ${}^{\gamma}\mathbf{T}^y_x f(x){=}{}^{\gamma}\mathbf{T}^x_y f(y)$ and the inequality $|{}^{\gamma}\mathbf{T}^y_x f(x)|{\leq}\sup\limits_{\mathbb{R}^n_+}|f(x)|$ (see [317], p. 124) and considering that $f\in S_{ev}$, we get

$$\left|\int\limits_{K^+}r^{\alpha-n-|\gamma|}(y)({}^{\gamma}\mathbf{T}^y_x f)(x)y^{\gamma}dy\right|\leq$$
$$C\int\limits_0^{\infty}\frac{\rho^{\alpha-1}}{(1+\rho^2)^{\frac{\alpha+1}{2}}}d\rho\int\limits_{S_1^+(n),|\sigma'|<\sigma_1}(\sigma_1^2-|\sigma'|^2)^{\frac{\alpha-n-|\gamma|}{2}}\sigma^{\gamma}dS<\infty,$$

for $\alpha>n+|\gamma|-2$. Similarly, (10.5) converges absolutely for $\alpha{>}n{+}|\gamma|{-}2$. So for $\alpha>n+|\gamma|-2$ integrals $(I^{\alpha}_{P\pm i0,\gamma}f)(x)$ converge absolutely. □

Corollary 22. *The integral in (10.2) also converges absolutely.*

We present here the Marcinkiewicz interpolation theorem in the following form (see [25]).

Theorem 128. *Let* $1 \leq p_i \leq q_i < \infty$ *(*$i = 1, 2$*),* $q_1 \neq q_2$*,* $0 < \tau < 1$*,* $\frac{1}{p}=\frac{1-\tau}{p_1}+\frac{\tau}{p_2}$*,* $\frac{1}{q}=\frac{1-\tau}{q_1}+\frac{\tau}{q_2}$*. If a linear operator* A *has simultaneously weak types* $(p_1, q_1)_\gamma$ *and* $(p_2, q_2)_\gamma$*, then an operator* A *has a strong type* $(p, q)_\gamma$ *and*

$$||Af||_{q,\gamma} \leq M||f||_{p,\gamma}, \tag{10.13}$$

where a constant $M = M(\gamma, \tau, \kappa, p_1, p_2, q_1, q_2)$ *and does not depend on* f *and* A*.*

Theorem 129. *Let* $n + |\gamma| - 2 < \alpha < n + |\gamma|$*,* $1 \leq p < \frac{n+|\gamma|}{\alpha}$*. For the estimate*

$$||I^\alpha_{P\pm i0,\gamma} f||_{q,\gamma} \leq C_{n,\gamma,p}||f||_{p,\gamma}, \qquad f(x) \in S_{ev} \tag{10.14}$$

to be valid, it is necessary and sufficient that $q = \frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$*. Constant* $C_{n,\gamma,p}$ *does not depend on* f*.*

Proof. Necessity. Let $n + |\gamma| - 2 < \alpha < n + |\gamma|$, let $1 < p < \frac{n+|\gamma|}{\alpha}$, and let for some q the inequality

$$||I^\alpha_{P\pm i0,\gamma} f||_{q,\gamma} \leq C_{n,\gamma,p}||f||_{p,\gamma}, \qquad f(x) \in S_{ev} \tag{10.15}$$

hold.

We show that inequality (10.15) is valid only for $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Let us obtain the required inequality for each term in the representation (10.6).

Let us consider the extension operator τ_δ: $(\tau_\delta f)(x) = f(\delta x)$, $\delta > 0$. We have

$$||\tau_\delta f||_{p,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} f^p(\delta x)x^\gamma dx\right)^{\frac{1}{p}} = \left(\delta^{-n-|\gamma|}\int\limits_{\mathbb{R}^n_+} f^p(y)y^\gamma dy\right)^{\frac{1}{p}}.$$

Therefore,

$$||\tau_\delta f||_{p,\gamma} = \delta^{-\frac{n+|\gamma|}{p}}||f||_{p,\gamma}. \tag{10.16}$$

For $(I^\alpha_{P_+,\gamma} f)(x)$, we obtain

$$(I^\alpha_{P_+,\gamma} f)(x) = \int\limits_{K^+} [y_1^2 - y_2^2 - ... - y_n^2]^{\frac{\alpha-n-|\gamma|}{2}} ({}^\gamma\mathbf{T}^y_x \tau_\delta f)(y)y^\gamma dy =$$

$$2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+} \frac{[y_1^2 - y_2^2 - ... - y_n^2]^{\frac{\alpha-n-|\gamma|}{2}} y^\gamma dy}{(xy)^{\gamma-1}}\times$$

$$\int\limits_{|x_1-y_1|}^{x_1+y_1} ... \int\limits_{|x_n-y_n|}^{x_n+y_n} f(\delta z)\prod_{i=1}^n z_i[(z_i^2 - (x_i - y_i)^2)((x_i + y_i)^2 - z_i^2)]^{\frac{\gamma_i}{2}-1}dz =$$

$$\{\delta z = s\} =$$

$$2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+}\frac{[y_1^2-y_2^2-...-y_n^2]^{\frac{\alpha-n-|\gamma|}{2}}y^\gamma dy}{(xy)^{\gamma-1}}\int\limits_{\delta|x_1-y_1|}^{\delta(x_1+y_1)}...\int\limits_{\delta|x_n-y_n|}^{\delta(x_n+y_n)}f(s)\delta^{-n}\times$$

$$\prod_{i=1}^n\frac{s_i}{\delta}\left[\left(\frac{s_i^2}{\delta^2}-(x_i-y_i)^2\right)\left((x_i+y_i)^2-\frac{s_i^2}{\delta^2}\right)\right]^{\frac{\gamma_i}{2}-1}ds =$$

$$\delta^{2n-2|\gamma|}2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+}\frac{[y_1^2-y_2^2-...-y_n^2]^{\frac{\alpha-n-|\gamma|}{2}}y^\gamma dy}{(xy)^{\gamma-1}}\times$$

$$\int\limits_{\delta|x_1-y_1|}^{\delta(x_1+y_1)}...\int\limits_{\delta|x_n-y_n|}^{\delta(x_n+y_n)}f(s)\prod_{i=1}^n s_i[(s_i^2-\delta^2(x_i-y_i)^2)(\delta^2(x_i+y_i)^2-s_i^2)]^{\frac{\gamma_i}{2}-1}ds =$$

$$\{\delta y = t\} =$$

$$\delta^{2n-2|\gamma|}2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+}\frac{\delta^{n+|\gamma|-\alpha}[t_1^2-t_2^2-...-t_n^2]^{\frac{\alpha-n-|\gamma|}{2}}\delta^{-n-|\gamma|}t^\gamma dt}{\delta^{n-|\gamma|}(xt)^{\gamma-1}}\times$$

$$\int\limits_{|\delta x_1-t_1|}^{\delta x_1+t_1}...\int\limits_{|\delta x_n-t_n|}^{\delta x_n+t_n}f(s)\prod_{i=1}^n s_i[(s_i^2-(\delta x_i-t_i)^2)((\delta x_i+t_i)^2-s_i^2)]^{\frac{\gamma_i}{2}-1}ds =$$

$$\delta^{-\alpha}2^{2n-|\gamma|}C(\gamma)\int\limits_{K^+}\frac{[t_1^2-t_2^2-...-t_n^2]^{\frac{\alpha-n-|\gamma|}{2}}t^\gamma dt}{\delta^{|\gamma|-n}(xt)^{\gamma-1}}\int\limits_{|\delta x_1-t_1|}^{\delta x_1+t_1}...\int\limits_{|\delta x_n-t_n|}^{\delta x_n+t_n}f(s)\times$$

$$\prod_{i=1}^n s_i[(s_i^2-(\delta x_i-t_i)^2)((\delta x_i+t_i)^2-s_i^2)]^{\frac{\gamma_i}{2}-1}ds =$$

$$\delta^{-\alpha}\int\limits_{K^+}({}^\gamma\mathbf{T}_t^{\delta x}f(t))[t_1^2-t_2^2-...-t_n^2]^{\frac{\alpha-n-|\gamma|}{2}}t^\gamma dt = \delta^{-\alpha}\tau_\delta(I^\alpha_{P_+,\gamma}f)(x).$$

Then

$$(I^\alpha_{P_+,\gamma}f)(x) = \delta^\alpha\tau_\delta^{-1}(I^\alpha_{P_+,\gamma}\tau_\delta f)(x). \tag{10.17}$$

Next, we have

$$||\tau_\delta^{-1}I^\alpha_{P_+,\gamma}f||_q^\gamma = \left(\int\limits_{\mathbb{R}^n_+}(\tau_\delta^{-1}(I^\alpha_{P_+,\gamma}f)(x))^q x^\gamma dx\right)^{\frac{1}{q}} =$$

$$\left(\int\limits_{\mathbb{R}^n_+}\left(\int\limits_{K^+}[y_1^2-y_2^2-...-y_n^2]^{\frac{\alpha-n-|\gamma|}{2}}({}^\gamma\mathbf{T}_y^{\frac{x}{\delta}}f)(y)y^\gamma dy\right)^q x^\gamma dx\right)^{\frac{1}{q}}=\left(\frac{x}{\delta}=t\right)=$$
$$\delta^{\frac{n+|\gamma|}{q}}||I^\alpha_{P_+,\gamma}f||^\gamma_q.$$

Hence,

$$||\tau_\delta^{-1}I^\alpha_{P_+,\gamma}f||^\gamma_q=\delta^{\frac{n+|\gamma|}{q}}||I^\alpha_{P_+,\gamma}f||^\gamma_q. \tag{10.18}$$

Using (10.16)–(10.18), we get

$$||I^\alpha_{P_+,\gamma}f||_{q,\gamma}=\delta^\alpha||\tau_\delta^{-1}I^\alpha_{P_+,\gamma}\tau_\delta f||_{q,\gamma}=$$
$$\delta^{\frac{n+|\gamma|}{q}+\alpha}||I^\alpha_{P_+,\gamma}\tau_\delta f||_{q,\gamma}\le C_{n,\gamma,p}\delta^{\frac{n+|\gamma|}{q}+\alpha}||\tau_\delta f||_{p,\gamma}=$$
$$C_{n,\gamma,p}\delta^{\frac{n+|\gamma|}{q}-\frac{n+|\gamma|}{p}+\alpha}||f||_{p,\gamma}$$

or

$$||I^\alpha_{P_+,\gamma}f(x)||_{q,\gamma}\le C_{n,\gamma,p}\ \delta^{\frac{n+|\gamma|}{q}-\frac{n+|\gamma|}{p}+\alpha}||f(x)||_{p,\gamma}. \tag{10.19}$$

If $\frac{n+|\gamma|}{q}-\frac{n+|\gamma|}{p}+\alpha>0$ or $\frac{n+|\gamma|}{q}-\frac{n+|\gamma|}{p}+\alpha<0$, then passing to the limit at $\delta\to 0$ or at $\delta\to\infty$ in (10.19) accordingly we obtain that for all functions $f\in L_p^\gamma$ the equality

$$||I^\alpha_{P_+,\gamma}f||_{q,\gamma}=0$$

holds, which is wrong. That means that inequality (10.19) is possible only if $\frac{n+|\gamma|}{q}-\frac{n+|\gamma|}{p}+\alpha=0$, i.e., for $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Necessity is proved.

Sufficiency. Let $x'=(x_2,...,x_n)$, $|x'|=\sqrt{x_2^2+...+x_n^2}$, $(x')^{\gamma'}=x_2^{\gamma_2}...x_n^{\gamma_n}$. Without loss of generality, we will assume that $f(x)\ge 0$ and $||f||_{p,\gamma}=1$.

Let $0<\delta<1$. We consider the operators

$$(I^\alpha_{P_+,\gamma,\delta}f)(x)=\int\limits_{\delta y_1^2\ge|y'|^2}r^{\alpha-n-|\gamma|}(y)({}^\gamma\mathbf{T}_x^y f)(y)y^\gamma dy$$

and

$$(I^\alpha_{P_-,\gamma,\delta}f)(x)=\int\limits_{y_1^2\le\delta|y'|^2}r^{\alpha-n-|\gamma|}(y)({}^\gamma\mathbf{T}_x^y f)(y)y^\gamma dy.$$

Let μ be some fixed real number. We introduce the notations

$$G^0_{\delta,\mu}=\{y\in\mathbb{R}^n_+:\delta y_1^2\ge|y'|^2,\ 0\le y_1\le\mu\},$$

$$G^{\infty}_{\delta,\mu} = \{y \in \mathbb{R}^n_+ : \delta y_1^2 \geq |y'|^2,\ \mu < y_1\},$$

$$K^+_{0,\delta}(y) = \begin{cases} r^{\alpha-n-|\gamma|}(y) & y \in G^0_{\delta,\mu}, \\ 0 & y \in \mathbb{R}^n_+ \setminus G^0_{\delta,\mu}, \end{cases}$$

$$K^+_{\infty,\delta}(y) = \begin{cases} r^{\alpha-n-|\gamma|}(y) & y \in G^{\infty}_{\delta,\mu}, \\ 0 & y \in \mathbb{R}^n_+ \setminus G^{\infty}_{\delta,\mu}, \end{cases}$$

$$H^0_{\delta,\mu} = \{y \in \mathbb{R}^n_+ : y_1^2 \leq \delta|y'|^2,\ |y'| \leq \mu\},$$

$$H^{\infty}_{\delta,\mu} = \{y \in \mathbb{R}^n_+ : y_1^2 \leq \delta|y'|^2,\ \mu < |y'|\},$$

$$M^+_{0,\delta}(y) = \begin{cases} r^{\alpha-n-|\gamma|}(y) & y \in H^0_{\delta,\mu}, \\ 0 & y \in \mathbb{R}^n_+ \setminus H^0_{\delta,\mu}, \end{cases}$$

$$M^+_{\infty,\delta}(y) = \begin{cases} r^{\alpha-n-|\gamma|}(y) & y \in H^{\infty}_{\delta,\mu}, \\ 0 & y \in \mathbb{R}^n_+ \setminus H^{\infty}_{\delta,\mu}. \end{cases}$$

In these notations we have

$$(I^{\alpha}_{P_+,\gamma,\delta}f)(x) = (K^+_{0,\delta} * f)_{\gamma} + (K^+_{\infty,\delta} * f)_{\gamma}, \tag{10.20}$$

$$(I^{\alpha}_{P_-,\gamma,\delta}f)(x) = (M^+_{0,\delta} * f)_{\gamma} + (M^+_{\infty,\delta} * f)_{\gamma}. \tag{10.21}$$

To apply Marcinkiewicz's theorem, we should prove that the operators $I^{\alpha}_{P_\pm,\gamma,\delta}$ have weak types $(p_1,q_1)_{\gamma}$ and $(p_2,q_2)_{\gamma}$, where p_1,q_1,p_2,q_2 are such that $\frac{1}{p} = \frac{1-\tau}{p_1} + \frac{\tau}{p_2}$, $\frac{1}{q} = \frac{1-\tau}{q_1} + \frac{\tau}{q_2}$, $0<\tau<1$. In order to do this we will be interested in the estimate of

$$\sup_{0<\lambda<\infty} \lambda(\mu_{\gamma}(I^{\alpha}_{P_\pm,\gamma,\delta}f,\lambda))^{1/p} =$$
$$\sup_{0<\lambda<\infty} \lambda \left(\mathrm{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : |(I^{\alpha}_{P_\pm,\gamma,\delta}f)(x)| > \lambda\}\right).$$

Considering (10.20) and (10.21) it is enough to estimate

$$\mathrm{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : |(K^+_{0,\delta} * f)_{\gamma}| > \lambda\},$$
$$\mathrm{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : |(K^+_{\infty,\delta} * f)_{\gamma}| > \lambda\},$$
$$\mathrm{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : |(M^+_{0,\delta} * f)_{\gamma}| > \lambda\},$$
$$\mathrm{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : |(M^+_{\infty,\delta} * f)_{\gamma}| > \lambda\}$$

and then to apply the inequality

$$\mathrm{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : |A+B| > \lambda\} \leq$$
$$\mathrm{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : |A| > \lambda\} + \mathrm{mes}_{\gamma}\{x \in \mathbb{R}^n_+ : |B| > \lambda\}.$$

To estimate the generalized convolution, we will use Young's inequality (3.178).

We have

$$||K_{0,\delta}^{+}||_{1,\gamma}=\int\limits_{\mathbb{R}^n_+}K_{0,\delta}^{+}(y)y^{\gamma}dy=\int\limits_{G^0_{\delta,\mu}}(y_1^2-y_2^2-...-y_n^2)^{\frac{\alpha-n-|\gamma|}{2}}y^{\gamma}dy=$$

$$\int\limits_0^{\mu}y_1^{\gamma_1}dy_1\int\limits_{|y'|^2\leq\delta y_1^2}(y_1^2-|y'|^2)^{\frac{\alpha-n-|\gamma|}{2}}(y')^{\gamma'}dy'=\{y'=y_1z',\ z'\in\mathbb{R}^{n-1}_+\}=$$

$$\int\limits_0^{\mu}y_1^{\alpha-1}dy_1\int\limits_{|z'|^2\leq\delta}(1-|z'|^2)^{\frac{\alpha-n-|\gamma|}{2}}(z')^{\gamma'}dz'\leq$$

$$\int\limits_0^{\mu}y_1^{\alpha-1}dy_1\int\limits_{|z'|^2\leq 1}(1-|z'|^2)^{\frac{\alpha-n-|\gamma|}{2}}(z')^{\gamma'}dz'=$$

$$\frac{\mu^{\alpha}}{\alpha}\int\limits_{|z'|\leq 1}(1-|z'|^2)^{\frac{\alpha-n-|\gamma|}{2}}(z')^{\gamma'}dz'=C^1_{\alpha,n,\gamma}\,\mu^{\alpha},$$

where $C^1_{\alpha,n,\gamma}=2^{1-n}\dfrac{\Gamma\left(\frac{\alpha-n-|\gamma|+2}{2}\right)\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\alpha\Gamma\left(\frac{\alpha-\gamma_1+1}{2}\right)}$ does not depend on δ. Therefore,

$$||K_{0,\delta}^{+}||_{1,\gamma}\leq C^1_{\alpha,n,\gamma}\,\mu^{\alpha} \tag{10.22}$$

and $K_{0,\delta}^{+}\in L_1^{\gamma}$.

Now let us consider $M_{0,\delta}^{+}$. We have

$$||M_{0,\delta}^{+}||_{1,\gamma}=\int\limits_{\mathbb{R}^n_+}M_{0,\delta}^{+}(y)y^{\gamma}dy=\int\limits_{H^0_{\delta,\mu}}(y_1^2-y_2^2-...-y_n^2)^{\frac{\alpha-n-|\gamma|}{2}}y^{\gamma}dy=$$

$$\int\limits_{|y'|\leq\mu}(y')^{\gamma'}dy'\int\limits_{y_1^2\leq\delta|y'|^2}(|y'|^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}}y_1^{\gamma_1}dy_1=\{y_1=|y'|z_1,\ z_1\in\mathbb{R}^1_+\}=$$

$$\int\limits_{|y'|\leq\mu}|y'|^{\alpha-n-|\gamma|+\gamma_1+1}(y')^{\gamma'}dy'\int\limits_{z_1^2\leq\delta}(1-z_1^2)^{\frac{\alpha-n-|\gamma|}{2}}z_1^{\gamma_1}dz_1\leq$$

$$D^1_{\alpha,n,\gamma}\int\limits_{|y'|\leq\mu}|y'|^{\alpha-n-|\gamma|+\gamma_1+1}(y')^{\gamma'}dy',$$

where $D^1_{\alpha,n,\gamma} = \int\limits_{z_1^2 \le 1} (1-z_1^2)^{\frac{\alpha-n-|\gamma|}{2}} z_1^{\gamma_1} dz_1$ does not depend on δ. Going over to spherical coordinates $y' = \rho\sigma$, we obtain

$$||M^+_{0,\delta}||_{1,\gamma} \le D^2_{\alpha,n,\gamma} \int\limits_0^{\mu} \rho^{\alpha-1} d\rho = D^3_{\alpha,n,\gamma} \mu^{\alpha},$$

where $D^3_{\alpha,n,\gamma} = \frac{1}{\alpha} \int\limits_{S_1^+(n-1)} \sigma^{\gamma'} dS.$

Now we estimate the norm $K^+_{\infty,\delta}$. Let us take p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. First we consider $||K^+_{\infty,\delta}||_{p',\gamma}$. Let $p \neq 1$ (i.e., $p' \neq \infty$). Then

$$||K^+_{\infty,\delta}||_{p',\gamma} = \left(\int\limits_{\mathbb{R}^n_+} |K^+_{0,\delta}(y)|^{p'} y^{\gamma} dy \right)^{1/p'} =$$

$$\left(\int\limits_{G^{\infty}_{\delta,\mu}} (y_1^2 - |y'|^2)^{\frac{\alpha-n-|\gamma|}{2}\,p'} y^{\gamma} dy \right)^{1/p'} =$$

$$\left(\int\limits_{\mu}^{\infty} y_1^{\gamma_1} dy_1 \int\limits_{|y'|^2 \le \delta y_1^2} (y_1^2 - |y'|^2)^{\frac{\alpha-n-|\gamma|}{2}\,p'} (y')^{\gamma'} dy' \right)^{1/p'} = \{y' = y_1 z',\, z' \in \mathbb{R}^{n-1}_+\} =$$

$$\left(\int\limits_{\mu}^{\infty} y_1^{(\alpha-n-|\gamma|)p'+n+|\gamma|-1} dy_1 \int\limits_{|z'|^2 \le \delta} (1 - |z'|^2)^{\frac{\alpha-n-|\gamma|}{2}\,p'} (z')^{\gamma'} dz' \right)^{1/p'} \le$$

$$\frac{\prod\limits_{i=2}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n \Gamma\left(\frac{n+|\gamma'|+1}{2}\right)} (1-\delta)^{\frac{\alpha-n-|\gamma|}{2}} \left(\int\limits_{\mu}^{\infty} y_1^{(\alpha-n-|\gamma|)p'+n+|\gamma|-1} dy_1 \right)^{1/p'} =$$

$$C^2_{\alpha,n,\gamma} (1-\delta)^{\frac{\alpha-n-|\gamma|}{2}} \mu^{-\frac{n+|\gamma|}{q}},$$

$$C^2_{\alpha,n,\gamma,p} = \frac{2^{-n} \prod\limits_{i=2}^{n} \Gamma\left(\frac{\gamma_i+1}{2}\right)}{((n+|\gamma|-\alpha)p' - n - |\gamma|)^{1/p'} \Gamma\left(\frac{n+|\gamma'|+1}{2}\right)}.$$

Here we take into account that $\alpha - n - |\gamma| < 0$, $p' = \frac{p}{p-1}$, $p < \frac{n+|\gamma|}{\alpha}$, and $q = \frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Then

$$||K^+_{\infty,\delta}||_{p',\gamma} \leq C^2_{\alpha,n,\gamma,p}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}}, \qquad \frac{1}{p}+\frac{1}{p'}=1, \tag{10.23}$$

and hence $K^+_{\infty,\delta} \in L^\gamma_{p'}$, $p' < \infty$.

Passing to the limit as $p' \to \infty$ in (10.23), we obtain

$$||K^+_{\infty,\delta}||_{\infty,\gamma} \leq C^2_{\alpha,n,\gamma,1}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}}, \tag{10.24}$$

$$C^2_{\alpha,n,\gamma,1} = \frac{e^{\frac{n+|\gamma'|}{n+|\gamma|-\alpha}}\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma'|+1}{2}\right)}.$$

Let us estimate the norm $||M^+_{\infty,\delta}||_{p',\gamma}$. Let p' be such that $\frac{1}{p}+\frac{1}{p'}=1$ and $p \neq 1$ (i.e., $p' \neq \infty$). Then

$$||M^+_{\infty,\delta}||_{p',\gamma} = \left(\int\limits_{\mathbb{R}^n_+}|M^+_{\infty,\delta}(y)|^{p'}y^\gamma dy\right)^{1/p'} =$$

$$\left(\int\limits_{H^\infty_{\delta,\mu}}(|y'|^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}p'}y^\gamma dy\right)^{1/p'} =$$

$$\left(\int\limits_{\mu\leq|y'|}(y')^{\gamma'}dy'\int\limits_{y_1^2\leq\delta|y'|^2}(|y'|^2-y_1^2)^{\frac{\alpha-n-|\gamma|}{2}p'}y_1^{\gamma_1}dy_1\right)^{1/p'} =$$

$$\{y_1=|y'|z_1,\, z_1\in\mathbb{R}^1_+\} =$$

$$\left(\int\limits_{\mu\leq|y'|}|y'|^{(\alpha-n-|\gamma|)p'+\gamma_1+1}(y')^{\gamma'}dy'\int\limits_{z_1^2\leq\delta}(1-z_1^2)^{\frac{\alpha-n-|\gamma|}{2}p'}z_1^{\gamma_1}dz_1\right)^{1/p'} \leq$$

$$D^4_{\alpha,n,\gamma}(1-\delta^2)^{\frac{\alpha-n-|\gamma|}{2}}\left(\int\limits_{\mu\leq|y'|}|y'|^{(\alpha-n-|\gamma|)p'+\gamma_1+1}(y')^{\gamma'}dy'\right)^{1/p'}.$$

Going over to spherical coordinates $y'=\rho\sigma$, we obtain

$$||M^+_{\infty,\delta}||_{1,\gamma} \leq D^5_{\alpha,n,\gamma}(1-\delta^2)^{\frac{\alpha-n-|\gamma|}{2}}\left(\int\limits_{\mu}^{\infty}\rho^{(\alpha-n-|\gamma|)p'+n+|\gamma|-1}d\rho\right)^{1/p'} =$$

$$D^5_{\alpha,n,\gamma}(1-\delta^2)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}}.$$

Here we take into account that $\alpha-n-|\gamma|<0$, $p'=\frac{p}{p-1}$, $p<\frac{n+|\gamma|}{\alpha}$, and $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Then

$$||M^+_{\infty,\delta}||_{p',\gamma}\leq D^5_{\alpha,n,\gamma,p}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}},\qquad \frac{1}{p}+\frac{1}{p'}=1, \tag{10.25}$$

and hence $M^+_{\infty,\delta}\in L^\gamma_{p'}$, $p'<\infty$.

Passing to the limit as $p'\to\infty$ in (10.25), we obtain

$$||M^+_{\infty,\delta}||_{\infty,\gamma}\leq D^6_{\alpha,n,\gamma,1}(1-\delta)^{\frac{\alpha-n-|\gamma|}{2}}\mu^{-\frac{n+|\gamma|}{q}}. \tag{10.26}$$

So we have

$$(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||K^+_{\infty,\delta}||_{p',\gamma}\leq C\mu^{-\frac{n+|\gamma|}{q}},\qquad 1\leq p'\leq\infty,$$

and

$$(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||M^+_{\infty,\delta}||_{p',\gamma}\leq C\mu^{-\frac{n+|\gamma|}{q}},\qquad 1\leq p'\leq\infty.$$

Then applying (3.179) we can write

$$(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||(K^+_{\infty,\delta}*f)_\gamma||_{\infty,\gamma}\leq(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||K^+_{\infty,\delta}||_{p',\gamma}\leq C\mu^{-\frac{n+|\gamma|}{q}}$$

and

$$(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||(M^+_{\infty,\delta}*f)_\gamma||_{\infty,\gamma}\leq(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||M^+_{\infty,\delta}||_{p',\gamma}\leq C\mu^{-\frac{n+|\gamma|}{q}}.$$

If we choose μ such that $C\mu^{-\frac{n+|\gamma|}{q}}=\lambda$, then

$$\text{mes}_\gamma\{x\in\mathbb{R}^n_+:(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(K^+_{\infty,\delta}*f)_\gamma|>\lambda\}=0$$

and

$$\text{mes}_\gamma\{x\in\mathbb{R}^n_+:(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(M^+_{\infty,\delta}*f)_\gamma|>\lambda\}=0.$$

Considering (10.20) and (10.21) and applying Young's inequality (3.178), we obtain

$$\begin{aligned}
&\text{mes}_\gamma\{x\in\mathbb{R}^n_+:(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(I^\alpha_{P_+,\delta}f)(x)|>2\lambda\}\leq\\
&\text{mes}_\gamma\{x\in\mathbb{R}^n_+:(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(K^+_{0,\delta}*f)_\gamma|>\lambda\}+\\
&\text{mes}_\gamma\{x\in\mathbb{R}^n_+:(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(K^+_{\infty,\delta}*f)_\gamma|>\lambda\}=\\
&\text{mes}_\gamma\{x\in\mathbb{R}^n_+:(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(K^+_{0,\delta}*f)_\gamma|>\lambda\}\leq\\
&(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}\frac{||(K^+_{0,\delta}*f)_\gamma||^p_{p,\gamma}}{\lambda^p}\leq
\end{aligned}$$

$$\frac{(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}\,p}||K^{+}_{0,\delta}||^{p}_{1,\gamma}||f||^{p}_{p,\gamma}}{\lambda^{p}}\leq$$
$$\frac{(C^{1}_{\alpha,n,\gamma})^{p}(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}\,p}\,\mu^{p\alpha}}{\lambda^{p}}=$$
$$C^{7}(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}\frac{1}{\lambda^{q}}.$$

Similarly,

$$\mathrm{mes}_{\gamma}\{x\in\mathbb{R}^{n}_{+}:(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}|(I^{\alpha}_{P_{-},\delta}f)(x)|>2\lambda\}\leq C^{7}(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}\frac{1}{\lambda^{q}}.$$

It was shown that the operators $I^{\alpha}_{P_{\pm},\gamma,\delta}$ have a weak type $(p,q)_{\gamma}$, where p and q are related by the equality $q=\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}$. Let $0<\tau<1$, $p_1=\frac{p(1-\tau)}{1-\tau p}$, $p_1\in\left[1,\frac{n+|\gamma|}{\alpha}\right)$. The operators $I^{\alpha}_{P_{\pm},\gamma,\delta}$ have a weak type $\left(1,\frac{n+|\gamma|}{n+|\gamma|-\alpha}\right)_{\gamma}$ and a weak type $\left(p_1,\frac{(n+|\gamma|)p_1}{n+|\gamma|-\alpha p_1}\right)_{\gamma}$. Then by Marcinkiewicz's theorem, Theorem 128, the operators $I^{\alpha}_{P_{\pm},\gamma,\delta}$ have a strong type $\left(p,\frac{(n+|\gamma|)p}{n+|\gamma|-\alpha p}\right)_{\gamma}$ and the inequality

$$||(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}(I^{\alpha}_{P_{\pm},\gamma,\delta}f)(x)||_{q,\gamma}\leq M(1-\delta)^{\frac{n+|\gamma|-\alpha}{2}}||f||_{p,\gamma}$$

is true. So

$$||(I^{\alpha}_{P_{\pm},\gamma,\delta}f)(x)||_{q,\gamma}\leq M||f||_{p,\gamma},\qquad 1\leq p<\frac{n+|\gamma|}{\alpha},\qquad n+|\gamma|-2<\alpha<n+|\gamma|. \tag{10.27}$$

Since $f(x)\geq 0$, for $0<\delta_1\leq\delta_2\leq...\leq\delta_m\leq...<1$ we have

$$(I^{\alpha}_{P_{\pm},\gamma,\delta_1}f)(x)\leq(I^{\alpha}_{P_{\pm},\gamma,\delta_2}f)(x)\leq...\leq(I^{\alpha}_{P_{\pm},\gamma,\delta_m}f)(x)\leq....$$

Due to the fact that

$$\lim_{\delta\to 1}(I^{\alpha}_{P_{\pm},\gamma,\delta}f)(x)=(I^{\alpha}_{P_{\pm},\gamma}f)(x),$$

passing to the limit as $\delta\to 1$ in (10.27), we obtain

$$||(I^{\alpha}_{P_{\pm},\gamma}f)(x)||_{q,\gamma}\leq M||f||_{p,\gamma},\qquad 1\leq p<\frac{n+|\gamma|}{\alpha},\qquad n+|\gamma|-2<\alpha<n+|\gamma|.$$

The theorem is proved. □

We will define further operators $I^{\alpha}_{P_{\pm},\gamma}$ on functions L^{γ}_{p} as continuations of operators (10.1) with preservation of boundedness. If integral (10.1) converges absolutely

for $f \in L_p^\gamma$, then these continuations are representable as

$$(I^\alpha_{P\pm i0,\gamma} f)(x) = \frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha)} \int\limits_{\mathbb{R}^n_+} (P\pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}} ({}^\gamma\mathbf{T}^y_x f)(x) y^\gamma dy, \quad y^\gamma = \prod_{i=1}^n y_i^{\gamma_i}.$$

Next we show how hyperbolic B-potentials are connected with the operator $(\Box_\gamma)^k$, $k \in \mathbb{N}$.

Theorem 130. *If $f \in S_{ev}$, $n+|\gamma|-2<\alpha$, and $k \in \mathbb{N}$, then*

$$(\Box_\gamma)^k I^{\alpha+2k}_{P\pm i0,\gamma} f = I^\alpha_{P\pm i0,\gamma} f, \tag{10.28}$$

where $\Box_\gamma = B_{\gamma_1} - \sum\limits_{i=2}^n B_{\gamma_i}$.

Proof. Using representation (10.1) and the property ${}^{\gamma_i}T^{y_i}_{x_i}(B_{\gamma_i})_{x_i} = (B_{\gamma_i})^{\gamma_i}_{x_i} T^{y_i}_{x_i}$ (see formula (1.8.3) from [242]), we obtain

$$(\Box_\gamma)^k (I^{\alpha+2k}_{P\pm i0,\gamma} f)(x) =$$
$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2k)} (\Box_\gamma)^k \int\limits_{\mathbb{R}^n_+} (P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} ({}^\gamma\mathbf{T}^y_x f)(x) y^\gamma dy =$$
$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha+2k)} \int\limits_{\mathbb{R}^n_+} \left({}^\gamma\mathbf{T}^y_x (\Box_\gamma)^k (P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right) f(y) y^\gamma dy.$$

For function $(P\pm i0)^\lambda_\gamma$ the following equality is true (see [505]):

$$(\Box_\gamma)^k (P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}} = 2^{2k} \frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+k+1\right)}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}\right)} (P\pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}}. \tag{10.29}$$

Since

$$2^{2k} \frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+k+1\right)}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)} \frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}\right)} \cdot \frac{1}{H_{n,\gamma}(\alpha+2k)} = \frac{1}{H_{n,\gamma}(\alpha)},$$

using (10.29) we obtain

$$(\Box_\gamma)^k (I^{\alpha+k}_{P\pm i0,\gamma} f)(x) =$$

$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{H_{n,\gamma}(\alpha)}\int\limits_{\mathbb{R}^n_+}\left({}^\gamma\mathbf{T}^y_x(P\pm i0)_\gamma^{\frac{\alpha-n-|\gamma|}{2}}\right)f(y)y^\gamma dy=(I^\alpha_{P\pm i0,\gamma}f)(x).$$

The proof is complete. □

10.1.3 Semigroup properties

Theorem 131. *If $f\in S_{ev}$, $n+|\gamma|-2<\alpha$, and $k\in\mathbb{N}$, then*

$$I^{\alpha+2k}_{P\pm i0,\gamma}(\square_\gamma)^k f=I^\alpha_{P\pm i0,\gamma}f, \tag{10.30}$$

where $\square_\gamma=B_{\gamma_1}-\sum\limits_{i=2}^n B_{\gamma_i}$ *and* $x_i^{\gamma_i}\frac{\partial}{\partial x_i}(\square_\gamma)^m f|_{x_i=0}=0$, $m=0,..,k-1$, $i=1,...,n$.

Proof. Using formula (1.8.3) from [242] of the form ${}^{\gamma_i}T^{y_i}_{x_i}(B_{\gamma_i})_{x_i}=(B_{\gamma_i})^{\gamma_i}_{x_i}T^{y_i}_{x_i}$, we get

$$(I^{\alpha+2k}_{P\pm i0,\gamma}\square^k_\gamma f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2k)}\int\limits_{\mathbb{R}^n_+}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}({}^\gamma\mathbf{T}^y_x(\square_\gamma)^k_x f)(x)y^\gamma dy=$$

$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2)}\int\limits_{\mathbb{R}^n_+}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[(\square_\gamma)_y\,{}^\gamma\mathbf{T}^y_x(\square_\gamma)^{k-1}_x f)(x)\right]y^\gamma dy=$$

$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2)}\int\limits_{\mathbb{R}^n_+}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\times$$

$$\left[\left((B_{\gamma_1})_{y_1}-\sum_{i=2}^n(B_{\gamma_i})_{y_i}\right){}^\gamma\mathbf{T}^y_x(\square_\gamma)^{k-1}_x f(x)\right]y^\gamma dy=$$

$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2)}\int\limits_{\mathbb{R}^n_+}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[(B_{\gamma_1})_{y_1}\,{}^\gamma\mathbf{T}^y_x(\square_\gamma)^{k-1}_x f(x)\right]y^\gamma dy-$$

$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2)}\sum_{i=2}^n\int\limits_{\mathbb{R}^n_+}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[(B_{\gamma_i})_{y_i}\,{}^\gamma\mathbf{T}^y_x(\square_\gamma)^{k-1}_x f(x)\right]y^\gamma dy. \tag{10.31}$$

Integrating by parts at $j=1,...,n$, we obtain

$$\int\limits_0^\infty(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}(B_{\gamma_j})_{y_j}\left[{}^\gamma\mathbf{T}^y_x(\square_\gamma)^{k-1}_x f(x)\right]y_j^{\gamma_j}dy_j=$$

$$\int\limits_0^\infty (P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[\frac{\partial}{\partial y_j}y_j^{\gamma_j}\frac{\partial}{\partial y_j}\,{}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}f(x)\right]dy_j=$$

$$\left\{u=(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}},\ dv=\frac{\partial}{\partial y_j}y_j^{\gamma_j}\frac{\partial}{\partial y_j}\,{}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}f(x)dy_j\right\}=$$

$$(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\ y_j^{\gamma_j}\frac{\partial}{\partial y_j}\,{}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}f(x)\Big|_{y_j=0}^{\infty}-$$

$$\int\limits_0^\infty y_j^{\gamma_j}\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[\frac{\partial}{\partial y_j}\,{}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}(\Box_\gamma)_x^{k-1}(\Box_\gamma)_x^{k-1}f(x)\right]dy_j=$$

$$-\int\limits_0^\infty y_j^{\gamma_j}\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\left[\frac{\partial}{\partial y_j}\,{}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}f(x)\right]dy_j=$$

$$\left\{u=y_j^{\gamma_j}\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}},\ dv=\frac{\partial}{\partial y_j}\,{}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}f(x)dy_j\right\}=$$

$$-y_j^{\gamma_j}\left[\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right]{}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}f(x)\Big|_{y_j=0}^{\infty}+$$

$$\int\limits_0^\infty\left[\frac{\partial}{\partial y_j}y_j^{\gamma_j}\frac{\partial}{\partial y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right]{}^\gamma\mathbf{T}_x^y f(x)dy_j=$$

$$\int\limits_0^\infty\left[(B_{\gamma_j})_{y_j}(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right]{}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}f(x)y_j^{\gamma_j}dy_j.$$

Returning to (10.31), we obtain

$$(I_{P\pm i0,\gamma}^{\alpha+2k}\Box_\gamma^k f)(x)=$$

$$\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2k)}\int\limits_{\mathbb{R}_+^n}\left[(\Box_\gamma)_y(P\pm i0)_\gamma^{\frac{\alpha+2k-n-|\gamma|}{2}}\right]({}^\gamma\mathbf{T}_x^y(\Box_\gamma)_x^{k-1}f)(x)y^\gamma dy.$$

Consistently applying these actions k times, we get

$$(I_{P\pm i0,\gamma}^{\alpha+2k}\Box_\gamma^k f)(x)=\frac{e^{\pm\frac{n-1+|\gamma'|}{2}i\pi}}{\gamma_{n,\gamma}(\alpha+2k)}\int\limits_{\mathbb{R}_+^n}\left[(\Box_\gamma)_y^k(P\pm i0)_\gamma^{\frac{\alpha+2-n-|\gamma|}{2}}\right]({}^\gamma\mathbf{T}_x^y f)(x)y^\gamma dy.$$

Now applying (10.29) we obtain the required statement,

$$(I_{P\pm i0,\gamma}^{\alpha+2k}\Box_\gamma^k f)(x)=(I_{P\pm i0,\gamma}^{\alpha}f)(x).$$

□

By virtue of the density S_{ev} in L_p^γ, equalities (10.28) and (10.30) are valid for functions from L_p^γ for $1<p<\frac{n+|\gamma|}{\alpha}$ when integrals $I^\alpha_{P\pm i0,\gamma}f$ converge absolutely for $f\in L_p^\gamma$.

10.1.4 Examples

Example 1. *Let $n{=}3$, $\gamma_1{=}2$, $\alpha{=}4$, $|\gamma'|{=}\gamma_2{+}\gamma_3{<}1$, $f(x){=}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)$, $|b|{=}1$, $x{\in}\mathbb{R}^3_+$. We have*

$$I^4_{P_+,\gamma}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)=$$
$$2^{-\frac{3}{2}}\prod_{i=2}^3\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\mathbf{j}_{\gamma'}(x';b)\int\limits_0^\infty e^{-y_1}\left({}^2T^{y_1}_{x_1}J_{\frac{1}{2}}(x_1)x_1^{\frac{1}{2}}\right)y_1^4dy_1=$$
$$2^{-\frac{3}{2}}\prod_{i=2}^3\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\frac{C(2)\sqrt{2}}{\sqrt{\pi}x_1}\mathbf{j}_{\gamma'}(x';b)\times$$
$$\left(\int\limits_0^\infty e^{-y_1}\left(\sin(x_1+y_1)-(x_1+y_1)\cos(x_1+y_1)\right)y_1^3dy_1-\right.$$
$$\left.\int\limits_0^\infty e^{-y_1}\left(\sin(|x_1-y_1|)-|x_1-y_1|\cos(|x_1-y_1|)\right)y_1^3dy_1\right)=$$
$$\prod_{i=2}^3\Gamma\left(\frac{\gamma_i{+}1}{2}\right)\Gamma\left(\frac{1{-}|\gamma'|}{2}\right)\frac{1}{4\sqrt{\pi}}\mathbf{j}_{\gamma'}(x';b)\left(\frac{6(\cos x_1{-}e^{x_1})}{x_1}{-}12{-}6x_1-x_1^2\right)$$

and

$$I^4_{P_-,\gamma}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)=2^{-\frac{3}{2}}\prod_{i=2}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\mathbf{j}_{\gamma'}(x';b)\int\limits_0^\infty e^{-y_1}\times$$
$$\left(\left({}^2T^{y_1}_{x_1}J_{\frac{1}{2}}(x_1)x_1^{\frac{1}{2}}\right)\cos\frac{3+|\gamma'|}{2}\pi+\left({}^2T^{y_1}_{x_1}J_{-\frac{1}{2}}(x_1)x_1^{\frac{1}{2}}\right)\sin\frac{1+|\gamma'|}{2}\pi\right)y_1^4dy_1=$$
$$2^{-\frac{3}{2}}\prod_{i=2}^3\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\frac{C(2)\sqrt{2}}{\sqrt{\pi}x_1}\mathbf{j}_{\gamma'}(x';b)\left(\cos\left(\frac{3+|\gamma'|}{2}\right)\pi\times\right.$$
$$\int\limits_0^\infty e^{-y_1}y_1^3\left(\sin(x_1+y_1)-(x_1+y_1)\cos(x_1+y_1)-\sin(|x_1-y_1|)+\right.$$
$$\left.|x_1-y_1|\cos(|x_1-y_1|)\right)dy_1+$$
$$\sin\left(\frac{1+|\gamma'|}{2}\right)\pi\int\limits_0^\infty e^{-y_1}y_1^3\left(\cos(x_1+y_1)+(x_1+y_1)\sin(x_1+y_1)-\right.$$

$$\cos(|x_1-y_1|)-|x_1-y_1|\sin(|x_1-y_1|)dy_1\Big)=$$
$$\prod_{i=2}^{3}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{1-|\gamma'|}{2}\right)\frac{1}{4\sqrt{\pi}}\mathbf{j}_{\gamma'}(x';b)\times$$
$$\left(\cos\left(\frac{3+|\gamma'|}{2}\right)\pi\left(\frac{6(\cos x_1-e^{-x_1})}{x_1}-12-6x_1-x_1^2\right)-\right.$$
$$\left.6\sin\left(\frac{1+|\gamma'|}{2}\right)\pi\,\frac{\sin x_1}{x_1}\right).$$

Considering that

$$H_{3,\gamma}(4)=\frac{\sqrt{\pi}\prod\limits_{i=2}^{3}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{|\gamma'|+1}{2}\right)},$$

we obtain

$$I^4_{P\pm i0,\gamma}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)=$$
$$\frac{e^{\pm\frac{2+|\gamma'|}{2}i\pi}}{H_{3,\gamma}(4)}\left[I^4_{P_+,\gamma}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)+e^{\mp\frac{|\gamma'|+1}{2}\pi i}I^4_{P_-,\gamma}x_1^2e^{-x_1}\mathbf{j}_{\gamma'}(x',b)\right]=$$
$$\frac{e^{\pm\frac{2+|\gamma'|}{2}i\pi}}{4\sin\left(\frac{1+|\gamma'|}{2}\right)\pi}\mathbf{j}_{\gamma'}(x';b)\left(\frac{6(\cos x_1-e^{x_1})}{x_1}-12-6x_1-x_1^2+\right.$$
$$e^{\mp\frac{|\gamma'|+1}{2}\pi i}\left(\cos\left(\frac{3+|\gamma'|}{2}\right)\pi\left(\frac{6(\cos x_1-e^{-x_1})}{x_1}-12-6x_1-x_1^2\right)-\right.$$
$$\left.\left.6\sin\left(\frac{1+|\gamma'|}{2}\right)\pi\cdot\frac{\sin x_1}{x_1}\right)=$$
$$\mathbf{j}_{\gamma'}(x';b)\,\frac{e^{-x_1}(6-6e^{(1+i)x_1}+x_1(12+x_1(6+x_1)))}{4x_1}.$$

10.2 Method of approximative inverse operators applied to inversion of the hyperbolic B-potentials

10.2.1 Method of approximative inverse operators

Here we describe one approach for inverting potential type operators, based on the idea of approximative inverse operators developed in [426,492].

The problem to invert a convolution operator $Af=a*f$ reduces to multiplication of some convenient integral transform of a function f by the reciprocal $\frac{1}{\hat{a}}$ of a chosen

integral transform of the kernel:

$$Af = a * f, \qquad \widehat{Af} = \hat{a} \cdot \hat{f}, \qquad \widehat{A^{-1}f} = \frac{1}{\hat{a}} \cdot \hat{f}.$$

Indeed, we have

$$g = Af, \qquad \widehat{A^{-1}g} = \frac{1}{\hat{a}} \cdot \hat{a} \cdot \hat{f} = \hat{f}.$$

However, in the case of potentials, the multiplier $\frac{1}{\hat{a}}$ is unbounded at infinity and, maybe, on some sets. In this case we use the multiplier m_ε, which is dependent on ε such that $\frac{m_\varepsilon}{\hat{a}}$ vanishes at those sets on which it is necessary and $\lim\limits_{\varepsilon\to 0} m_\varepsilon = 1$. So we can construct $\widehat{A_\varepsilon^{-1} f} = \frac{m_\varepsilon}{\hat{a}} \cdot \hat{f}$. Applying the inverse integral transform and passing to the limit $\varepsilon \to 0$, we obtain A^{-1}. Next, it is necessary to prove that the resulting operator will be inverse to the operator A in some appropriate space. Therefore, the factor m_ε should be chosen so that inverse integral transform of $\frac{m_\varepsilon}{\hat{a}} \cdot \hat{f}$ provides a fairly good class of functions.

In our case, we take the Hankel transform. Considering that

$$\mathbf{F}_\gamma I^\alpha_{P\pm i0,\gamma} f = (P \mp i0)_\gamma^{-\frac{\alpha}{2}} \mathbf{F}_\gamma f,$$

where $f \in \Phi_V^\gamma$, $V = \{x \in \mathbb{R}^n_+ : P(x) = 0\}$, we take

$$M_{\varepsilon,\delta} = \frac{(P \mp i0)^m e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}.$$

So we should prove that left inverse operators to $I^\alpha_{P\pm i0,\gamma}$ are

$$(I^\alpha_{P\pm i0,\gamma})^{-1} f) = \overset{L_p^\gamma}{\lim_{\delta\to 0}} \overset{L_2^\gamma}{\lim_{\varepsilon\to 0}} \left(\left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m} \right)(x) * f(x) \right)_\gamma .$$

We denote

$$(I^\alpha_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} f = \left(\left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m} \right)(x) * f(x) \right)_\gamma =$$

$$\int\limits_{\mathbb{R}^n_+} {}^{\mp}g^\alpha_{\varepsilon,\delta}(y) ({}^\gamma \mathbf{T}^y_x f(x)) y^\gamma \, dy,$$

where

$${}^{\mp}g^\alpha_{\varepsilon,\delta}(x) = \left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m} \right)(x) =$$

$$\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi,$$

$m\geq n+|\gamma|-\frac{\alpha}{2}$, $n+|\gamma|-2<\alpha<n+|\gamma|$.

10.2.2 General Poisson kernel

In this section, we consider a certain function used for solving the problem of inverting a hyperbolic B-potential. Based on the type and properties of this function, we will call it the general Poisson kernel.

We first prove an auxiliary lemma.

Lemma 39. *The Hankel transform of $e^{-\delta|x|}$ is*

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\frac{2^{|\gamma|}\delta\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}. \tag{10.32}$$

Proof. We have

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\int\limits_{\mathbb{R}^n_+}e^{-\delta|x|}\mathbf{j}_\gamma(x;\xi)x^\gamma\,dx=\{x=\rho\sigma\}=$$

$$\int\limits_0^\infty e^{-\delta\rho}\rho^{n+|\gamma|-1}d\rho\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(\rho\sigma;\xi)\sigma^\gamma\,dS.$$

Applying formula (3.140), we obtain

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi)=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty e^{-\delta\rho}j_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{n+|\gamma|-1}d\rho=$$

$$\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{\frac{n-|\gamma|}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}}\int\limits_0^\infty e^{-\delta\rho}J_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{\frac{n+|\gamma|}{2}}d\rho.$$

Applying formula (2.12.8.4) from [456], p. 164, of the form

$$\int\limits_0^\infty x^{\nu+2}e^{-px}J_\nu(cx)dx=\frac{2p(2c)^\nu\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}(p^2+c^2)^{\nu+\frac{3}{2}}},\qquad \mathrm{Re}\,\nu>-1,$$

we obtain

$$\int\limits_0^\infty e^{-\delta\rho} J_{\frac{n+|\gamma|}{2}-1}(\rho|\xi|)\rho^{\frac{n+|\gamma|}{2}}d\rho = \frac{2\delta(2|\xi|)^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}},$$

and therefore

$$\mathbf{F}_\gamma[e^{-\delta|x|}](\xi) = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{\frac{n-|\gamma|}{2}}|\xi|^{\frac{n+|\gamma|}{2}-1}} \frac{2\delta(2|\xi|)^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}} =$$

$$\frac{2^{|\gamma|}\delta \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}. \qquad \square$$

We give the following formula from [246], which will be used later:

$$\int\limits_{S_1^+(n)} \mathcal{P}_\xi^\gamma f(\langle\xi,x\rangle)x^\gamma d\omega_x = \frac{\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\sqrt{\pi}2^{n-1}\Gamma\left(\frac{|\gamma|+n-1}{2}\right)} \int\limits_{-1}^1 f(|\xi|p)(1-p^2)^{\frac{n+|\gamma|-3}{2}}dp, \tag{10.33}$$

where $f(t)(1-t^2)^{\frac{n+|\gamma|-3}{2}} \in L_1(-1,1)$.

Definition 51. *The function*

$$P_\gamma(x,\delta) = \frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n \Gamma\left(\frac{\gamma_j+1}{2}\right)}\,\delta\,(\delta^2+|x|^2)^{-\frac{n+|\gamma|+1}{2}}, \qquad \delta>0, \tag{10.34}$$

is called the ***general Poisson kernel****.*

Lemma 40. *For $P_\gamma(x,\delta)$ the following properties are valid:*

1. $\mathbf{F}_\gamma[P_\gamma(x,\delta)](\xi) = e^{-\delta|\xi|}$,
2. $\int\limits_{\mathbb{R}^n_+} P_\gamma(x,\delta)x^\gamma dx = \int\limits_{\mathbb{R}^n_+} P_\gamma(x,1)x^\gamma dx = 1$,
3. $P_\gamma(x,\delta) \in L_p^\gamma$, $1 \le p \le \infty$.

Proof. 1. From Lemma 39 we get

$$\mathbf{F}_\gamma^{-1}[e^{-\delta|x|}](\xi) = \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\mathbf{F}_\gamma[e^{-\delta|x|}](\xi) =$$

$$\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\;\frac{2^{|\gamma|}\delta\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}=$$

$$\frac{2^n\delta\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\;\frac{1}{(\delta^2+|\xi|^2)^{\frac{n+|\gamma|+1}{2}}}=P_\gamma(x,\delta).$$

Hence we obtain $\mathbf{F}_\gamma[P_\gamma(x,\delta)](\xi)=e^{-\delta|\xi|}$.

2. Consider the integral $\int\limits_{\mathbb{R}^n_+}P_\gamma(x,\delta)x^\gamma dx$. We have

$$\int\limits_{\mathbb{R}^n_+}P_\gamma(x,\delta)x^\gamma dx=\frac{2^n\delta\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{x^\gamma\,dx}{(\delta^2+|x|^2)^{\frac{n+|\gamma|+1}{2}}}=\{x=\delta y\}=$$

$$\frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}}=\int\limits_{\mathbb{R}^n_+}P_\gamma(x,1)x^\gamma dx.$$

Let us show now that $\int\limits_{\mathbb{R}^n_+}P_\gamma(x,1)x^\gamma dx=1$. Going over to spherical coordinates and using (1.107), we obtain

$$\int\limits_{\mathbb{R}^n_+}\frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}}=\{y=\rho\sigma\}=\int\limits_0^\infty\frac{\rho^{n+|\gamma|-1}\,d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}}\int\limits_{S_1^+(n)}\sigma^\gamma\,dS=$$

$$\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty\frac{\rho^{n+|\gamma|-1}\,d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}}=\{\rho^2=r\}=$$

$$\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\int\limits_0^\infty\frac{r^{\frac{n+|\gamma|}{2}-1}}{(1+r)^{\frac{n+|\gamma|+1}{2}}}dr.$$

Using formula (2.2.5.24) from [455], p. 239, of the form

$$\int\limits_0^\infty\frac{x^{\alpha-1}}{(x+z)^\beta}dx=z^{\alpha-\beta}B(\alpha,\beta-\alpha),\qquad 0<\mathrm{Re}\,\alpha<\mathrm{Re}\,\beta,$$

we obtain

$$2\int\limits_0^\infty \frac{\rho^{n+|\gamma|-1}\,d\rho}{(1+\rho^2)^{\frac{n+|\gamma|+1}{2}}}=\int\limits_0^\infty \frac{r^{\frac{n+|\gamma|}{2}-1}}{(1+r)^{\frac{n+|\gamma|+1}{2}}}dr=\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\Gamma\left(\frac{n+|\gamma|+1}{2}\right)} \tag{10.35}$$

and

$$\int\limits_{\mathbb{R}^n_+}\frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}}=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|}{2}\right)}\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}=\frac{\sqrt{\pi}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}.$$

Finally,

$$\int\limits_{\mathbb{R}^n_+}P_\gamma(x,1)x^\gamma\,dx=\frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n\Gamma\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{y^\gamma\,dy}{(1+|y|^2)^{\frac{n+|\gamma|+1}{2}}}=$$

$$\frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^n\Gamma\left(\frac{\gamma_j+1}{2}\right)}\,\frac{\sqrt{\pi}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}=1.$$

3. We finally prove that $P_\gamma(x,\delta)\in L_p^\gamma$, $1\leq p\leq\infty$. We have

$$\int\limits_{\mathbb{R}^n_+}\frac{x^\gamma\,dx}{(\delta^2+|x|^2)^{p\,\frac{n+|\gamma|+1}{2}}}=\delta^{(n+|\gamma|)(1-p)-p}\int\limits_{\mathbb{R}^n_+}\frac{x^\gamma\,dx}{(|x|^2+1)^{p\,\frac{n+|\gamma|+1}{2}}}=$$

$$\{x=\rho\sigma,\,|x|=\rho\}=\delta^{(n+|\gamma|)(1-p)-p}\int\limits_0^\infty\frac{\rho^{n+|\gamma|-1}d\rho}{(\rho^2+1)^{p\,\frac{n+|\gamma|+1}{2}}}\int\limits_{S_1^+(n)}\sigma^\gamma\,dS.$$

Applying (1.107) and (10.35) for $1\leq p<\infty$, we obtain

$$||P_\gamma(x,\delta)||_{p,\gamma}=\left(\delta^{(n+|\gamma|)(1-p)-p}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+|\gamma|}{2}\right)}{2\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}\,\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\right)^{\frac{1}{p}}$$

$$=\left(\delta^{(n+|\gamma|)(1-p)-p}\frac{\sqrt{\pi}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}\right)^{\frac{1}{p}}<\infty.$$

For $p=\infty$, using (1.47), we get the inequality $||P_\gamma(x,\delta)||_{\infty,\gamma}<\infty$. □

Following [543] we prove that a generalized convolution of a function with the Poisson kernel tends to a function in L_p^γ.

Let

$$(\mathbf{P}_{\gamma,\delta} f)(x) = (f(x) * P_\gamma(x,\delta))_\gamma. \tag{10.36}$$

Lemma 41. *If $f \in L_p^\gamma$, $1 \le p \le \infty$ or $f \in C_0 \subset L_\infty^\gamma$, then*

$$||(\mathbf{P}_{\gamma,\delta} f)(x) - f(x)||_{p,\gamma} \to 0 \qquad \textit{if} \qquad \delta \to 0.$$

Proof. Considering the property 2 from Lemma 40, we can write

$$(f(x) * P_\gamma(x,\delta))_\gamma - f(x) = \int\limits_{\mathbb{R}_+^n} [{}^\gamma\mathbf{T}_x^y f(x) - f(y)] P_\gamma(y,\delta) y^\gamma dy.$$

Hence, applying the generalized Minkowski inequality, we obtain

$$||(f(x) * P_\gamma(x,\delta))_\gamma - f(x)||_{p,\gamma} \le$$

$$\int\limits_{\mathbb{R}_+^n} \left(\int\limits_{\mathbb{R}_+^n} [{}^\gamma\mathbf{T}_x^y f(x) - f(x)]^p x^\gamma dx \right)^{\frac{1}{p}} |P_\gamma(y,\delta)| y^\gamma dy = \{y = \delta t\} =$$

$$\int\limits_{\mathbb{R}_+^n} \left(\int\limits_{\mathbb{R}_+^n} [{}^\gamma\mathbf{T}_x^{\delta t} f(x) - f(x)]^p x^\gamma dx \right)^{\frac{1}{p}} |P_\gamma(t,1)| t^\gamma dt. \tag{10.37}$$

From [441], p. 166, Lemma 3.6, it follows that for $f \in L_p^\gamma$

$$||{}^\gamma\mathbf{T}_x^{\delta t} f(x) - f(x)||_{p,\gamma} \le c||f(x)||_{p,\gamma},$$

and from [442], p. 182, Preposition 4.1, and [445], p. 50, it follows that

$$\lim_{\delta\to 0} \left(\int\limits_{\mathbb{R}_+^n} [{}^\gamma\mathbf{T}_x^{\delta t} f(x) - f(x)]^p x^\gamma dx \right)^{\frac{1}{p}} = 0.$$

Then, by the Lebesgue theorem on dominated convergence, the integral (10.37) tends to zero when $\delta \to 0$, since the integrand is majorized by the integrable function $c||f||_{p,\gamma}|P_\gamma(t,1)|t^\gamma$. □

10.2.3 Representation of the kernel ${}^{\mp}g_{\varepsilon,\delta}^{\alpha}$

In this section we get the integral kernel representation ${}^{\mp}g_{\varepsilon,\delta}^{\alpha}$.

Theorem 132. *The function ${}^{\mp}g^{\alpha}_{\varepsilon,\delta}$ can be presented in the form*

$$
{}^{\mp}g^{\alpha}_{\varepsilon,\delta}(x)=\frac{2^{2-|\gamma|}}{\delta^{n+|\gamma|+\alpha}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times
$$

$$
\int\limits_0^{\infty} r^{n+|\gamma'|-2}\frac{(1-r^2\mp i0)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m}\times
$$

$$
F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right)dr,
$$

where $\beta=n+|\gamma|+\alpha$ $F_4(a,b,c_1,c_2;x,y)$ is the Appell hypergeometric function (1.36).

Proof. We represent the function ${}^{\mp}g^{\alpha}_{\varepsilon,\delta}(t)$ as the sum

$$
{}^{\mp}g^{\alpha}_{\varepsilon,\delta}(x)=\mathbf{F}^{-1}_{\gamma}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}=
$$

$$
\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\frac{(P\mp i0)^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi=
$$

$$
\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\left[\int\limits_{\{P(\xi)>0\}^+}\frac{P^{m+\frac{\alpha}{2}}(\xi)e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi+\right.
$$

$$
\left.e^{\mp(m+\frac{\alpha}{2})\pi i}\int\limits_{\{P(\xi)<0\}^+}\frac{|P(\xi)|^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi\right].
$$

Let

$$
J_1=\int\limits_{\{P(\xi)>0\}^+}\frac{P^{m+\frac{\alpha}{2}}(\xi)e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi,
$$

$$
J_2=\int\limits_{\{P(\xi)<0\}^+}\frac{|P(\xi)|^{m+\frac{\alpha}{2}}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi.
$$

Going in J_1 over to spherical coordinates $\xi'=\rho\sigma$, $\sigma\in\mathbb{R}^{n-1}_+$, $\rho=|\xi'|$, we obtain

$$
J_1=\int\limits_0^{\infty} j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1}\,d\xi_1\times
$$

$$\int\limits_{|\xi'|^2<\xi_1^2} \frac{(\xi_1^2-|\xi'|^2)^{m+\frac{\alpha}{2}} e^{-\delta\sqrt{\xi_1^2+|\xi'|^2}}}{(\xi_1^2-|\xi'|^2+i\varepsilon(\xi_1^2+|\xi'|^2))^m}\mathbf{j}_{\gamma}(x',\xi')(\xi')^{\gamma'}\,d\xi' =$$

$$\int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1}\,d\xi_1 \int\limits_0^{\xi_1} \rho^{n+|\gamma'|-2}\frac{(\xi_1^2-\rho^2)^{m+\frac{\alpha}{2}} e^{-\delta\sqrt{\xi_1^2+\rho^2}}}{(\xi_1^2-\rho^2+i\varepsilon(\xi_1^2+\rho^2))^m}d\rho\times$$

$$\int\limits_{S_1^+(n-1)} \mathbf{j}_{\gamma}(x',\rho\sigma)(\sigma)^{\gamma'}\,dS.$$

The formula

$$\int\limits_{S_1^+(n-1)} \mathbf{j}_{\gamma}(x',\rho\sigma)(\sigma)^{\gamma'}\,dS = \frac{\prod\limits_{i=2}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}\, j_{\frac{n-1+|\gamma'|}{2}-1}(\rho|x'|),$$

is valid (see (3.140)). Therefore,

$$J_1 = \frac{\prod\limits_{i=2}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)} \int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{\gamma_1}\,d\xi_1\times$$

$$\int\limits_0^{\xi_1} \rho^{n+|\gamma'|-2} j_{\frac{n-1+|\gamma'|}{2}-1}(\rho|x'|)\frac{(\xi_1^2-\rho^2)^{m+\frac{\alpha}{2}} e^{-\delta\sqrt{\xi_1^2+\rho^2}}}{(\xi_1^2-\rho^2+i\varepsilon(\xi_1^2+\rho^2))^m}d\rho = \{\rho=\xi_1 r\} =$$

$$\frac{\prod\limits_{i=2}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)} \int\limits_0^\infty j_{\frac{\gamma_1-1}{2}}(x_1\xi_1)\xi_1^{n+|\gamma|-1+\alpha}\,d\xi_1\times$$

$$\int\limits_0^1 r^{n+|\gamma'|-2} j_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\frac{(1-r^2)^{m+\frac{\alpha}{2}} e^{-\delta\xi_1\sqrt{1+r^2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr =$$

$$\frac{\prod\limits_{i=2}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)} \frac{2^{\frac{\gamma_1-1}{2}}\Gamma\left(\frac{\gamma_1+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}}} \frac{2^{\frac{n-1+|\gamma'|}{2}-1}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}{|x'|^{\frac{n-1+|\gamma'|}{2}-1}}\times$$

$$\int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m}dr\times$$

$$\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1} e^{-\delta\xi_1\sqrt{1+r^2}} J_{\frac{\gamma_1-1}{2}}(x_1\xi_1) J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\,d\xi_1 =$$

$$\frac{2^{\frac{|\gamma|-n}{2}} \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}} |x'|^{\frac{n-1+|\gamma'|}{2}-1}} \int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}} \frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m} dr\times$$

$$\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1} e^{-\delta\xi_1\sqrt{1+r^2}} J_{\frac{\gamma_1-1}{2}}(x_1\xi_1) J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\,d\xi_1.$$

To calculate the internal integral, we apply formula (2.12.38.2) from [456], p. 194, of the form

$$\int\limits_0^\infty x^{a-1} e^{-px} J_\mu(bx) J_\nu(cx)dx = \frac{b^\mu c^\nu}{2^{\mu+\nu} p^{a+\mu+\nu}} \frac{\Gamma(a+\mu+\nu)}{\Gamma(\mu+1)\Gamma(\nu+1)}\times$$

$$F_4\left(\frac{a+\mu+\nu}{2}, \frac{a+\mu+\nu+1}{2}; \mu+1, \nu+1; -\frac{b^2}{p^2}, -\frac{c^2}{p^2}\right),$$

$$\mathrm{Re}\,(a+\mu+\nu)>0,\ \mathrm{Re}\,p>0.$$

We have

$$a=\frac{n+|\gamma|}{2}+\alpha+2,\ \ p=\delta\sqrt{1+r^2},\ \ \mu=\frac{\gamma_1-1}{2},$$

$$\nu=\frac{n-1+|\gamma'|}{2}-1,\ \ b=x_1,\ \ c=r|x'|$$

and

$$\int\limits_0^\infty \xi_1^{\frac{n+|\gamma|}{2}+\alpha+1} e^{-\delta\xi_1\sqrt{1+r^2}} J_{\frac{\gamma_1-1}{2}}(x_1\xi_1) J_{\frac{n-1+|\gamma'|}{2}-1}(r\xi_1|x'|)\,d\xi_1 =$$

$$\frac{x_1^{\frac{\gamma_1-1}{2}} (r|x'|)^{\frac{n+|\gamma'|-3}{2}}}{2^{\frac{n+|\gamma|}{2}-2}(\delta\sqrt{1+r^2})^{n+|\gamma|+\alpha}} \frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$F_4\left(\frac{\beta}{2}, \frac{\beta+1}{2}; \frac{\gamma_1+1}{2}, \frac{n+|\gamma'|-1}{2}; -\frac{x_1^2}{\delta^2(1+r^2)}, -\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right),$$

where $\beta=n+|\gamma|+\alpha$. Then

$$J_1 = \frac{2^{\frac{|\gamma|-n}{2}} \prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)}{x_1^{\frac{\gamma_1-1}{2}} |x'|^{\frac{n-1+|\gamma'|}{2}-1}} \int\limits_0^1 r^{\frac{n+|\gamma'|-1}{2}} \frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1-r^2+i\varepsilon(1+r^2))^m} dr\times$$

$$\frac{x_1^{\frac{\gamma_1-1}{2}}(r|x'|)^{\frac{n+|\gamma'|-3}{2}}}{2^{\frac{n+|\gamma|}{2}-2}(\delta\sqrt{1+r^2})^{n+|\gamma|+\alpha}}\frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right)=$$

$$\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\delta^{\beta}}\frac{\Gamma(n+|\gamma|+\alpha)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\int\limits_0^1 r^{n+|\gamma'|-2}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m}\times$$

$$F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right)dr.$$

Similarly, we find

$$J_2=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\delta^{\beta}}\frac{\Gamma(\beta)}{\Gamma\left(\frac{\gamma_1+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-1}{2}\right)}\times$$

$$\int\limits_1^{\infty} r^{n+|\gamma'|-2}\frac{(1-r^2)^{m+\frac{\alpha}{2}}}{(1+r^2)^{\frac{n+|\gamma|+\alpha}{2}}(1-r^2+i\varepsilon(1+r^2))^m}\times$$

$$F_4\left(\frac{\beta}{2},\frac{\beta+1}{2};\frac{\gamma_1+1}{2},\frac{n+|\gamma'|-1}{2};-\frac{x_1^2}{\delta^2(1+r^2)},-\frac{(r|x'|)^2}{\delta^2(1+r^2)}\right)dr.$$

Multiplying by the corresponding constants, adding $J_1(x)$ to $J_2(x)$, and taking into account that

$$(1-r^2\mp i0)^{m+\frac{\alpha}{2}}=(1-r^2)_+^{m+\frac{\alpha}{2}}+e^{\mp(m+\frac{\alpha}{2})\pi i}(1-r^2)_-^{m+\frac{\alpha}{2}},$$

we obtain the statement of the theorem. □

10.2.4 Inversion of the hyperbolic B-potentials

Consider a convolution operator

$$Af=(T*f)_\gamma,\qquad f\in S_{ev}. \tag{10.38}$$

In the images of the Hankel transform we can write

$$\mathbf{F}_\gamma[Af]=\mathbf{F}_\gamma[T]\cdot\mathbf{F}_\gamma[f].$$

Definition 52. *Let $M \in S'_{ev}$. The weighted generalized function is called* ***B-multiplier in*** L_p^γ *if for all* $f \in S_{ev}$ *the generalized convolution* $(\mathbf{F}_\gamma^{-1} M * f)_\gamma$ *belongs to* L_p^γ *and the supremum*

$$\sup_{||f||_{p,\gamma}=1} ||(\mathbf{F}_\gamma^{-1} M * f)_\gamma||_{p,\gamma} \tag{10.39}$$

is finite. Linear space of all such M *is denoted by* $M_{p,\gamma} = M_{p,\gamma}(\mathbb{R}_+^n)$. *The norm in* $M_{p,\gamma}$ *is the supremum* (10.39).

Consider a singular differential operator

$$(D_B)_{x_i}^{\beta_i} = \begin{cases} B_{\gamma_i}^{\frac{\beta_i}{2}} & \beta = 0, 2, 4, \ldots, \\ D_{x_i} B_{\gamma_i}^{\frac{\beta_i - 1}{2}} & \beta = 1, 3, 5, \ldots, \end{cases}$$

where $B_{\gamma_i} = \frac{\partial^2}{\partial x_i} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}$.

In the article [348] the following criterion of the Mikhlin type for a B-multiplier is proved.

Theorem 133. *Let* $M(\xi) \in C_{ev}^k(\mathbb{R}_+^n) \backslash \{0\}$, *where* k *is an even number greater than* $\frac{n+|\gamma|}{2}$ *and there is a constant* A *which does not depend on* $\beta = (\beta_1, \ldots, \beta_m)$, $|\beta| < k$, *such that for* $\xi \neq 0$, $\xi \in \overline{\mathbb{R}}_+^n$ *the condition*

$$\left| \xi^\beta \, (D_B)_\xi^\beta \, M(\xi) \right| \leq A$$

is valid. Then $M(\xi)$ *is a B-multiplier for* $1 < p < \infty$.

Lemma 42. *Let* $\varepsilon, \delta > 0$ *be fixed numbers and* $m \geq n + |\gamma| - \frac{\alpha}{2}$. *The function*

$$M_{\alpha,\varepsilon,\delta}^{\mp}(\xi) = \begin{cases} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m} & P(\xi) \neq 0, \\ 0 & P(\xi) = 0 \end{cases}$$

is B-multiplier for $1 < p < \infty$.

Proof. We prove the estimate

$$\left| \xi_1^{\beta_1} \ldots \xi_n^{\beta_n} (D_B)_{\xi_1}^{\beta_1} \ldots (D_B)_{\xi_n}^{\beta_n} \, M_{\alpha,\varepsilon,\delta}^{\mp}(\xi) \right| \leq C(\varepsilon, \delta). \tag{10.40}$$

For $\xi \notin V = \{\xi \in \mathbb{R}_+^n : P(\xi) = 0\}$, we have

$$|(D_B)_\xi^j (P \mp i0)^{m+\frac{\alpha}{2}}| \leq C_1 |\xi^j| \cdot |P(\xi)|^{m+\frac{\alpha}{2}-|j|},$$

$$|(D_B)_\xi^k (P(\xi) + i\varepsilon|\xi|^2)^{-m}| \leq C_2 |\xi^k| \cdot |P^2(\xi) + \varepsilon^2 |\xi|^4|^{-\frac{m+|k|}{2}},$$

$$|(D_B)_\xi^r e^{-\delta|\xi|}| \leq C_3 |\xi^r| \cdot \frac{e^{-\delta|\xi|}}{|\xi|^{2r-1}}.$$

Using these estimates and the Leibniz type formula for B-differentiation of the form (see [67])

$$B_i^l(u\,v) = \sum_{k=0}^{2l} C_{2l}^k \left(D_{B_i}^{2l-k} u\right)\left(D_{B_i}^k v\right) + \sum_{m=1}^{2l-2} \frac{1}{x_i^m} \mathbf{P}_{2l-m}\left(D_{B_i} v;\, D_{B_i} u\right),$$

where

$$\mathbf{P}_{2l-m}\left(D_{B_i} v;\, D_{B_i} u\right) = \sum_{j=1}^{2l-v-1} a_{2l-m-j,j}(\gamma_j)\left(D_{B_i}^{2l-m-j} u\right)\left(D_{B_i}^j v\right),$$

we get the required estimate (10.40).

If $\xi \in V$, then the estimate (10.40) follows from the continuity of the function $M_{\alpha,\varepsilon,\delta}^{\mp}(\xi)$ and its derivatives on V. □

Lemma 43. *The function* ${}^{\mp}g_{\varepsilon,\delta}^{\alpha}(x)$ *belongs to space* L_p^γ, $1 < p < \infty$.

Proof. Since the function

$${}^{\mp}g_{\varepsilon,\delta}^{\alpha}(t) = \mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|x|}}{(P(x) + i\varepsilon|x|^2)^m}$$

is representable by an operator generated by the B-multiplier $M_{\alpha,\varepsilon,\delta}^{\mp}(\xi)$ in L_p^γ, we have ${}^{\mp}g_{\varepsilon,\delta}^{\alpha} \in L_p^\gamma$. □

Lemma 44. *Let* $f \in S_{ev}$. *The operator*

$$(I_{P\pm i0,\gamma}^{\alpha})_{\varepsilon,\delta}^{-1} f(x) = \int\limits_{\mathbb{R}_+^n} {}^{\mp}g_{\varepsilon,\delta}^{\alpha}(t)({}^\gamma\mathbf{T}_x^t f(x))t^\gamma\,dt$$

is bounded in L_p^γ, $1 < p < \infty$.

Proof. By the definition of the operator

$$(I_{P\pm i0,\gamma}^{\alpha})_{\varepsilon,\delta}^{-1} f = \left(\left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}\right)(x) * f(x)\right)_\gamma,$$

it is a generalized convolution $(\mathbf{F}_\gamma^{-1} M_{\alpha,\varepsilon,\delta}^{\mp} * f)_\gamma$ with the B-multiplier $M_{\alpha,\varepsilon,\delta}^{\mp}(\xi)$ and therefore belongs to L_p^γ. □

Lemma 45. *Let* $f \in \Phi_V^\gamma$, $V=\{\xi\in\mathbb{R}_+^n : P(\xi)=0\}$. *Then*

$$((I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} I^{\alpha}_{P\pm i0,\gamma} f)(x) =$$
$$(\mathbf{P}_{\gamma,\delta} f)(x) + \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n} \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \sum_{k=0}^{m} C_m^k (-i\varepsilon)^k (\mathbf{A}_k^{\gamma,\delta,\varepsilon} f)(x),$$

where $(\mathbf{P}_{\gamma,\delta} f)(x)$ *is a generalized convolution with the Poisson kernel* (10.36)

$$(\mathbf{A}_k^{\gamma,\delta,\varepsilon} f)(x) = (A_k^{\gamma,\delta,\varepsilon}(x) * f(x))_\gamma,$$
$$A_k^{\gamma,\delta,\varepsilon}(x) = \int\limits_{\mathbb{R}^n_+} \frac{|\xi|^{2k} e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k} \mathbf{j}_\gamma(x,\xi)\xi^\gamma \, d\xi.$$

Proof. Let $I^{\alpha}_{P\pm i0,\gamma} f = g$. We have

$$\mathbf{F}_\gamma((I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} g)(x) = \mathbf{F}_\gamma\left(\left(\mathbf{F}^{-1}_\gamma \frac{(P\mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m}\right)(x) * g(x)\right)_\gamma =$$
$$\frac{(P\mp i0)_\gamma^{m+\frac{\alpha}{2}} e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} \cdot \mathbf{F}_\gamma g = \frac{(P\mp i0)_\gamma^{m+\frac{\alpha}{2}} e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} \cdot (P\mp i0)_\gamma^{-\frac{\alpha}{2}} \mathbf{F}_\gamma f =$$
$$\frac{(P\mp i0)_\gamma^{m} e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} \cdot \mathbf{F}_\gamma f.$$

Then

$$((I^{\alpha}_{P\pm i0,\gamma})^{-1}_{\varepsilon,\delta} I^{\alpha}_{P\pm i0,\gamma} f)(x) = \mathbf{F}^{-1}_\gamma\left(\frac{(P\mp i0)_\gamma^{m} e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} \cdot \mathbf{F}_\gamma f\right) =$$
$$\left(\mathbf{F}^{-1}_\gamma \frac{(P\mp i0)_\gamma^{m} e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} * f\right)_\gamma. \tag{10.41}$$

Applying to $\mathbf{F}^{-1}_\gamma \frac{(P\mp i0)_\gamma^{m} e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}$ the Newton binomial formula, we obtain

$$\mathbf{F}^{-1}_\gamma \frac{(P\mp i0)_\gamma^{m} e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m} =$$
$$\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n} \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \left[\int\limits_{\{\xi_1>|\xi'|\}^+} \frac{(\xi_1^2-|\xi'|^2)^m e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma \, d\xi + \right.$$
$$\left. e^{\mp m\pi i} \int\limits_{\{\xi_1<|\xi'|\}^+} \frac{(|\xi'|^2-\xi_1^2)^m e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^m} \mathbf{j}_\gamma(x,\xi)\xi^\gamma \, d\xi \right] =$$

$$\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\left[\int\limits_{\{\xi_1>|\xi'|\}^+}\left((1-\frac{i\varepsilon|\xi|^2}{P(\xi)+i\varepsilon|\xi|^2}\right)^m e^{-\delta|\xi|}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi+\right.$$

$$\left.e^{\mp m\pi i}(-1)^m\int\limits_{\{\xi_1<|\xi'|\}^+}\left((1-\frac{i\varepsilon|\xi|^2}{P(\xi)+i\varepsilon|\xi|^2}\right)^m e^{-\delta|\xi|}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi\right]=$$

$$\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^{m}C_m^k(-i\varepsilon)^k\left[\int\limits_{\{\xi_1>|\xi'|\}^+}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi+\right.$$

$$\left.\int\limits_{\{\xi_1<|\xi'|\}^+}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi\right]=$$

$$\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^{m}C_m^k(-i\varepsilon)^k\int\limits_{\mathbb{R}_+^n}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi.$$

For $m=0$, application of (10.32) gives

$$(\mathbf{F}_\gamma^{-1}e^{-\delta|\xi|})(x)=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\frac{2^{|\gamma|}\delta\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}(\delta^2+|x|^2)^{\frac{n+|\gamma|+1}{2}}}=$$

$$\frac{2^n\Gamma\left(\frac{n+|\gamma|+1}{2}\right)}{\sqrt{\pi}\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)}\delta\,(\delta^2+|x|^2)^{-\frac{n+|\gamma|+1}{2}}=P_\gamma(x,\delta). \tag{10.42}$$

Here $P_\gamma(x,\delta)$ is the general Poisson kernel (10.34). By Lemma 40, $P_\gamma(x,\delta)\in L_p^\gamma$.

Introducing the notation

$$A_k^{\gamma,\delta,\varepsilon}(x)=\int\limits_{\mathbb{R}_+^n}\frac{|\xi|^{2k}e^{-\delta|\xi|}}{(P(\xi)+i\varepsilon|\xi|^2)^k}\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\xi=\mathbf{F}_\gamma\frac{|x|^{2k}e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^k}$$

for $m>0$, we get

$$\mathbf{F}_\gamma^{-1}\frac{(P\mp i0)_\gamma^m e^{-\delta|x|}}{(P(x)+i\varepsilon|x|^2)^m}=\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\sum_{k=0}^{m}C_m^k(-i\varepsilon)^k A_k^{\gamma,\delta,\varepsilon}(x). \tag{10.43}$$

Substituting (10.42) and (10.43) in (10.41), we obtain the statement of the theorem for $f \in \Phi_V^\gamma$. ☐

Theorem 134. *Let* $f \in \Phi_V^\gamma$, $V = \{\xi \in \mathbb{R}_+^n : P(\xi) = 0\}$, $1 < p < \frac{n+|\gamma|}{\alpha}$, $p \leq 2$, $n + |\gamma| - 2 < \alpha < n + |\gamma|$. *Then*

$$((I^\alpha_{P\pm i0,\gamma})^{-1} I^\alpha_{P\pm i0,\gamma} f)(x) = f(x),$$

where

$$(I^\alpha_{P\pm i0,\gamma})^{-1} f) = \overset{L_p^\gamma}{\lim_{\delta\to 0}} \overset{L_2^\gamma}{\lim_{\varepsilon\to 0}} \left(\left(\mathbf{F}_\gamma^{-1} \frac{(P \mp i0)^{m+\frac{\alpha}{2}} e^{-\delta|\xi|}}{(P(\xi) + i\varepsilon|\xi|^2)^m}\right)(x) * f(x)\right)_\gamma,$$

where the limit by ε *is understood by the norm* L_2^γ *and the limit by* δ *is understood by the norm* L_p^γ.

Proof. From Lemma 45 it follows that it is enough to show

$$\overset{L_p^\gamma}{\lim_{\delta\to 0}} \overset{L_2^\gamma}{\lim_{\varepsilon\to 0}} \left[(\mathbf{P}_{\gamma,\delta} f)(x) + \frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \sum_{k=0}^m C_m^k (-i\varepsilon)^k (\mathbf{A}_k^{\gamma,\delta,\varepsilon} f)(x)\right] = f(x).$$

We find the limit for ε in L_2^γ. We have

$$(\mathbf{A}_k^{\gamma,\delta,\varepsilon} f)(x) = (A_k^{\gamma,\delta,\varepsilon}(x) * f(x))_\gamma =$$
$$\int\limits_{\mathbb{R}_+^n} \mathbf{F}_\gamma \left[\frac{|x|^{2k} e^{-\delta|x|}}{(P(x) + i\varepsilon|x|^2)^k}\right](y)({}^\gamma\mathbf{T}_x^y f)(x) y^\gamma \, dy =$$
$$\int\limits_{\mathbb{R}_+^n} \mathbf{F}_\gamma \left[\frac{|x|^{2k} e^{-\frac{\delta}{2}|x|}}{(P(x) + i\varepsilon|x|^2)^k} e^{-\frac{\delta}{2}|x|}\right](y)({}^\gamma\mathbf{T}_x^y f)(x) y^\gamma \, dy =$$
$$\int\limits_{\mathbb{R}_+^n} \mathbf{F}_\gamma \left[\frac{|x|^{2k} e^{-\frac{\delta}{2}|x|}}{(P(x) + i\varepsilon|x|^2)^k} \mathbf{F}_\gamma \left[P_\gamma\left(z, \frac{\delta}{2}\right)\right](x)\right](y)({}^\gamma\mathbf{T}_x^y f)(x) y^\gamma \, dy.$$

Applying the Parseval equation to the Hankel transform (see [242], p. 20), we obtain

$$||(-i\varepsilon)^k (\mathbf{A}_k^{\gamma,\delta,\varepsilon} f)(x)||^2_{2,\gamma} = ||(A_k^{\gamma,\delta,\varepsilon}(x) * f(x))_\gamma||^2_{2,\gamma} =$$
$$||\mathbf{F}_\gamma A_k^{\gamma,\delta,\varepsilon}(x) \cdot \mathbf{F}_\gamma f(x))_\gamma||^2_{2,\gamma} =$$
$$\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^n \Gamma^2\left(\frac{\gamma_j+1}{2}\right)} \int\limits_{\mathbb{R}_+^n} \left|\frac{(-i\varepsilon)^k |x|^{2k} e^{-\frac{\delta}{2}|x|}}{(P(x) + i\varepsilon|x|^2)^k} \mathbf{F}_\gamma \left[P_\gamma\left(x, \frac{\delta}{2}\right)\right] \mathbf{F}_\gamma f(x)\right|^2 x^\gamma dx =$$

$$\frac{2^{n-|\gamma|}}{\prod\limits_{j=1}^{n}\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^n_+}\left|\frac{(-i\varepsilon)^k|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2 x^\gamma dx.$$

Considering that

$$\left|\frac{(-i\varepsilon)^k|x|^{2k}e^{-\frac{\delta}{2}|x|}}{(P(x)+i\varepsilon|x|^2)^k}\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2 \leq e^{-\delta|x|}\left|\mathbf{F}_\gamma\left[(\mathbf{P}_{\gamma,\delta}f)(x)\right]\right|^2$$

and $e^{-\delta|x|}\left|\mathbf{F}_\gamma\left[P_\gamma\left(x,\frac{\delta}{2}\right)\right]\right|^2 \in L_1^\gamma$ on the basis of the Lebesgue dominated convergence theorem, we obtain

$$(-i\varepsilon)^k(\mathbf{A}_k^{\gamma,\delta,\varepsilon}f)(x)\to 0 \qquad \text{for} \qquad \varepsilon\to 0 \qquad \text{in} \qquad L_2^\gamma.$$

The fact that

$$||(\mathbf{P}_{\gamma,\delta}f)(x)-f(x)||_{p,\gamma}\to 0 \qquad \text{for} \qquad \delta\to 0$$

was proved in Lemma 41. Thus, the theorem is proved. □

10.3 Mixed hyperbolic Riesz B-potentials

In this section we consider the so-called *mixed hyperbolic Riesz B-potential*. This potential is the negative real power of the hyperbolic operator

$$\frac{\partial^2}{\partial t^2}-\sum_{k=1}^{n}(B_{\gamma_i})_{x_i}, \tag{10.44}$$

where $\gamma_1>0, ..., \gamma_n>0$ and $(B_{\gamma_i})_{x_i}=\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ is the singular differential Bessel operator.

10.3.1 Definition and basic properties of the mixed hyperbolic Riesz B-potential

Let $|x|=\sqrt{x_1^2+...+x_n^2}$. First for $(t,x)\in\mathbb{R}^{n+1}_+$, $\lambda\in C$ we define the function s^λ by the formula

$$s^\lambda(t,x)=\begin{cases}\frac{(t^2-|x|^2)^\lambda}{N(\alpha,\gamma,n)} & \text{when } t^2\geq|x|^2 \text{ and } t\geq 0,\\ 0 & \text{when } t^2<|x|^2 \text{ or } t<0,\end{cases} \tag{10.45}$$

where

$$N(\alpha,\gamma,n)=\frac{2^{\alpha-n-1}}{\sqrt{\pi}}\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{10.46}$$

We will denote the regular weighted generalized function corresponding to (10.45) by s_+^λ.

We introduce the *mixed hyperbolic Riesz B-potential* $I^\alpha_{s,\gamma}$ *of order* $\alpha>0$ as a mixed generalized convolution product (3.174) with a weighted generalized function $s_+^{\frac{\alpha-n-|\gamma|-1}{2}}$ and $f\in\mathfrak{S}_{ev}$:

$$(I^\alpha_{s,\gamma}f)(t,x)=\langle s_+^{\frac{\alpha-n-|\gamma|-1}{2}}*f\rangle_\gamma(t,x). \tag{10.47}$$

The precise definition of the constant $N(\alpha,\gamma,n)$ allows to obtain the semigroup property or index low of the potential (10.47).

We can rewrite formula (10.47) as

$$(I^\alpha_{s,\gamma}f)(t,x)=\int\limits_{\mathbb{R}^{n+1}_+}s_+^{\frac{\alpha-n-|\gamma|-1}{2}}(\tau,y)({}^\gamma\mathbf{T}^y_x)f(t-\tau,x)y^\gamma d\tau dy. \tag{10.48}$$

Since

$$(I^\alpha_{s,\gamma}f)(t,x)=\int\limits_{\mathbb{R}^{n+1}_+}s_+^{\frac{\alpha-n-|\gamma|-1}{2}}(\tau,y)({}^\gamma\mathbf{T}^y_x)f(t-\tau,x)y^\gamma d\tau dy=$$

$$\frac{1}{N(\alpha,\gamma,n)}\int\limits_{-\infty}^{+\infty}d\tau\int\limits_{|y|<\tau}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}({}^\gamma\mathbf{T}^y_x)f(t-\tau,x)y^\gamma dy=$$

$$\{y=r\theta\}=\frac{1}{N(\alpha,\gamma,n)}\int\limits_{-\infty}^{+\infty}d\tau\int\limits_0^\tau(\tau^2-r^2)^{\frac{\alpha-n-|\gamma|-1}{2}}r^{n+|\gamma|-1}dr\times$$

$$\int\limits_{S_1^+(n)}({}^\gamma\mathbf{T}^{r\theta}_x)f(t-\tau,x)\theta^\gamma dS,$$

using the weighted spherical mean (3.183) we obtain

$$(I^\alpha_{s,\gamma}f)(t,x)=\frac{2^{2-\alpha}\sqrt{\pi}}{\Gamma\left(\frac{n+|\gamma|}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_{-\infty}^{+\infty}d\tau\times$$

$$\int\limits_0^\tau(M_r^\gamma)_xf(t-\tau,x)(\tau^2-r^2)^{\frac{\alpha-n-|\gamma|-1}{2}}r^{n+|\gamma|-1}dr. \tag{10.49}$$

Theorem 135. *Let $n+|\gamma|-1<\alpha<n+|\gamma|+1$, $1\leq p<\frac{n+|\gamma|+1}{\alpha}$. For the estimate*

$$||I^{\alpha}_{s,\gamma}f||_{q,\gamma}\leq M||f||_{p,\gamma},\qquad f\in\mathfrak{S}_{ev},\tag{10.50}$$

to be valid it is necessary and sufficient that $q=\frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p}$. Constant M does not depend on f.

Remark 23. *By virtue of (10.50) there is a unique extension of $I^{\alpha}_{s,\gamma}$ to all $\mathcal{L}^{\gamma}_{p}$, $1<p<\frac{n+|\gamma|+1}{\alpha}$, preserving boundedness when $n+|\gamma|-1<\alpha<n+|\gamma|$. It follows that this extension is introduced by the integral (10.48) from its absolute convergence.*

Theorem 136. *For $f\in\mathfrak{S}_{ev}$ the Fourier–Hankel transform of mixed hyperbolic Riesz potential $I^{\alpha}_{s,\gamma}f$ is defined by the formula*

$$\mathcal{F}_{\gamma}[I^{\alpha}_{s,\gamma}f](\tau,\xi)=q\left|\tau^2-|\xi|^2\right|^{-\frac{\alpha}{2}}\cdot\mathcal{F}_{\gamma}[f(t,x)](\tau,\xi),\tag{10.51}$$

where

$$q=\begin{cases}1 & |\xi|^2\geq\tau^2,\\ e^{-\frac{\alpha\pi}{2}i} & |\xi|^2<\tau^2,\ \tau\geq 0,\\ e^{\frac{\alpha\pi}{2}i} & |\xi|^2<\tau^2,\ \tau<0.\end{cases}$$

10.3.2 Homogenizing kernel

Now we introduce the homogenizing kernel $N_{\gamma}(t,x,\varepsilon)$, which is defined as follows:

$$N_{\gamma}(t,x,\varepsilon)=\frac{C(n,\gamma,\varepsilon)}{(t^2+\varepsilon^2)(|x|^2+\varepsilon^2)^{\frac{n+|\gamma|}{2}}},$$

where $x=(x_1,...,x_n)$, $\varepsilon>0$,

$$C(n,\gamma,\varepsilon)=\frac{2^n\varepsilon^2\Gamma\left(\frac{n+1+|\gamma|}{2}\right)}{\pi^{\frac{3}{2}}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.$$

We give properties for the function $N_{\gamma}(t,x,\varepsilon)$ proved in Lemma 40 for a more general case.

Theorem 137. *The homogenizing kernel $N_{\gamma}(t,x,\varepsilon)$ has the following properties:*

1. $\mathcal{F}_{\gamma}[N_{\gamma}(t,x,\varepsilon)](\xi)=e^{-\varepsilon\tau-\varepsilon|\xi|}$,

2. $\int\limits_{\mathbb{R}^{n+1}_{+}}N_{\gamma}(t,x,\varepsilon)x^{\gamma}dtdx=\int\limits_{\mathbb{R}^{n+1}_{+}}N_{\gamma}(t,x,1)x^{\gamma}dtdx=1$,

3. $N_{\gamma}(t,x,\varepsilon)\in L^{\gamma}_{p}$, $1\leq p\leq\infty$.

Let us define the operator $N_{\gamma,\varepsilon}$ by the formula

$$(N_{\gamma,\varepsilon} f)(\tau, y) = \int\limits_{\mathbb{R}_+^{n+1}} N_\gamma(t, x, \varepsilon)\, {}^\gamma\mathbf{T}_x^y f(\tau - t, x) x^\gamma\, dt dx.$$

Theorem 138. *Let $f \in L_p^\gamma$. Then $(N_{\gamma,\varepsilon} f)(\tau, y)$ converges to $f(\tau, y)$ almost everywhere for ε tending to zero:*

$$\lim_{\varepsilon\to 0} (N_{\gamma,\varepsilon} f)(\tau, y) = f(\tau, y), \qquad a.e.$$

Proof. Let us consider

$$(N_{\gamma,\varepsilon} f)(\tau, y) = \int\limits_{\mathbb{R}_+^{n+1}} N_\gamma(t, x, \varepsilon)\, {}^\gamma\mathbf{T}_x^y f(\tau - t, x) x^\gamma\, dt dx =$$

$$C(n, \gamma, \varepsilon) \int\limits_{-\infty}^{+\infty} \frac{dt}{(t^2 + \varepsilon^2)} \int\limits_{R_+^n} \frac{{}^\gamma\mathbf{T}_x^y f(\tau - t, x) x^\gamma\, dx}{(|x|^2 + \varepsilon^2)^{\frac{n+|\gamma|}{2}}}.$$

The inner integral is estimated using [210], p. 315, Theorem 2 (see also [208]), and this estimate has the form

$$(N_{\gamma,\varepsilon} f)(\tau, y) \le C(n, \gamma, \varepsilon) \int\limits_{-\infty}^{+\infty} \frac{(M_x f)(\tau - t, \xi) dt}{(t^2 + \varepsilon^2)},$$

where M_x is the maximal function (see [210], p. 313). Applying now Stein's theorem (see [543], p. 77, Theorem 2), we obtain the following estimate:

$$\sup_{\varepsilon>0} |(N_{\gamma,\varepsilon} f)(\tau, y)| \le A (M_C f)(\tau, \xi),$$

where

$$(M_C f)(\tau, \xi) = \sup_{\beta>0} \sup_{r>0} \frac{1}{2\beta |B_\xi^+(r)|_\gamma} \int\limits_{|t-\tau|<\beta} dt \int\limits_{B_\xi^+(r)} |f(t, x)| x^\gamma\, dx$$

is the maximum function with respect to the cylinders in $\mathbb{R}_+^{n+1}$ with centers at the point ξ. Now the existence of the limit is proved, as also in the book [543], p. 58. □

10.4 Inversion of the mixed hyperbolic Riesz B-potentials

In this section we obtain an inverse operator for the mixed hyperbolic Riesz B-potential which generalizes the classical, presented in [425,426]. For the inversion

of the potential (10.47) we will use an approach based on the idea of approximative inverse operators as in Section 10.2. This method gives an inverse operator as a limit of regularized operators. Namely, taking into account formula (10.51), we will construct an inverse operator for the potential (10.47) in the form

$$(I^{\alpha}_{s,\gamma})^{-1}f=\lim_{\varepsilon\to 0}\left\langle \mathcal{F}^{-1}_{\gamma}(q|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|-\varepsilon|\xi|})*f\right\rangle_{\gamma},$$

where the limit is understood in the norm L^{γ}_{p} or almost everywhere. Let

$$g_{\alpha,\gamma,\varepsilon}(t,x)=\mathcal{F}^{-1}_{\gamma}(q^{-1}|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|-\varepsilon|\xi|})(t,x).$$

10.4.1 Auxiliary lemma

In order to prove that $g_{\alpha,\gamma,\varepsilon}$ belongs to $\mathcal{L}^{\gamma}_{r}(\mathbb{R}^{n+1}_{+})$, $1<r<\infty$, we first prove the following lemma.

Lemma 46. *If $m\in\mathbb{N}$, $2(m-1)<\alpha$,*

$$I^{+}_{0,\gamma}(\tau,x)=\int\limits_{0}^{\tau|x|}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}e^{-\frac{\varepsilon\rho}{|x|}}\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|}{2}-1}(\rho)\,d\rho, \tag{10.52}$$

then

$$I^{+}_{0,\gamma}(\tau,x)=\int\limits_{0}^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-m}J_{\frac{n+|\gamma|-2}{2}+m}(\rho)\times$$
$$\sum_{s=0}^{m}c_{m,s}\left(\rho\frac{d}{d\rho}\right)^{s}\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]d\rho, \tag{10.53}$$

where

$$\begin{aligned}
&c_{0,0}=0, \qquad c_{m,0}=0, \qquad m\geq 1,\\
&c_{1,1}=-1, \qquad c_{m,1}=2(m-1)c_{m-1,1}, \qquad m\geq 2,\\
&c_{m,s}=2(m-1)c_{m-1,s}-c_{m-1,s-1}, \qquad s=2,3,...,m-1, \qquad m\geq 2,\\
&c_{m,m}=-c_{m-1,m-1}, \qquad m\geq 2.
\end{aligned}$$

Moreover,

$$\left(\rho\frac{d}{d\rho}\right)^{s}\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]=$$
$$\sum_{l=0}^{p}\tau^{-2l}Q_{p+l}\left(\frac{\rho}{|x|}\right)\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}-l}e^{-\frac{\varepsilon\rho}{|x|}}, \tag{10.54}$$

where $Q_{p+l}(y)=\sum\limits_{j=2l}^{p+l} a_j(\alpha,\varepsilon,l)y^j$ *is a polynomial of degree* $p+l$. *So*

$$I^+_{0,\gamma}(\tau,x)=\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-m}\,J_{\frac{n+|\gamma|-2}{2}+m}(\rho)\sum_{s=1}^{m}c_{m,s}\sum_{l=0}^{p}\tau^{-2l}Q_{p+l}\left(\frac{\rho}{|x|}\right)\times$$
$$\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}-l}e^{-\frac{\varepsilon\rho}{|x|}}\,d\rho. \tag{10.55}$$

Proof. Following [425], we prove (10.53) using the method of mathematical induction.

First we show that formula (10.53) is true for $m=1$. Applying the formula

$$\frac{d}{dz}[z^\nu J_\nu(z)]=z^\nu J_{\nu-1}(z) \tag{10.56}$$

to (10.52), we obtain

$$I^+_{0,\gamma}(\tau,x)=\int\limits_0^{\tau|x|}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}e^{-\frac{\varepsilon\rho}{|x|}}\frac{d}{d\rho}\left[\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|}{2}}(\rho)\right]d\rho.$$

Integrating by parts, we have

$$u=e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}},\qquad du=\frac{d}{d\rho}\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]d\rho,$$
$$dv=\frac{d}{d\rho}\left[\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|}{2}}(\rho)\right]d\rho,\qquad v=\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|}{2}}(\rho)$$

and

$$I^+_{0,\gamma}(\tau,x)=e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|}{2}}(\rho)\Bigg|_{\rho=0}^{\tau|x|}-$$
$$\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|}{2}}(\rho)\frac{d}{d\rho}\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]d\rho=$$
$$c_{1,1}\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-1}J_{\frac{n+|\gamma|}{2}}(\rho)\left(\rho\frac{d}{d\rho}\right)\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]d\rho, \tag{10.57}$$

where $c_{1,1}=-1$. For $m=1$, formula (10.53) is proved.

Suppose that (10.53) is true for $m=k$. Then

$$I^{+}_{0,\gamma}(\tau,x)=$$
$$\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-k}\,J_{\frac{n+|\gamma|}{2}+k-1}(\rho)\sum_{s=0}^{k}c_{k,s}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]d\rho. \tag{10.58}$$

Applying (10.56) to (10.58), we can write

$$I^{+}_{0,\gamma}(\tau,x)=$$
$$\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}+k}\,J_{\frac{n+|\gamma|}{2}+k-1}(\rho)\sum_{s=1}^{k}c_{k,s}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\rho^{-2k}d\rho=$$
$$I^{+}_{0,\gamma}(\tau,x)=$$
$$\int\limits_0^{\tau|x|}\frac{d}{d\rho}\left[\rho^{\frac{n+|\gamma|}{2}+k}\,J_{\frac{n+|\gamma|}{2}+k}(\rho)\right]\sum_{s=0}^{k}c_{k,s}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\rho^{-2k}d\rho.$$

Integrating by parts, we obtain

$$u=\sum_{s=0}^{k}c_{k,s}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\rho^{-2k},$$
$$du=\sum_{s=0}^{k}c_{k,s}\left\{-2k\rho^{-2k-1}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]+\right.$$
$$\left.\rho^{-2k}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\right\}d\rho=$$
$$\sum_{s=0}^{k}c_{k,s}\rho^{-2k-1}\left\{-2k\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]+\right.$$
$$\left.\left(\rho\frac{d}{d\rho}\right)^{s+1}\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\right\}d\rho,$$
$$dv=\frac{d}{d\rho}\left[\rho^{\frac{n+|\gamma|}{2}+k}\,J_{\frac{n+|\gamma|}{2}+k}(\rho)\right]d\rho,$$
$$v=\rho^{\frac{n+|\gamma|}{2}+k}\,J_{\frac{n+|\gamma|}{2}+k}(\rho).$$

Therefore,

$$I^+_{0,\gamma}(\tau,x)=\rho^{\frac{n+|\gamma|}{2}+k}\,J_{\frac{n+|\gamma|}{2}+k}(\rho)\times$$
$$\sum_{s=0}^{k}c_{k,s}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\rho^{-2k}\Big|_{\rho=0}^{\tau|x|}-$$
$$\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}+k}\,J_{\frac{n+|\gamma|}{2}+k}(\rho)\sum_{s=0}^{k}c_{k,s}\,\rho^{-2k-1}\Bigg\{-2k\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]+$$
$$\left(\rho\frac{d}{d\rho}\right)^{s+1}\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\Bigg\}d\rho.$$

For $2k<\alpha$ we have

$$I^+_{0,\gamma}(\tau,x)=$$
$$-\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-(k+1)}\,J_{\frac{n+|\gamma|}{2}+k}(\rho)\Bigg\{\sum_{s=0}^{k}(-2k)c_{k,s}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]+$$
$$\sum_{s=0}^{k}c_{k,s}\left(\rho\frac{d}{d\rho}\right)^{s+1}\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\Bigg\}d\rho=$$
$$-\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-(k+1)}\,J_{\frac{n+|\gamma|}{2}+k}(\rho)\Bigg\{\sum_{s=0}^{k}(-2k)c_{k,s}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]+$$
$$\sum_{s=1}^{k+1}c_{k,s-1}\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\Bigg\}d\rho=$$
$$\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-(k+1)}\,J_{\frac{n+|\gamma|}{2}+k}(\rho)\left\{2kc_{k,1}\left(\rho\frac{d}{d\rho}\right)\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]+\right.$$
$$\sum_{s=1}^{k}[2kc_{k,s}-c_{k,s-1}]\left(\rho\frac{d}{d\rho}\right)^s\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]-$$
$$\left.c_{k,k}\left(\rho\frac{d}{d\rho}\right)^{k+1}\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right]\right\}d\rho.$$

Taking into account that

$$c_{k+1,1}=2kc_{k,1},\qquad k\geq 2,$$
$$c_{k+1,s}=2kc_{k,s}-c_{k,s-1},\qquad s=2,3,...,k,\qquad k\geq 2,$$
$$c_{k+1,k+1}=-c_{k,k},\qquad k\geq 2,$$

we obtain

$$I^+_{0,\gamma}(\tau,x)=\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-(k+1)}\,J_{\frac{n+|\gamma|}{2}+k}(\rho)\sum_{s=0}^{k+1}c_{k+1,s}\left(\rho\frac{d}{d\rho}\right)^s\times$$
$$\left[e^{-\frac{\varepsilon\rho}{|x|}}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}\right].$$

The proof is complete. □

10.4.2 Property of $\mathcal{L}_r^\gamma$-boundedness of the function $g_{\alpha,\gamma,\varepsilon}$

Using Lemma 46 we prove that $g_{\alpha,\gamma,\varepsilon}(t,x)$ belongs to $\mathcal{L}_r^\gamma(\mathbb{R}^{n+1}_+)$.

Theorem 139. *The function $g_{\alpha,\gamma,\varepsilon}(t,x)$ belongs to the space $\mathcal{L}_{r,\gamma}$, $1<r<\infty$, with the additional restriction $\frac{2(n+|\gamma|)-1}{2(n+|\gamma|)-2}<r$ for $n+|\gamma|-1<\alpha<n+|\gamma|$ when $n+|\gamma|+1$ is odd.*

Proof. Let us write the function $g_{\alpha,\gamma,\varepsilon}(t,x)$ as a sum

$$g_{\alpha,\gamma,\varepsilon}(t,x)=\frac{2^{n-|\gamma|-1}}{\pi\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\times$$
$$\int\limits_{\mathbb{R}^{n+1}_+}q\,|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}\,e^{-\varepsilon|\tau|-\varepsilon|\xi|-i\tau t}\,\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\tau d\xi=$$
$$\frac{2^{n-|\gamma|-1}}{\pi\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\times$$
$$\left[\int\limits_{\{\tau^2>|\xi|^2\}^+}e^{-\frac{\alpha\pi i}{2}\operatorname{sgn}\tau}\,|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}\,e^{-\varepsilon|\tau|-\varepsilon|\xi|-i\tau t}\,\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\tau d\xi+\right.$$
$$\left.\int\limits_{\{\tau^2<|\xi|^2\}^+}|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}\,e^{-\varepsilon|\tau|-\varepsilon|\xi|-i\tau t}\,\mathbf{j}_\gamma(x,\xi)\xi^\gamma\,d\tau d\xi\right]=$$
$$\frac{2^{n-|\gamma|-1}}{\pi\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}[J_1(t,x)+J_2(t,x)].$$

Consider the first integral $J_1(t,x)$:

$$J_1(t,x)=\int\limits_{-\infty}^{\infty}e^{-\frac{\alpha\pi i}{2}\operatorname{sgn}\tau}e^{-\varepsilon|\tau|-i\tau t}|\tau|^{\alpha}d\tau\int\limits_{\tau^2>|\xi|^2}\left(1-\frac{|\xi|^2}{\tau^2}\right)^{\frac{\alpha}{2}}e^{-\varepsilon|\xi|}\mathbf{j}_{\gamma}(x,\xi)\xi^{\gamma}\,d\xi.$$

By replacing variables $\xi=|\tau|y$ and passing to spherical coordinates $y{=}r\sigma$, $|y|{=}r$, $\sigma{=}(\sigma_1,...,\sigma_n)$ in the internal integral, we obtain

$$J_1(x)=\int\limits_{-\infty}^{\infty}e^{-\frac{\alpha\pi i}{2}\operatorname{sgn}\tau}e^{-\varepsilon|\tau|-i\tau t}|\tau|^{n+|\gamma|+\alpha}d\tau\times$$
$$\int\limits_{\{1>|y|\}^+}\left(1-|y|^2\right)^{\frac{\alpha}{2}}e^{-\varepsilon|\tau||y|}\mathbf{j}_{\gamma}(|\tau|y,x)y^{\gamma}\,dy=$$
$$\int\limits_{-\infty}^{\infty}e^{-\frac{\alpha\pi i}{2}\operatorname{sgn}\tau}e^{-\varepsilon|\tau|-i\tau t}|\tau|^{n+|\gamma|+\alpha}d\tau\int\limits_{0}^{1}\left(1-r^2\right)^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|r}\,r^{n+|\gamma|-1}dr\times$$
$$\int\limits_{S_1^+(n)}\mathbf{j}_{\gamma}(|\tau|r\sigma,x)\sigma^{\gamma}\,dS.$$

Integral over $S_1^+(n)$ is (3.140). Then

$$J_1(t,x)=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\times$$
$$\int\limits_{-\infty}^{\infty}e^{-\frac{\alpha\pi i}{2}\operatorname{sgn}\tau}e^{-\varepsilon|\tau|-i\tau}|\tau|^{n+|\gamma|+\alpha}d\tau\times$$
$$\int\limits_{0}^{1}\left(1-r^2\right)^{\frac{\alpha}{2}}e^{-\varepsilon|t|r}\,r^{n+|\gamma|-1}\,j_{\frac{n+|\gamma|-2}{2}}(|\tau||x|r)dr=$$
$$\frac{2^{\frac{n+|\gamma|-2}{2}}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}|x|^{\frac{n+|\gamma|-2}{2}}}\int\limits_{-\infty}^{\infty}e^{-\frac{\alpha\pi i}{2}\operatorname{sgn}\tau}e^{-\varepsilon|\tau|-i\tau t}|\tau|^{\frac{n+|\gamma|}{2}+\alpha+1}d\tau\times$$
$$\int\limits_{0}^{1}\left(1-r^2\right)^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|r}\,r^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|-2}{2}}(|\tau||x|r)dr.$$

Doing the same with the second integral $J_1(t,x)$, we can write the function $g_{\alpha,\varepsilon}$ as

$$g_{\alpha,\varepsilon}(x)=\frac{2^{\frac{n-2}{2}}}{\pi\prod\limits_{j=1}^{n}\Gamma\left(\frac{\gamma_j+1}{2}\right)|x|^{\frac{n+|\gamma|-2}{2}}}\left[e^{-\frac{\alpha\pi i}{2}}J^{+}_{0,\gamma}+e^{\frac{\alpha\pi i}{2}}J^{-}_{0,\gamma}+J^{+}_{\infty,\gamma}+J^{-}_{\infty,\gamma}\right],$$

where

$$J^{\pm}_{0,\gamma}(t,x)=|x|^{\frac{2-n-|\gamma|}{2}}\int\limits_0^\infty e^{-\tau(\varepsilon\pm it)}\tau^{\frac{n+|\gamma|}{2}+\alpha+1}d\tau\times$$

$$\int\limits_0^1\left(1-r^2\right)^{\frac{\alpha}{2}}e^{-\varepsilon\tau r}r^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|-2}{2}}(\tau|x|r)dr,$$

$$J^{\pm}_{\infty,\gamma}(t,x)=\int\limits_0^\infty e^{-\tau(\varepsilon\pm it)}\tau^{\frac{n+|\gamma|}{2}+\alpha+1}d\tau\int\limits_1^\infty\left(1-r^2\right)^{\frac{\alpha}{2}}e^{-\varepsilon\tau r}r^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|-2}{2}}(\tau|x|r)dr.$$

Let us consider $J^{+}_{0,\gamma}(t,x)$. By replacing variables $\tau|x|r=\rho$ in $J^{+}_{0,\gamma}(t,x)$, we obtain

$$J^{+}_{0,\gamma}(t,x)=|x|^{-n-|\gamma|}\int\limits_0^\infty e^{-\tau(\varepsilon+it)}\tau^{\alpha}I^{+}_{0,\gamma}(\tau,x)d\tau,$$

where

$$I^{+}_{0,\gamma}(\tau,x)=\int\limits_0^{\tau|x|}\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}}e^{-\varepsilon\frac{\rho}{|x|}}\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|-2}{2}}(\rho)d\rho.$$

Applying Lemma 46 to $I^{+}_{0,\gamma}(\tau,x)$, we obtain

$$I^{+}_{0,\gamma}(\tau,x)=\int\limits_0^{\tau|x|}\rho^{\frac{n+|\gamma|}{2}-m}J_{\frac{n+|\gamma|-2}{2}+m}(\rho)\sum_{s=0}^{m}c_{m,s}\sum_{l=0}^{s}\tau^{-2l}\sum_{j=2l}^{s+l}a_j(\alpha,\varepsilon,l)\times$$

$$\left(\frac{\rho}{|x|}\right)^j\left(1-\frac{\rho^2}{\tau^2|x|^2}\right)^{\frac{\alpha}{2}-l}e^{-\frac{\varepsilon\rho}{|x|}}d\rho.$$

So

$$J^{+}_{0,\gamma}(t,x)=\sum_{s=0}^{m}c_{m,s}\sum_{l=0}^{s}\sum_{j=2l}^{s+l}a_j(\alpha,\varepsilon,l)\frac{1}{|x|^{\frac{n+|\gamma|}{2}+m-1}}\times$$

$$\int\limits_0^1 \left(1-r^2\right)^{\frac{\alpha}{2}-l} r^{\frac{n+|\gamma|}{2}-m+j} d\rho \times$$

$$\int\limits_0^\infty \tau^{\frac{n+|\gamma|}{2}-m+\alpha-2l+j+1} e^{-\tau(\varepsilon+\varepsilon r+it)} J_{\frac{n+|\gamma|-2}{2}+m}(\tau|x|r)d\tau.$$

Using formula (2.12.8.4) from [456], we obtain

$$J^+_{0,\gamma}(t,x) = \sum_{s=0}^m c_{m,s} \sum_{l=0}^s \sum_{j=2l}^{s+l} a_j(\alpha,\varepsilon,l) \frac{\Gamma(n+|\gamma|+\alpha-2l+j+1)}{2^{\frac{n+|\gamma|-2}{2}+m}\Gamma\left(\frac{n+|\gamma|}{2}+m\right)} \times$$

$$\int\limits_0^1 \frac{\left(1-r^2\right)^{\frac{\alpha}{2}-l} r^{n+|\gamma|+j-1}}{(\varepsilon+\varepsilon r+it)^{n+|\gamma|+\alpha-2l+j+1}} \times$$

$${}_2F_1\left(\begin{array}{c} \frac{n+|\gamma|+\alpha-2l+j+1}{2}, \frac{n+|\gamma|+\alpha-2l+j+2}{2} \\ \frac{n+|\gamma|}{2}+m \end{array} \middle| -\frac{|x|^2r^2}{(\varepsilon+\varepsilon r+it)^2} \right) dr.$$

Let $b(m,l,j,\alpha) = \frac{\Gamma(n+|\gamma|+\alpha-2l+j+1)}{2^{\frac{n+|\gamma|-2}{2}+m}\Gamma\left(\frac{n+|\gamma|}{2}+m\right)}$. We estimate the weight norm $||J^+_{0,\gamma}||_{\mathcal{L}^\gamma_r(\mathbb{R})}$ applying Minkowski's inequality as

$$||J^+_{0,\gamma}||_{\mathcal{L}^\gamma_r(\mathbb{R})} = \sum_{s=0}^m |c_{m,s}| \sum_{l=0}^s \sum_{j=2l}^{s+l} |a_j(\alpha,\varepsilon,l) b(m,l,j,\alpha)| \times$$

$$\int\limits_0^1 \left(1-\rho^2\right)^{\frac{\alpha}{2}-l} \rho^{n+|\gamma|+j-1} d\rho \times$$

$$\left\{ \int\limits_{\mathbb{R}^{n+1}_+} \frac{\left| {}_2F_1\left(\begin{array}{c} \frac{n+|\gamma|+\alpha-2l+j+1}{2}, \frac{n+|\gamma|+\alpha-2l+j+2}{2} \\ \frac{n+|\gamma|}{2}+m \end{array} \middle| -\frac{|x|^2r^2}{(\varepsilon+\varepsilon r+it)^2} \right) \right|^r}{((\varepsilon+\varepsilon\rho)^2+t^2)^{\frac{n+|\gamma|+\alpha-2l+j+1}{2}r}} x^\gamma dt dx \right\}^{1/r}.$$

The substitution $t \to (\varepsilon+\varepsilon\rho)t$, $x \to \frac{\varepsilon+\varepsilon\rho}{\rho}x$ converts $||J^+_{0,\gamma}||_{r,\gamma}$ to

$$||J^+_{0,\gamma}||_{r,\gamma} = \sum_{s=0}^m |c_{m,s}| \sum_{l=0}^s \sum_{j=2l}^{s+l} |a_j(\alpha,\varepsilon,l) b(m,l,j,\alpha)| \times$$

$$\int\limits_0^1 \frac{\left(1-\rho^2\right)^{\frac{\alpha}{2}-l} \rho^{n+|\gamma|+j-1-\frac{n+|\gamma|}{r}}}{(\varepsilon+\varepsilon\rho)^{n+1+|\gamma|+\alpha-2l+j-\frac{n+1+|\gamma|}{r}}} d\rho \times$$

$$\left\{ \int\limits_{\mathbb{R}^{n+1}_+} \frac{\left| {}_2F_1 \left(\begin{array}{c} \frac{n+|\gamma|+\alpha-2l+j+1}{2}, \frac{n+|\gamma|+\alpha-2l+j+2}{2} \\ \frac{n+|\gamma|}{2}+m \end{array} \middle| -\frac{|x|^2}{(1+it)^2} \right) \right|^r}{((1+t^2)^{\frac{n+|\gamma|+\alpha-2l+j+1}{2}r}} x^\gamma \, dt dx \right\}^{1/r}.$$

Noting that the functions $\left(1-\rho^2\right)^{\frac{\alpha}{2}-l}$, $l \le m$, are integrable when $\rho = 1$ and $n+|\gamma|+j-1-\frac{n+|\gamma|}{r} > -1$ if $r > 1$, we obtain

$$\int\limits_0^1 \frac{\left(1-\rho^2\right)^{\frac{\alpha}{2}-l} \rho^{n+|\gamma|+j-1-\frac{n+|\gamma|}{r}}}{(\varepsilon+\varepsilon\rho)^{n+1+|\gamma|+\alpha-2l+j-\frac{n+1+|\gamma|}{r}}} d\rho < \infty.$$

Assuming

$$I^{j,l,m,r} = \int\limits_{\mathbb{R}^{n+1}_+} \frac{\left| {}_2F_1 \left(\begin{array}{c} \frac{n+|\gamma|+\alpha-2l+j+1}{2}, \frac{n+|\gamma|+\alpha-2l+j+2}{2} \\ \frac{n+|\gamma|}{2}+m \end{array} \middle| -\frac{|x|^2}{(1+it)^2} \right) \right|^r}{((1+t^2)^{\frac{n+|\gamma|+\alpha-2l+j+1}{2}r}} x^\gamma \, dt dx,$$

we can show that the integral $I^{j,l,m,r}$ converges. We have

$$I^{j,l,m,r} = \mathbf{J}_1^{j,l,m,r} + \mathbf{J}_2^{j,l,m,r} + \mathbf{J}_3^{j,l,m,r},$$

where

$$\mathbf{J}_1^{j,l,m,r} = |S_1^+(n)|_\gamma \int\limits_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^{\frac{n+|\gamma|+\alpha-2l+j+1}{2}r}} \int\limits_0^{\sqrt{\frac{1+t^2}{2}}} \rho^{n+|\gamma|-1} \times$$
$$\left| {}_2F_1 \left(\begin{array}{c} \frac{n+|\gamma|+\alpha-2l+j+1}{2}, \frac{n+|\gamma|+\alpha-2l+j+2}{2} \\ \frac{n+|\gamma|}{2}+m \end{array} \middle| -\frac{\rho^2}{(1+it)^2} \right) \right|^r d\rho,$$

$$\mathbf{J}_2^{j,l,m,r} = |S_1^+(n)|_\gamma \int\limits_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^{\frac{n+|\gamma|+\alpha-2l+j+1}{2}r}} \int\limits_{\sqrt{\frac{1+t^2}{2}}}^{\sqrt{2(1+t^2)}} \rho^{n+|\gamma|-1} \times$$
$$\left| {}_2F_1 \left(\begin{array}{c} \frac{n+|\gamma|+\alpha-2l+j+1}{2}, \frac{n+|\gamma|+\alpha-2l+j+2}{2} \\ \frac{n+|\gamma|}{2}+m \end{array} \middle| -\frac{\rho^2}{(1+it)^2} \right) \right|^r d\rho,$$

$$\mathbf{J}_3^{j,l,m,r} = |S_1^+(n)|_\gamma \int\limits_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^{\frac{n+|\gamma|+\alpha-2l+j+1}{2}r}} \int\limits_{\sqrt{2(1+t^2)}}^{+\infty} \rho^{n+|\gamma|-1} \times$$

$$\left|{}_2F_1\left(\begin{array}{c|c} \frac{n+|\gamma|+\alpha-2l+j+1}{2}, \frac{n+|\gamma|+\alpha-2l+j+2}{2} \\ \frac{n+|\gamma|}{2}+m \end{array} - \frac{\rho^2}{(1+it)^2}\right)\right|^r d\rho.$$

We estimate each of the integrals $\mathbf{J}_1^{j,l,m,r}$, $\mathbf{J}_2^{j,l,m,r}$, $\mathbf{J}_3^{j,l,m,r}$. The hypergeometric series ${}_2F_1(a,b;c;z)$ absolutely converges when $|z|<1$. Therefore,

$$\mathbf{J}_1^{j,l,m,r} =$$

$$C_1(l,j,m)) \int\limits_{-\infty}^{+\infty} \frac{dt}{(1+t^2)^{\frac{n+|\gamma|+\alpha-2l+j+1}{2}r}} \int\limits_0^{\sqrt{\frac{1+t^2}{2}}} \rho^{n+|\gamma|-1} d\rho =$$

$$C_2(l,j,m)) \int\limits_{-\infty}^{+\infty} (1+t^2)^{n+|\gamma|-\frac{n+|\gamma|+\alpha-2l+j+1}{2}r} dr < \infty$$

for $r>1$.

Consider now $\mathbf{J}_2^{j,l,m,r}$. Since $\left|\frac{\rho^2}{(1+it)^2}\right| \geq 1$, it can no longer be argued that the hypergeometric series in this expression converges absolutely. Transforming the hypergeometric series using formulas (15.3.3) and (15.3.7) from [2] we get a convergent series. So

$$\mathbf{J}_2^{j,l,m,r} \leq 2C_4(m,j,l) \int\limits_0^{+\infty} (1+t^2)^{\left(3l-m+1+\frac{\alpha+j}{2}\right)r} dt \times$$

$$\int\limits_{\sqrt{\frac{1+t^2}{2}}}^{\sqrt{2(1+t^2)}} \rho^{n+|\gamma|-1}((1-t^2+\rho^2)^2+4t^2)^{-\left(\frac{n+|\gamma|+3}{2}-m+\alpha+2l+j\right)\frac{r}{2}} d\rho.$$

The substitution $\rho^2 = 2t\eta + t^2 - 1$ gives

$$\mathbf{J}_2^{j,l,m,r} \leq \mathcal{I}_1^{j,m,l,r} + \mathcal{I}_2^{j,m,l,r},$$

where

$$\mathcal{I}_1^{j,m,l,r} = C_1(m,j,l) \int\limits_0^1 (1+t^2)^{\left(3l-m+1+\frac{\alpha+j}{2}\right)r} t^{1-\left(\frac{n+|\gamma|+3}{2}-m+\alpha+2l+j\right)r} dt \times$$

$$\int\limits_{\frac{3-t^2}{4t}}^{\frac{3+t^2}{4t}} (2t\eta+t^2-1)^{\frac{n+|\gamma|-2}{2}} (1+\eta^2)^{-\left(\frac{n+|\gamma|+3}{2}-m+\alpha+2l+j\right)\frac{r}{2}} d\eta,$$

$$\mathcal{I}_2^{j,m,l,r} = C_1(m,j,l) \int\limits_1^{+\infty} (1+t^2)^{\left(3l-m+1+\frac{\alpha+j}{2}\right)r} t^{1-\left(\frac{n+|\gamma|+3}{2}-m+\alpha+2l+j\right)r} dt \times$$

$$\int\limits_{\frac{3-t^2}{4t}}^{\frac{3+t^2}{4t}} (2t\eta+t^2-1)^{\frac{n+|\gamma|-2}{2}} (1+\eta^2)^{-\left(\frac{n+|\gamma|+3}{2}-m+\alpha+2l+j\right)\frac{r}{2}} d\eta.$$

Evaluating $\mathcal{I}_1^{j,m,l,r}$, we obtain

$$\mathcal{I}_1^{j,m,l,r} \leq C_2(m,j,l) \int\limits_0^1 (1+t^2)^{\frac{n+|\gamma|-2}{2}+\left(\frac{\alpha+j}{2}+1-l-m\right)r} dt, \qquad r>1.$$

Since $m=\left[\frac{n+1+|\gamma|}{2}\right]$ for $n-1+|\gamma|<\alpha<n+1+|\gamma|$ for noninteger or even $n+|\gamma|$ and for $n+|\gamma|\leq\alpha<n+1+|\gamma|$ for odd $n+|\gamma|$ we get

$$\mathcal{I}_2^{j,m,l,r} \leq C_3(m,j,l) \int\limits_1^{+\infty} (1+t^2)^{\left(3l-m+1+\frac{\alpha+j}{2}\right)r} t^{1-\left(\frac{n+|\gamma|+3}{2}-m+\alpha+2l+j\right)r} dt \times$$

$$\int\limits_{-\infty}^{+\infty} (1+\eta^2)^{-\left(\frac{n+|\gamma|+3}{2}-m+\alpha-2l+j\right)\frac{r}{2}} d\eta < \infty$$

if $\frac{2(n+|\gamma|)}{2(n+|\gamma|)-1}<r<+\infty$ for $n+|\gamma|-1<\alpha<n+|\gamma|$ and if $n+|\gamma|+1$ is odd and $1<r<+\infty$ in other cases.

It is similarly shown that the integral $\mathbf{J}_3^{j,l,m,r}$ converges. So $I_{0,\gamma}^{+}\in\mathcal{L}_r^{\gamma}(\mathbb{R}_+^n)$. Boundedness of the $\mathcal{L}_r^{\gamma}(\mathbb{R}_+^n)$-norm of $I_{0,\gamma}^{-}$, $I_{\infty,\gamma}^{+}$, and $I_{\infty,\gamma}^{-}$ is proved in the same way. □

10.4.3 Inversion theorems

Theorem 140. *Let $n+|\gamma|-1<\alpha<n+1+|\gamma|$, $1<p<\frac{n+1+|\gamma|}{\alpha}$ with the additional restriction $p<\frac{2(n+1+|\gamma|)(n+|\gamma|)}{n+|\gamma|+3\alpha(n+|\gamma|)}$ when $n+|\gamma|-1<\alpha<n+|\gamma|$ and n is odd. Then for $f(x)\in\mathcal{L}_p^{\gamma}$ the following representation is true:*

$$((I_{s,\gamma}^{\alpha})_{\varepsilon}^{-1} I_{s,\gamma}^{\alpha} f)(t,x) = (N_{\gamma,\varepsilon} f)(t,x), \tag{10.59}$$

where $(I_{s,\gamma}^{\alpha})_{\varepsilon}^{-1} f = \left(\mathcal{F}_{\gamma}^{-1}(q|\tau^2-|\xi|^2|^{\frac{\alpha}{2}} e^{-\varepsilon|\tau|-\varepsilon|\xi|}) * f\right)_{\gamma}.$

Proof. Let us consider the operator $(I^\alpha_{s,\gamma})^{-1}_\varepsilon I^\alpha_{s,\gamma}$. The following Young inequality for generalized convolution is well known (see, for example, [419]):

$$||(f*g)_\gamma||_{r,\gamma}\leq||f||_{p,\gamma}||g||_{q,\gamma},\qquad 1\leq p,q,r\leq\infty,\qquad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1. \tag{10.60}$$

Using (10.60) we obtain that the operator $I^\alpha_{s,\gamma}$ is bounded from $\mathcal{L}_{p,\gamma}$ to $\mathcal{L}_{\mathfrak{l},\gamma}$, where $\frac{1}{\mathfrak{l}}=\frac{1}{r}+\frac{1}{q}-1$, $r=\frac{(n+1+|\gamma|)p}{n+1+|\gamma|-\alpha p}$, and q satisfies the conditions of Theorem 135. The operator $N_{\gamma,\varepsilon}$ is bounded in $\mathcal{L}_{p,\gamma}$. Therefore, it suffices to verify equality on a weighted Lizorkin space $\widetilde{\Phi}_{V,\gamma}$ which is dense in $\mathcal{L}_{p,\gamma}$ (see [343,346]), where $V=\{\tau^2-|\xi|^2=0\}\cup\{\tau=0\}\cup\{\xi_i=0, i{=}1,...,n\}$, $(\tau,\xi)\in\mathbb{R}^{n+1}_+$. Thus

$$\mathcal{F}_\gamma\left[\left((I^\alpha_{s,\gamma})^{-1}_\varepsilon\right)I^\alpha_{s,\gamma}f\right](\tau,\xi){=}q^{-1}|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|-\varepsilon|\xi|}\mathcal{F}_\gamma[I^\alpha_{s,\gamma}f](\tau,\xi){=}$$
$$e^{-\varepsilon|\tau|-\varepsilon|\xi|}\mathcal{F}_\gamma[f](\tau,\xi)=\mathcal{F}_\gamma\left[\left(\mathcal{F}^{-1}_\gamma e^{-\varepsilon|\tau|-\varepsilon|\xi|}*f\right)_\gamma\right](\tau,\xi)$$

and

$$\mathcal{F}_\gamma\left[\left((I^\alpha_{s,\gamma})^{-1}_\varepsilon\right)I^\alpha_{s,\gamma}f\right](\tau,\xi){=}\mathcal{F}_\gamma\left[\left(\mathcal{F}^{-1}_\gamma e^{-\varepsilon|\tau|-\varepsilon|\xi|}*f\right)_\gamma\right](\tau,\xi). \tag{10.61}$$

Applying the inverse Fourier–Hankel transform to (10.61), we get (10.59).

□

Theorem 141. *Let $n+|\gamma|-1<\alpha<n+1+|\gamma|$, $1<p<\frac{n+1+|\gamma|}{\alpha}$ with the additional restriction $p<\frac{2(n+1+|\gamma|)(n+|\gamma|)}{n+1+|\gamma|+2\alpha(n+|\gamma|)}$ when $n+|\gamma|-1<\alpha<n+|\gamma|$ and n is odd. Then*

$$((I^\alpha_{s,\gamma})^{-1}I^\alpha_{s,\gamma}f)(t,x)=f(t,x),\qquad f(t,x)\in L^\gamma_p,$$

where $(I^\alpha_{s,\gamma})^{-1}f=\lim\limits_{\varepsilon\to0}(I^\alpha_{s,\gamma})^{-1}_\varepsilon f$.

The theorem follows from Theorems 140 and 138.

Fractional differential equations with singular coefficients

11

11.1 Meijer transform method for the solution to homogeneous fractional equations with left-sided fractional Bessel derivatives on semiaxes of Gerasimov–Caputo type

11.1.1 *General case*

Using the Meijer transform method we will solve the equation

$$(\mathcal{B}^{\alpha}_{\gamma,0+}f)(x)=\lambda f(x),\qquad \alpha>0,\qquad \lambda\in\mathbb{R},\tag{11.1}$$

with the left-sided fractional Bessel derivatives on semiaxes of Gerasimov–Caputo type with constant coefficient when $\gamma\neq 1$.

Let $\frac{m-1}{2}<\alpha\leq\frac{m}{2}$, $m\in\mathbb{N}$. To Eq. (11.1) we should add m conditions which for $0\leq\gamma<1$ have the form

$$(B^k_{\gamma,0+}f)(0+)=a_{2k},\qquad \lim_{x\to 0+}x^{\gamma}\frac{d}{dx}B^k_{\gamma,0+}f(x)=a_{2k+1},\qquad a_{2k},a_{2k+1}\in\mathbb{R},\tag{11.2}$$

and for the case when $\gamma>1$ we should consider the conditions

$$(B^k_{\gamma,0+}f)(0+)=b_{2k},\qquad \lim_{x\to 0+}x\frac{d}{dx}B^k_{\gamma,0+}f(x)=b_{2k+1},\qquad b_{2k},b_{2k+1}\in\mathbb{R},\tag{11.3}$$

where $k\in\mathbb{N}\cup\{0\}$, such that the following inequalities are true:

$$0\leq 2k\leq m-1,\qquad 1\leq 2k+1\leq m-2\qquad \text{if } m \text{ is odd}$$

and

$$1\leq 2k+1\leq m-1,\qquad 0\leq 2k\leq m-2\qquad \text{if } m \text{ is even}.$$

That means that for odd m the last condition is $(B^k_{\gamma,0+}f)(0+)=a_{m-1}$ or $(B^k_{\gamma,0+}f)(0+)=b_{m-1}$ and for even m the last condition is $\lim\limits_{x\to 0+}x^{\gamma}\frac{d}{dx}B^k_{\gamma,0+}f(x)=a_{m-1}$ or $\lim\limits_{x\to 0+}x\frac{d}{dx}B^k_{\gamma,0+}f(x)=b_{m-1}$.

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Theorem 142. *When $0 \le \gamma < 1$ the solution to (11.1)–(11.2)*

for the case when m is odd is

$$f(x) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^{\frac{m-1}{2}} a_{2k}\, x^{2k}\, {}_2\Psi_2\left[\begin{array}{c} \left(k+1+\frac{\gamma}{2}, \alpha\right), (1,1) \\ (k+1,\alpha), (2k+\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right] +$$
$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^{\frac{m-3}{2}} a_{2k+1}\, x^{2k+1-\gamma}\, {}_2\Psi_2\left[\begin{array}{c} \left(k+\frac{3}{2}, \alpha\right), (1,1) \\ \left(k+\frac{3-\gamma}{2}, \alpha\right), (2k+2, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right], \tag{11.4}$$

where the second sum vanishes if $\frac{m-3}{2} < 0$, i.e., when $m = 1$,

and for the case when m is even, it is

$$f(x) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^{\frac{m-2}{2}} a_{2k}\, x^{2k}\, {}_2\Psi_2\left[\begin{array}{c} \left(k+1+\frac{\gamma}{2}, \alpha\right), (1,1) \\ (k+1,\alpha), (2k+\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right] +$$
$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^{\frac{m-2}{2}} a_{2k+1}\, x^{2k+1-\gamma}\, {}_2\Psi_2\left[\begin{array}{c} \left(k+\frac{3}{2}, \alpha\right), (1,1) \\ \left(k+\frac{3-\gamma}{2}, \alpha\right), (2k+2, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right]. \tag{11.5}$$

When $\gamma > 1$ the solution to (11.1)–(11.3) for the case when m is odd is

$$f(x) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^{\frac{m-1}{2}} b_{2k}\, x^{2k}\, {}_2\Psi_2\left[\begin{array}{c} \left(k+1+\frac{\gamma}{2}, \alpha\right), (1,1) \\ (k+1,\alpha), (2k+\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right] +$$
$$\frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}(\gamma-1)} \sum_{k=0}^{\frac{m-3}{2}} b_{2k+1}\, x^{2k}\, {}_2\Psi_2\left[\begin{array}{c} \left(k+1+\frac{\gamma}{2}, \alpha\right), (1,1) \\ (k+1,\alpha), (2k+\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right], \tag{11.6}$$

where the second sum vanishes if $\frac{m-3}{2} < 0$, i.e., when $m = 1$,

and for the case when m is even, it is

$$f(x) = \frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \sum_{k=0}^{\frac{m-2}{2}} b_{2k}\, x^{2k}\, {}_2\Psi_2\left[\begin{array}{c} \left(k+1+\frac{\gamma}{2}, \alpha\right), (1,1) \\ (k+1,\alpha), (2k+\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right] +$$
$$\frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}(\gamma-1)} \sum_{k=0}^{\frac{m-2}{2}} b_{2k+1}\, x^{2k}\, {}_2\Psi_2\left[\begin{array}{c} \left(k+1+\frac{\gamma}{2}, \alpha\right), (1,1) \\ (k+1,\alpha), (2k+\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha}\right]. \tag{11.7}$$

Here ${}_p\Psi_q(z)$ is the Fox–Wright function (1.38).

Proof. First we consider the case when $0 \le \gamma < 1$. Applying the Meijer transform (1.58) to both parts of (11.1) and using (9.44), we obtain

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi) - \sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}B_\gamma^k f(0+) -$$
$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\lim_{x\to 0+}\sum_{k=0}^{n-1}\xi^{2\alpha-2k-2}x^\gamma\frac{d}{dx}[B_\gamma^k f(x)] = \lambda\mathcal{K}_\gamma[f](\xi),$$

where $n \in \mathbb{N}$, $n-1 < \alpha \le n$. Taking into account the conditions in (11.2),

for the case when m is odd we obtain

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi) - \sum_{k=0}^{\frac{m-1}{2}}a_{2k}\xi^{2\alpha-2k-1-\gamma} - \frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\sum_{k=0}^{\frac{m-3}{2}}a_{2k+1}\xi^{2\alpha-2k-2} =$$
$$\lambda\mathcal{K}_\gamma[f](\xi),$$

where the second sum vanishes if $\frac{m-3}{2} < 0$, i.e., when $m = 1$,

and for the case when m is even, we obtain

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi) - \sum_{k=0}^{\frac{m-2}{2}}a_{2k}\xi^{2\alpha-2k-1-\gamma} - \frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\sum_{k=0}^{\frac{m-2}{2}}a_{2k-1}\xi^{2\alpha-2k-2} =$$
$$\lambda\mathcal{K}_\gamma[f](\xi).$$

Therefore,

for the case when m is odd,

$$f(x) = \sum_{k=0}^{\frac{m-1}{2}}a_{2k}\,\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x)+$$
$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\sum_{k=0}^{\frac{m-3}{2}}a_{2k+1}\,\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-2}}{\xi^{2\alpha}-\lambda}\right](x),$$

and for the case when m is even,

$$f(x) = \sum_{k=0}^{\frac{m-2}{2}}a_{2k}\,\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x)+$$
$$\frac{\Gamma\left(\frac{1-\gamma}{2}\right)}{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}\sum_{k=0}^{\frac{m-2}{2}}a_{2k+1}\,\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-2}}{\xi^{2\alpha}-\lambda}\right](x).$$

In order to find an explicit expression for f we will use the formula for inversion of the Laplace transform. So first let us find inverse Laplace transforms taking into account formula (1.55):

$$\mathcal{L}^{-1}\left[\frac{\xi^{2\alpha-2k-2}}{\xi^{2\alpha}-\lambda}\right](x)=x^{2k+1}E_{2\alpha,2k+2}(\lambda x^{2\alpha}),$$
$$\mathcal{L}^{-1}\left[\frac{\xi^{2\alpha-2k-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x)=x^{2k+\gamma}E_{2\alpha,2k+\gamma+1}(\lambda x^{2\alpha}).$$

Now let us find $(\mathcal{P}_x^\gamma)^{-1}x^{\beta-\gamma}E_{2\alpha,\beta}(\lambda x^{2\alpha})$. Using (3.124) we can write

$$(\mathcal{P}_x^\gamma)^{-1}x^{\beta-\gamma}E_{2\alpha,\beta}(\lambda x^{2\alpha})=$$
$$\frac{2\sqrt{\pi}x}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(p-\frac{\gamma}{2}\right)}\left(\frac{d}{2xdx}\right)^p\int\limits_0^x z^\beta E_{2\alpha,\beta}(\lambda z^{2\alpha})(x^2-z^2)^{p-\frac{\gamma}{2}-1}dz,$$

where $p=\left[\frac{\gamma}{2}\right]+1$. We have

$$E_{2\alpha,\beta}(\lambda z^{2\alpha})=\sum_{m=0}^\infty\frac{\lambda^m z^{2m\alpha}}{\Gamma(2\alpha m+\beta)}$$

and

$$\int\limits_0^x z^\beta E_{2\alpha,\beta}(\lambda z^{2\alpha})(x^2-z^2)^{p-\frac{\gamma}{2}-1}dz=$$
$$\sum_{m=0}^\infty\frac{\lambda^m}{\Gamma(2\alpha m+\beta)}\int\limits_0^x z^{2m\alpha+\beta}(x^2-z^2)^{p-\frac{\gamma}{2}-1}dz=$$
$$\sum_{m=0}^\infty\frac{\lambda^m}{\Gamma(2\alpha m+\beta)}\frac{\Gamma\left(m\alpha+\frac{\beta+1}{2}\right)\Gamma\left(p-\frac{\gamma}{2}\right)}{2\Gamma\left(m\alpha+p+\frac{\beta-\gamma+1}{2}\right)}x^{2m\alpha+2p+\beta-\gamma-1}.$$

Therefore,

$$(\mathcal{P}_x^\gamma)^{-1}x^{\beta-\gamma}E_{2\alpha,\beta}(\lambda x^{2\alpha})=$$
$$\frac{\sqrt{\pi}x}{\Gamma\left(\frac{\gamma+1}{2}\right)}\left(\frac{d}{2xdx}\right)^p\sum_{m=0}^\infty\frac{\lambda^m}{\Gamma(2\alpha m+\beta)}\frac{\Gamma\left(m\alpha+\frac{\beta+1}{2}\right)}{\Gamma\left(m\alpha+p+\frac{\beta-\gamma+1}{2}\right)}x^{2m\alpha+2p+\beta-\gamma-1}.$$

Using the formula

$$\left(\frac{d}{2xdx}\right)^n x^{2\mu+2n}=\frac{\Gamma(\mu+n+1)}{\Gamma(\mu+1)}x^{2\mu},$$

we get

$$(\mathcal{P}_x^\gamma)^{-1}x^{\beta-\gamma}E_{2\alpha,\beta}(\lambda x^{2\alpha})=\frac{\sqrt{\pi}x^{\beta-\gamma}}{\Gamma\left(\frac{\gamma+1}{2}\right)}\sum_{m=0}^{\infty}\frac{\Gamma\left(m\alpha+\frac{\beta+1}{2}\right)}{\Gamma(2\alpha m+\beta)\Gamma\left(m\alpha+\frac{\beta-\gamma+1}{2}\right)}(\lambda x^{2\alpha})^m.$$

Using the Fox–Wright function (1.38), we can write

$$(\mathcal{P}_x^\gamma)^{-1}x^{\beta-\gamma}E_{2\alpha,\beta}(\lambda x^{2\alpha})=\frac{\sqrt{\pi}x^{\beta-\gamma}}{\Gamma\left(\frac{\gamma+1}{2}\right)}\,{}_2\Psi_2\left[\begin{array}{c}\left(\frac{\beta+1}{2},\alpha\right),(1,1)\\ \left(\frac{\beta-\gamma+1}{2},\alpha\right),(\beta,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right].$$

So

$$\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-2}}{\xi^{2\alpha}-\lambda}\right](x)=\frac{1}{A_\gamma x}(\mathcal{P}_x^\gamma)^{-1}x^{1-\gamma}\left(\mathcal{L}^{-1}\left[\frac{\xi^{2\alpha-2k-2}}{\xi^{2\alpha}-\lambda}\right]\right)(x)=$$
$$\frac{1}{A_\gamma x}(\mathcal{P}_x^\gamma)^{-1}x^{2k+2-\gamma}E_{2\alpha,2k+2}(\lambda x^{2\alpha})=$$
$$\frac{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}x^{2k+1-\gamma}\,{}_2\Psi_2\left[\begin{array}{c}\left(k+\frac{3}{2},\alpha\right),(1,1)\\ \left(k+\frac{3-\gamma}{2},\alpha\right),(2k+2,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right],$$
$$\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x)=$$
$$\frac{2^\gamma\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}x^{2k}\,{}_2\Psi_2\left[\begin{array}{c}\left(k+1+\frac{\gamma}{2},\alpha\right),(1,1)\\ (k+1,\alpha),(2k+\gamma+1,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right]. \tag{11.8}$$

Then for the case when m is odd we get (11.4) and for the case when m is even we get (11.5).

For $\gamma>1$, applying the Meijer transform (1.58) to both parts of (11.1) and using (9.46), we obtain

$$\xi^{2\alpha}\mathcal{K}_\gamma[f](\xi)-\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}B_\gamma^k f(0+)-$$
$$\frac{1}{\gamma-1}\lim_{x\to0+}\sum_{k=0}^{n-1}\xi^{2\alpha-2k-1-\gamma}x\frac{d}{dx}[B_\gamma^k f(x)]=\lambda\mathcal{K}_\gamma[f](\xi),$$

where $n\in\mathbb{N}$, $n-1<\alpha\le n$. Taking into account the conditions in (11.3), for the case when m is odd we obtain

$$f(x)=\sum_{k=0}^{\frac{m-1}{2}}b_{2k}\,\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x)+$$

$$\frac{1}{\gamma-1}\sum_{k=0}^{\frac{m-3}{2}} b_{2k+1}\,\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x),$$

and for the case when m is even, we obtain

$$f(x)=\sum_{k=0}^{\frac{m-2}{2}} b_{2k}\,\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x)+$$

$$\frac{1}{\gamma-1}\sum_{k=0}^{\frac{m-2}{2}} b_{2k+1}\,\mathcal{K}_\gamma^{-1}\left[\frac{\xi^{2\alpha-2k-1-\gamma}}{\xi^{2\alpha}-\lambda}\right](x).$$

Therefore, applying (11.8) we obtain (11.6) and (11.7). □

11.1.2 Particular cases and examples

In this subsection first we consider Eq. (11.1) when the conditions of Remark 21 are valid. Then we give some examples.

Theorem 143. *Let $k,m\in\mathbb{N}$, $\frac{m-1}{2}<\alpha\leq\frac{m}{2}$, and let $\frac{d}{dx}[B_\gamma^k f(x)]$ be bounded for $0<\gamma$, $\gamma\neq 1$, and $\frac{d}{dx}[B_\gamma^k f(x)]\sim x^\beta$, $\beta>0$ when $x\to 0+$. Then the solution to the equation*

$$(\mathcal{B}^\alpha_{\gamma,0+}f)(x)=\lambda f(x),\qquad \alpha>0,\qquad \lambda\in\mathbb{R}, \tag{11.9}$$

with m conditions for $0\leq\gamma<1$ of the form

$$(B^k_{\gamma,0+}f)(0+)=a_{2k},\qquad \lim_{x\to 0+} x^\gamma\frac{d}{dx}B^k_{\gamma,0+}f(x)=0, \tag{11.10}$$

with m conditions for $\gamma=1$ of the form

$$(B^k_{\gamma,0+}f)(0+)=a_{2k},\qquad \lim_{x\to 0+}\ln x\xi\frac{d}{dx}[B_\gamma^k f(x)]=0, \tag{11.11}$$

and with m conditions for $\gamma>1$ of the form

$$(B^k_{\gamma,0+}f)(0+)=a_{2k},\qquad \lim_{x\to 0+} x\frac{d}{dx}B^k_{\gamma,0+}f(x)=0, \tag{11.12}$$

where $a_{2k}\in\mathbb{R}$, $k\in\mathbb{N}\cup\{0\}$, such that

$$0\leq 2k\leq m-1,\qquad 1\leq 2k+1\leq m-2\qquad \textit{if } m \textit{ is odd}$$

and

$$1\leq 2k+1\leq m-1,\qquad 0\leq 2k\leq m-2\qquad \textit{if } m \textit{ is even},$$

for the case $m=1$ is $f(x)=0$, for the case of odd $m\geq 3$ is

$$f(x)=\frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}\sum_{k=0}^{\frac{m-1}{2}}a_{2k}\,x^{2k}\,{}_2\Psi_2\left[\begin{array}{c}\left(k+1+\frac{\gamma}{2},\alpha\right),(1,1)\\(k+1,\alpha),(2k+\gamma+1,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right],\tag{11.13}$$

and for the case of even m is

$$f(x)=\frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}\sum_{k=0}^{\frac{m-2}{2}}a_{2k}\,x^{2k}\,{}_2\Psi_2\left[\begin{array}{c}\left(k+1+\frac{\gamma}{2},\alpha\right),(1,1)\\(k+1,\alpha),(2k+\gamma+1,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right].\tag{11.14}$$

Here ${}_p\Psi_q(z)$ is the Fox–Wright function (1.38).

Example 1. *Let us consider the general case of the problem (11.1)–(11.2) when $0<\alpha\leq\frac{1}{2}$, $0\leq\gamma<1$. In this case $m=1$, $2k=0$ and using (11.4), we obtain that the solution to the problem*

$$(\mathcal{B}^{\alpha}_{\gamma,0+}f)(x)=\lambda f(x),\qquad \alpha>0,\qquad \lambda\in\mathbb{R},$$
$$f(0+)=a_0,\qquad a_1\in\mathbb{R}$$

is

$$f(x)=\frac{2^{\gamma}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}}a_0\,{}_2\Psi_2\left[\begin{array}{c}\left(1+\frac{\gamma}{2},\alpha\right),(1,1)\\(1,\alpha),(\gamma+1,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right].\tag{11.15}$$

It is easy to see that for $\gamma>1$ the solution has the same form when $0<\alpha\leq\frac{1}{2}$.

When $\gamma=0$, we obtain

$$({}^{GC}D^{2\alpha}_{0+}f)(x)=\lambda f(x),\qquad 0<2\alpha\leq 1,\qquad \lambda\in\mathbb{R},$$
$$f(0+)=a_1,\qquad a_1\in\mathbb{R},$$

and using (1.40) we obtain

$$f(x)=a_0\,{}_2\Psi_2\left[\begin{array}{c}(1,\alpha),(1,1)\\(1,\alpha),(1,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right]=$$
$$a_0\,{}_1\Psi_1\left[\begin{array}{c}(1,1)\\(1,2\alpha)\end{array}\middle|\lambda x^{2\alpha}\right]=a_0\,E_{2\alpha,1}(\lambda x^{2\alpha}),$$

which coincides with (2.50) if $l=1$ and 2α is taken instead of α.

Example 2. *Let us consider the case presented in Theorem 143 when $\alpha=1$, $b_0=1$, $\lambda=-1$. In this case $m=2$, $2k=0$, $2k+1=1$, which means $k=0$, and using (11.14) we obtain that the solution to the problem*

$$B_\gamma f(x) = -f(x), \qquad \lambda \in \mathbb{R},$$
$$f(0+) = 1, \qquad f'(0+) = 0$$

is

$$f(x) = \frac{2^\gamma \Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} {}_2\Psi_2 \left[\begin{array}{c} \left(1+\frac{\gamma}{2}, 1\right), (1,1) \\ (1,1), (\gamma+1, 2) \end{array} \middle| -x^2 \right]$$
$$= \frac{2^\gamma \Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} {}_2\Psi_2 \left[\begin{array}{c} \left(1+\frac{\gamma}{2}, 1\right) \\ (\gamma+1, 2) \end{array} \middle| -x^2 \right]$$
$$= \frac{2^\gamma \Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}} \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(1+\frac{\gamma}{2}+m\right)}{\Gamma(\gamma+1+2m)} \frac{x^{2m}}{m!}.$$

Using the Legendre duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z+\frac{1}{2}\right),$$

we obtain

$$f(x) = 2^\gamma \Gamma\left(\frac{\gamma+1}{2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(1+\frac{\gamma}{2}+m\right)}{2^{\gamma+2m}\Gamma\left(1+\frac{\gamma}{2}+m\right)\Gamma\left(\frac{\gamma+1}{2}+m\right)} \frac{x^{2m}}{m!} =$$
$$\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)}{x^{\frac{\gamma-1}{2}}} \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma\left(\frac{\gamma+1}{2}+m\right)} \frac{1}{m!} \left(\frac{x}{2}\right)^{2m+\frac{\gamma-1}{2}} = j_{\frac{\gamma-1}{2}}(x), \tag{11.16}$$

where

$$j_\nu(x) = \frac{2^\nu \Gamma(\nu+1)}{x^\nu} J_\nu(x).$$

For $j_{\frac{\gamma-1}{2}}(x)$ *we have*

$$B_\gamma j_{\frac{\gamma-1}{2}}(\tau x) = -\tau^2 j_{\frac{\gamma-1}{2}}(\tau x).$$

Therefore, the function

$${}_2\Psi_2 \left[\begin{array}{c} \left(1+\frac{\gamma}{2}, \alpha\right), (1,1) \\ (1,\alpha), (\gamma+1, 2\alpha) \end{array} \middle| \lambda x^{2\alpha} \right]$$

can be considered as a generalization of $j_{\frac{\gamma-1}{2}}$.

11.2 Mellin transform method

11.2.1 Ordinary linear nonhomogeneous differential equations of fractional order on semiaxes

Here, following [241], we apply the one-dimensional direct and inverse Mellin integral transforms to derive particular solutions to such ordinary linear nonhomogeneous differential equations as

$$\sum_{k=0}^{m} A_k x^{2(\alpha+k)}(DB_{\gamma,-}^{\alpha+k} f)(x) = h(x), \qquad x>0, \qquad \alpha>0 \tag{11.17}$$

with constants $A_k \in \mathbb{R}$, $k=0, ..., 1$.

The Mellin transform method for solving Eq. (11.17) is based on the following relation:

$$\begin{aligned}&(x^{2(\alpha+k)}(DB_{\gamma,-}^{\alpha+k} f)(x))^*(s) = \\ &2^{2(\alpha+k)}\,\Gamma\left[\begin{array}{cc} \frac{s}{2}+\alpha+k, & \frac{s-\gamma+1}{2}+\alpha+k \\ \frac{s}{2}, & \frac{s-\gamma+1}{2}\end{array}\right] f^*(s),\end{aligned} \tag{11.18}$$

which we obtained from (9.30).

Applying the Mellin transform to (11.17) and using (11.18), we obtain

$$\sum_{k=0}^{m} A_k 2^{2(\alpha+k)}\,\Gamma\left[\begin{array}{cc} \frac{s}{2}+\alpha+k, & \frac{s-\gamma+1}{2}+\alpha+k \\ \frac{s}{2}, & \frac{s-\gamma+1}{2}\end{array}\right] f^*(s) = h^*(s). \tag{11.19}$$

Let

$$P_{\alpha,\gamma,m}(s) = \sum_{k=0}^{m} A_k 2^{2(\alpha+k)}\,\Gamma\left[\begin{array}{cc} \frac{s}{2}+\alpha+k, & \frac{s-\gamma+1}{2}+\alpha+k \\ \frac{s}{2}, & \frac{s-\gamma+1}{2}\end{array}\right].$$

Using the inverse Mellin transform we derive the following solution of (11.17):

$$f(x) = \left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha,\gamma,m}(s)}(\mathcal{M}h)(s)\right]\right)(x).$$

We introduce the Mellin fractional analogue of the Green function:

$$G_{\alpha,\gamma,m}(x) = \left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha,\gamma,m}(s)}\right]\right)(x).$$

So we can write

$$f(x) = \left(\mathcal{M}^{-1}\left[(\mathcal{M}G_{\alpha,\gamma,m})(s)(\mathcal{M}h)(s)\right]\right)(x).$$

Applying the property of the Mellin convolution

$$\left(\mathcal{M}\int_0^\infty u\left(\frac{x}{t}\right)v(t)\frac{dt}{t}\right)(s)=(\mathcal{M}u)(s)(\mathcal{M}v)(s),$$

we obtain that the solution to (11.17) provided that $P_{\alpha,\gamma,m}(s)\neq 0$ has the form

$$f(x)=\int_0^\infty G_{\alpha,\gamma,m}\left(\frac{x}{t}\right)h(t)\frac{dt}{t}.$$

11.2.2 Example

Example 1. *Let $m=0$. We are looking for a solution to the equation*

$$A_0x^{2\alpha}(DB^\alpha_{\gamma,-}f)(x)=h(x),\qquad x>0,\qquad \alpha>0.$$

We have

$$P_{\alpha,\gamma,0}(s)=A_0 2^{2\alpha}\,\Gamma\left[\begin{array}{cc}\frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha\\ \frac{s}{2}, & \frac{s-\gamma+1}{2}\end{array}\right].$$

Now we can find

$$G_{\alpha,\gamma,0}(x)=\left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha,\gamma,0}(s)}\right]\right)(x)$$
$$=\frac{1}{A_0 2^{2\alpha}}\left(\mathcal{M}^{-1}\left[\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1-\gamma}{2}\right)}{\Gamma\left(\frac{s}{2}+\alpha\right)\Gamma\left(\frac{s}{2}+\frac{1-\gamma}{2}+\alpha\right)}\right]\right)(x).$$

Using the formula

$$g\left(\frac{s}{a}\right)=a\int_0^\infty f(x^a)x^{s-1}dx=a(\mathcal{M}f)(x^a)$$

and Wolfram Alpha, we obtain

$$\left(\mathcal{M}^{-1}\left[\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1-\gamma}{2}\right)}{\Gamma\left(\frac{s}{2}+\alpha\right)\Gamma\left(\frac{s}{2}+\frac{1-\gamma}{2}+\alpha\right)}\right]\right)(x)=$$
$$2\left(\mathcal{M}^{-1}\left[\frac{\Gamma(s)\Gamma\left(s+\frac{1-\gamma}{2}\right)}{\Gamma(s+\alpha)\Gamma\left(s+\frac{1-\gamma}{2}+\alpha\right)}\right]\right)(x^2)=$$

$$\frac{2}{\Gamma(2\alpha)}(1-x^2)^{2\alpha-1}(1-\theta(|x|-1))\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-x^2\right),$$

where $\theta(z)$ is the Heaviside theta function

$$\theta(z)=\begin{cases}1 & z>0,\\ 0 & z\le 0.\end{cases}$$

Therefore,

$$\left(\mathcal{M}^{-1}\left[\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1-\gamma}{2}\right)}{\Gamma\left(\frac{s}{2}+\alpha\right)\Gamma\left(\frac{s}{2}+\frac{1-\gamma}{2}+\alpha\right)}\right]\right)(x)=$$
$$\frac{2}{\Gamma(2\alpha)}\begin{cases}(1-x^2)^{2\alpha-1}\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-x^2\right) & |x|<1,\\ 0 & |x|\ge 1.\end{cases}$$

Finally,

$$G_{\alpha,\gamma,0}\left(\frac{x}{t}\right)=$$
$$\frac{2^{1-2\alpha}}{A_0\Gamma(2\alpha)}\begin{cases}2\left(\frac{t^2-x^2}{t^2}\right)^{2\alpha-1}\,{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{t^2}\right) & 0<x<t,\\ 0 & 0<t\le x\end{cases}$$

and

$$f(x)=\int\limits_0^\infty G_{\alpha,\gamma,0}\left(\frac{x}{t}\right)h(t)\frac{dt}{t}=$$
$$\frac{1}{A_0 2^{2\alpha-1}\Gamma(2\alpha)}\int\limits_x^\infty\left(\frac{t^2-x^2}{t^2}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{t^2}\right)h(t)\frac{dt}{t}=$$
$$\frac{1}{A_0\Gamma(2\alpha)}\int\limits_x^\infty\left(\frac{t^2-x^2}{2t}\right)^{2\alpha-1}{}_2F_1\left(\alpha+\frac{\gamma-1}{2},\alpha;2\alpha;1-\frac{x^2}{t^2}\right)\frac{h(t)}{t^{2\alpha}}\,dt.$$

Using the designation of the right-sided fractional Bessel integral on a semiaxis from Definition 47, we get

$$f(x)=\frac{1}{A_0}(IB^{\alpha}_{-,\gamma}t^{-2\alpha}h(t))(x).$$

Example 2. *We consider Eq. (11.17) with $m=1$. It is easy to see that without loss of generality we can put $A_1=1$. Let $x>0$, $\alpha>0$, $A_0\in\mathbb{R}$. We have*

$$x^{2(\alpha+1)}(DB^{\alpha+1}_{\gamma,-}f)(x)+A_0x^{2\alpha}(DB^{\alpha}_{\gamma,-}f)(x)=h(x). \tag{11.20}$$

Let

$$\gamma \neq 2k-1, \qquad k \in \mathbb{N},$$
$$\frac{1}{2}(\gamma - 1 - 4\alpha - \sqrt{(\gamma-1)^2 - 4A_0}) \neq -n, \qquad n \in \mathbb{N},$$
$$\frac{1}{2}(\gamma - 1 - 4\alpha + \sqrt{(\gamma-1)^2 - 4A_0}) \neq -n, \qquad n \in \mathbb{N}.$$

We have

$$P_{\alpha,\gamma,1} = 2^{2\alpha}\Gamma\left[\begin{array}{cc} \frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha \\ \frac{s}{2}, & \frac{s-\gamma+1}{2} \end{array}\right]((s+2\alpha)(s-\gamma+1+2\alpha)+A_0)\,.$$

So we obtain

$$G_{\alpha,\gamma,1}(x) = \left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha,\gamma,1}(s)}\right]\right)(x) =$$
$$\frac{1}{2^{2\alpha}}\left(\mathcal{M}^{-1}\left[\frac{1}{((s+2\alpha)(s-\gamma+1+2\alpha)+A_0)}\Gamma\left[\begin{array}{cc} \frac{s}{2}, & \frac{s-\gamma+1}{2} \\ \frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha \end{array}\right]\right]\right)(x) =$$
$$\frac{1}{2^{2\alpha+1}\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}\frac{1}{((s+2\alpha)(s-\gamma+1+2\alpha)+A_0)}\Gamma\left[\begin{array}{cc} \frac{s}{2}, & \frac{s-\gamma+1}{2} \\ \frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha \end{array}\right]x^{-s}ds.$$

Let us show that $G_{\alpha,\gamma,1}(x) = 0$ *for* $x > 1$ *and for* $\operatorname{Re} s > 0$. *We have*

$$(\mathcal{M}G_{\alpha,\gamma,1})(s) =$$
$$\frac{1}{2^{2\alpha}}\frac{1}{((s+2\alpha)(s-\gamma+1+2\alpha)+A_0)}\Gamma\left[\begin{array}{cc} \frac{s}{2}, & \frac{s-\gamma+1}{2} \\ \frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha \end{array}\right] =$$
$$\frac{1}{2^{2\alpha}}\frac{1}{s-s_1}\frac{1}{s-s_2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}+\alpha\right)}\frac{\Gamma\left(\frac{s-\gamma+1}{2}\right)}{\Gamma\left(\frac{s-\gamma+1}{2}+\alpha\right)} =$$
$$\frac{1}{2^{2\alpha}}(\mathcal{M}G_1)(s)(\mathcal{M}G_2)(s)(\mathcal{M}G_3)(s)(\mathcal{M}G_4)(s),$$

where s_1 *and* s_2 *are solutions of the quadratic equation*

$$s^2 + (4\alpha+1-\gamma)s + A_0 - 2\alpha\gamma + 2\alpha + 4\alpha^2 = 0,$$
$$(\mathcal{M}G_1)(s) = \frac{1}{s-s_1}, \qquad (\mathcal{M}G_2)(s) = \frac{1}{s-s_2}, \qquad (\mathcal{M}G_3)(s) = \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}+\alpha\right)},$$
$$(\mathcal{M}G_4)(s) = \frac{\Gamma\left(\frac{s-\gamma+1}{2}\right)}{\Gamma\left(\frac{s-\gamma+1}{2}+\alpha\right)}.$$

Then

$$G_1(x) = \begin{cases} x^{-s_1} & 0 < x < 1, \\ 0 & x > 1, \end{cases} \quad \textit{for} \quad \operatorname{Re} s > \operatorname{Re} s_1,$$

$$G_2(x) = \begin{cases} x^{-s_2} & 0 < x < 1, \\ 0 & x > 1, \end{cases} \quad \textit{for} \quad \operatorname{Re} s > \operatorname{Re} s_2,$$

$$G_3(x) = \begin{cases} 2\frac{(1-x^2)^{\alpha-1}}{\Gamma(\alpha)} & 0 < x < 1, \\ 0 & x > 1 \end{cases}$$

when $\operatorname{Re} s > 0$*,*

$$G_3(x) = \begin{cases} \frac{2}{\Gamma(\alpha)}\left((1-x^2)^{\alpha-1} - \sum\limits_{m=0}^{n-1} \frac{(1-\alpha)_m}{m!} x^m\right) & 0 < x < 1, \\ -\sum\limits_{m=0}^{n-1} \frac{(1-\alpha)_m}{m!} x^m & x > 1 \end{cases}$$

when $-n < \operatorname{Re} s < 1-n$*,*

$$G_4(x) = \begin{cases} 2\frac{(1-x^2)^{\alpha+\frac{1-\gamma}{2}-1}}{\Gamma(\alpha+\frac{1-\gamma}{2})} & 0 < x < 1, \\ 0 & x > 1 \end{cases}$$

when $\operatorname{Re} s > 0$*,* $\alpha + \frac{1-\gamma}{2} > 0$*, and*

$$G_4(x) = \begin{cases} \frac{2}{\Gamma(\alpha+\frac{1-\gamma}{2})}\left((1-x^2)^{\alpha+\frac{1-\gamma}{2}-1} - \sum\limits_{m=0}^{n-1} \frac{(1-\alpha-\frac{1-\gamma}{2})_m}{m!} x^m\right) & 0 < x < 1, \\ -\sum\limits_{m=0}^{n-1} \frac{(1-\alpha-\frac{1-\gamma}{2})_m}{m!} x^m & x > 1 \end{cases}$$

when $-n < \operatorname{Re} s < 1-n$*,* $\alpha + \frac{1-\gamma}{2} > 0$*.*

Let us consider

$$G_{\alpha,\gamma,1}(x) = \left(\mathcal{M}^{-1}\left[\frac{1}{P_{\alpha,\gamma,1}(s)}\right]\right)(x) =$$

$$\frac{1}{2^{2\alpha+1}\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{((s+2\alpha)(s-\gamma+1+2\alpha)+A_0)}\Gamma\left[\begin{matrix} \frac{s}{2}, & \frac{s-\gamma+1}{2} \\ \frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha \end{matrix}\right] x^{-s}\, ds.$$

We have four group of poles:

$$s = s_1 = \frac{1}{2}(\gamma - 1 - 4\alpha - \sqrt{(\gamma-1)^2 - 4A_0}),$$

$$s = s_2 = \frac{1}{2}(\gamma - 1 - 4\alpha + \sqrt{(\gamma-1)^2 - 4A_0}),$$

$$s = -2n, \qquad n = 0, 1, 2, ...,$$
$$s = \gamma - 2k - 1, \qquad \gamma \in \mathbb{N}, \qquad k = 0, 1, 2, ...$$

Let all poles be simple. If we choose

$$\gamma > \max\left\{\frac{1}{2}(\gamma - 1 - 4\alpha + \sqrt{(\gamma-1)^2 - 4A_0}), 0\right\},$$

$\gamma \neq 2k+1$, *then evaluation of the residues at the above poles yields*

$$G_{\alpha,\gamma,1}(x) = \frac{1}{2^{2\alpha}}\left(\operatorname*{res}_{s=s_1}\left[\frac{1}{(s-s_1)(s-s_2)}\Gamma\left[\begin{array}{cc}\frac{s}{2}, & \frac{s-\gamma+1}{2}\\ \frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha\end{array}\right]x^{-s}\right]+\right.$$
$$\operatorname*{res}_{s=s_2}\left[\frac{1}{(s-s_1)(s-s_2)}\Gamma\left[\begin{array}{cc}\frac{s}{2}, & \frac{s-\gamma+1}{2}\\ \frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha\end{array}\right]x^{-s}\right]+$$
$$\left.\sum_{n=0}^{\infty}\operatorname*{res}_{s=-2n}\left[\frac{1}{(s-s_1)(s-s_2)}\Gamma\left[\begin{array}{cc}\frac{s}{2}, & \frac{s-\gamma+1}{2}\\ \frac{s}{2}+\alpha, & \frac{s-\gamma+1}{2}+\alpha\end{array}\right]x^{-s}\right]\right)=$$
$$\frac{1}{2^{2\alpha}}\left(\left[\frac{1}{(s_1-s_2)}\Gamma\left[\begin{array}{cc}\frac{s_1}{2}, & \frac{s_1-\gamma+1}{2}\\ \frac{s_1}{2}+\alpha, & \frac{s_1-\gamma+1}{2}+\alpha\end{array}\right]x^{-s_1}\right]+\right.$$
$$\left[\frac{1}{(s_2-s_1)}\Gamma\left[\begin{array}{cc}\frac{s_2}{2}, & \frac{s_2-\gamma+1}{2}\\ \frac{s_2}{2}+\alpha, & \frac{s_2-\gamma+1}{2}+\alpha\end{array}\right]x^{-s_2}\right]+$$
$$\left.\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!(-2n-s_1)(-2n-s_2)}\frac{\Gamma\left(\frac{1-\gamma}{2}-n\right)}{\Gamma\left(\alpha-n\right)\Gamma\left(\alpha+\frac{1-\gamma}{2}-n\right)}x^{2n}\right]\right)=$$
$$\frac{1}{2^{2\alpha}}\left(\left[\frac{1}{(s_1-s_2)}\Gamma\left[\begin{array}{cc}\frac{s_1}{2}, & \frac{s_1-\gamma+1}{2}\\ \frac{s_1}{2}+\alpha, & \frac{s_1-\gamma+1}{2}+\alpha\end{array}\right]x^{-s_1}\right]+\right.$$
$$\left[\frac{1}{(s_2-s_1)}\Gamma\left[\begin{array}{cc}\frac{s_2}{2}, & \frac{s_2-\gamma+1}{2}\\ \frac{s_2}{2}+\alpha, & \frac{s_2-\gamma+1}{2}+\alpha\end{array}\right]x^{-s_2}\right]+$$
$$\left.\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!(2n+s_1)(2n+s_2)}\frac{\Gamma\left(\frac{1-\gamma}{2}-n\right)}{\Gamma\left(\alpha-n\right)\Gamma\left(\alpha+\frac{1-\gamma}{2}-n\right)}x^{2n}\right]\right)=$$
$$\frac{1}{2^{2\alpha}}\left(\left[\frac{1}{(s_1-s_2)}\Gamma\left[\begin{array}{cc}\frac{s_1}{2}, & \frac{s_1-\gamma+1}{2}\\ \frac{s_1}{2}+\alpha, & \frac{s_1-\gamma+1}{2}+\alpha\end{array}\right]x^{-s_1}\right]+\right.$$
$$\left[\frac{1}{(s_2-s_1)}\Gamma\left[\begin{array}{cc}\frac{s_2}{2}, & \frac{s_2-\gamma+1}{2}\\ \frac{s_2}{2}+\alpha, & \frac{s_2-\gamma+1}{2}+\alpha\end{array}\right]x^{-s_2}\right]+$$
$$\left.\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\frac{\Gamma(s_1+2n)\Gamma(s_2+2n)\Gamma\left(\frac{1-\gamma}{2}-n\right)}{\Gamma(s_1+2n+1)\Gamma(s_2+2n+1)\Gamma\left(\alpha-n\right)\Gamma\left(\alpha+\frac{1-\gamma}{2}-n\right)}x^{2n}\right]\right),$$

where

$$s_1 = \frac{1}{2}(\gamma - 1 - 4\alpha - \sqrt{(\gamma-1)^2 - 4A_0}),$$
$$s_2 = \frac{1}{2}(\gamma - 1 - 4\alpha + \sqrt{(\gamma-1)^2 - 4A_0}).$$

Let us consider the last term in $G_{\alpha,\gamma,1}(x)$ as a Wright function (1.38):

$$\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\frac{\Gamma(s_1+2n)\Gamma(s_2+2n)\Gamma\left(\frac{1-\gamma}{2}-n\right)}{\Gamma(s_1+2n+1)\Gamma(s_2+2n+1)\Gamma(\alpha-n)\Gamma\left(\alpha+\frac{1-\gamma}{2}-n\right)}x^{2n}\right]. \tag{11.21}$$

We obtain $p=3$, $q=4$,

$$a_1 = s_1,\ a_2 = s_2,\ a_3 = \frac{1-\gamma}{2},\ \alpha_1 = 2,\ \alpha_2 = 2,\ \alpha_3 = -1,$$
$$b_1 = s_1+1,\ b_2 = s_2+1,\ b_3 = \alpha,\ b_4 = \alpha + \frac{1-\gamma}{2},$$
$$\beta_1 = 2,\ \beta_2 = 2,\ \beta_3 = -1,\ \beta_4 = -1,$$
$$\sum_{j=1}^{4}\beta_j - \sum_{l=1}^{3}\alpha_l = 2+2-1-1-2-2+1 = -1,$$
$$\delta = 1.$$

So the series (11.21) is absolutely convergent for $|x|<1$ and

$$\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{n!}\frac{\Gamma(s_1+2n)\Gamma(s_2+2n)\Gamma\left(\frac{1-\gamma}{2}-n\right)}{\Gamma(s_1+2n+1)\Gamma(s_2+2n+1)\Gamma(\alpha-n)\Gamma\left(\alpha+\frac{1-\gamma}{2}-n\right)}x^{2n}\right] =$$
$${}_3\Psi_4\left[\begin{array}{c}(s_1, s_2, \frac{1-\gamma}{2}; 2, 2, -1)\\ (s_1+1, s_2+1, \alpha, \alpha+\frac{1-\gamma}{2}; 2, 2, -1, -1)\end{array}\middle| -x^2\right].$$

Therefore,

$$G_{\alpha,\gamma,1}(x) = \frac{1}{2^{2\alpha}}\left(\left[\frac{1}{(s_1-s_2)}\Gamma\left[\begin{array}{cc}\frac{s_1}{2}, & \frac{s_1-\gamma+1}{2}\\ \frac{s_1}{2}+\alpha, & \frac{s_1-\gamma+1}{2}+\alpha\end{array}\right]x^{-s_1}\right]+\right.$$
$$\left[\frac{1}{(s_2-s_1)}\Gamma\left[\begin{array}{cc}\frac{s_2}{2}, & \frac{s_2-\gamma+1}{2}\\ \frac{s_2}{2}+\alpha, & \frac{s_2-\gamma+1}{2}+\alpha\end{array}\right]x^{-s_2}\right]+$$
$$\left.{}_3\Psi_4\left[\begin{array}{c}(s_1, s_2, \frac{1-\gamma}{2}; 2, 2, -1)\\ (s_1+1, s_2+1, \alpha, \alpha+\frac{1-\gamma}{2}; 2, 2, -1, -1)\end{array}\middle| -x^2\right]\right),$$

where

$$s_1 = \frac{1}{2}(\gamma - 1 - 4\alpha - \sqrt{(\gamma-1)^2 - 4A_0}),$$
$$s_2 = \frac{1}{2}(\gamma - 1 - 4\alpha + \sqrt{(\gamma-1)^2 - 4A_0}).$$

So for $x < t$,

$$G_{\alpha,\gamma,1}\left(\frac{x}{t}\right) = \frac{1}{2^{2\alpha}}\left(\left[\frac{1}{(s_1-s_2)}\Gamma\left[\begin{array}{cc} \frac{s_1}{2}, & \frac{s_1-\gamma+1}{2} \\ \frac{s_1}{2}+\alpha, & \frac{s_1-\gamma+1}{2}+\alpha \end{array}\right]\left(\frac{x}{t}\right)^{-s_1}\right]+\right.$$
$$\left[\frac{1}{(s_2-s_1)}\Gamma\left[\begin{array}{cc} \frac{s_2}{2}, & \frac{s_2-\gamma+1}{2} \\ \frac{s_2}{2}+\alpha, & \frac{s_2-\gamma+1}{2}+\alpha \end{array}\right]\left(\frac{x}{t}\right)^{-s_2}\right]+$$
$$\left.{}_3\Psi_4\left[\begin{array}{c} (s_1, s_2, \frac{1-\gamma}{2}; 2, 2, -1) \\ (s_1+1, s_2+1, \alpha, \alpha+\frac{1-\gamma}{2}; 2, 2, -1, -1) \end{array}\middle| -\frac{x^2}{t^2}\right]\right),$$

and

$$f(x) = \int_x^\infty G_{\alpha,\gamma,1}\left(\frac{x}{t}\right) h(t)\frac{dt}{t}.$$

11.3 Hyperbolic Riesz B-potential and its connection with the solution of an iterated B-hyperbolic equation

M. Riesz [472,475] has created a new method for solution to nonhomogeneous linear equations by generalization of the fractional Riemann–Liouville integral. We generalize and apply this method to solution to some linear equations with Bessel operators acting by all variables. This method overcomes difficulties within the theory of differential equations which are due to the occurrence of divergent integrals. Namely, in some cases (for example, for hyperbolic equations) it is necessary to use the analytical continuation of a potential which depends analytically on a parameter.

11.3.1 General algorithm

Let us start from the presentation of the general algorithm of construction of the solution to the nonhomogeneous equation $Lu = f$ with some linear operator L.

An algorithm for constructing the Riesz potential generalized by the operator L and application to the solution of differential equations with this operator L follow next.

1. An integral transform $\mathscr{F}_L$ convenient for working with operator L is chosen (for example, $\mathscr{F}_L$ is the Fourier transform when $L = \square$, $\mathscr{F}_L$ is the Hankel transform when $L = \square_\gamma$). For suitable functions f we have $\mathscr{F}_L Lf = P\mathscr{F}_L f$, where P is a symbol of operator L.

2. The fractional negative power of L or the Riesz potential is constructed by the formula $L^{-\frac{\alpha}{2}}f=\mathscr{F}_L^{-1}P^{-\frac{\alpha}{2}}\mathscr{F}_L f$. Here $P^{-\frac{\alpha}{2}}$ can be a generalized function, for example when P is an indefinite quadratic form.
3. An integral representation of the Riesz potential for operator L is realized in the form of the convolution $I^\alpha f=(\mathscr{F}_L P^{-\frac{\alpha}{2}} * f)_L$. The convolution $(\cdot * \cdot)_L$ must correspond to the chosen integral transform $\mathscr{F}_L$.
4. The obtained integral $I^\alpha f$ is studied for absolute convergence for some class of functions f. It is examined at what values of α this integral converges absolutely. Other properties, such as boundedness, semigroup property, etc., can also be studied.
5. Additional conditions on the function f for which the equality $I^{\alpha+k}Lf=I^\alpha f$ for some natural k (for example, $k=2$ when P is a quadratic form) is true are clarified.
6. By constructing an analytic continuation (or without it if possible) it should be shown that for $\alpha=0$ the potential $I^\alpha f$ is the identity operator $I^0 f=f$ for some class of functions.
7. Using obtained results one can easily write a solution to the equation $Lu=f$ for some class of functions f. It is just necessary to apply $I^{\alpha+k}$ to both sides of $Lu=f$: $I^{\alpha+k}Lu=I^\alpha u=I^{\alpha+k}f$. Then putting $\alpha=0$, we get $u=I^k f$. Here an analytic continuation $I^\alpha f$ is used if needed.

It is easy to see that using this scheme, we can also obtain a solution to the equation $L^m u=f$ with the iterated operator L.

The algorithm for constructing the Riesz potential is close to the composition method (see Chapter 6) developed by S. M. Sitnik (see [146,230,231,535]).

Remark 24. *It we would like to construct a solution to $Lu=f$ with initial conditions (for example, in parabolic or hyperbolic cases), it is better to start from the case when the time variable acts by the first or the second derivative. Then, we apply a transmutation operator preserving initial conditions and obtain a solution to the problem under consideration.*

11.3.2 Definition

Consider the potential generalizing the Riesz potential of the form

$$(I^\alpha_{\square_\gamma}f)(x)=\frac{1}{\mathscr{H}_{n,\gamma}(\alpha)}\int\limits_{K^+}(y_1^2-y_2^2-\ldots-y_n^2)^{\frac{\alpha-n-|\gamma|}{2}}({}^\gamma\mathbf{T}^y f)(x)y^\gamma dy, \qquad (11.22)$$

where $y^\gamma=\prod\limits_{i=1}^n y_i^{\gamma_i}$,

$$\mathscr{H}_{n,\gamma}(\alpha)=\frac{2^{\alpha-n}}{\pi}\sin\left(\frac{\gamma_1+1}{2}\pi\right)\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right),$$

$K^+=\{y\in\mathbb{R}^n_+ : y_1^2\geq y_2^2+...+y_n^2\}$, $\gamma_1\neq 2k-1$, $k\in\mathbb{N}$, and $({}^\gamma\mathbf{T}^y f)(x)=({}^{\gamma_1}T^{y_1}_{x_1}...{}^{\gamma_n}T^{y_n}_{x_n} f)(x)$ is the multi-dimensional generalized translation (3.169). Operator (11.22) will be called **hyperbolic B-Riesz potential**. Up to a constant, this operator coincides with the first term in formula (10.3); therefore, for (11.22) the same statements about absolute convergence (Theorem 127) and about boundedness (Theorem 129) are true.

The reason for considering this operator is its convenience in finding a solution to the iterated nonhomogeneous general Euler–Poisson–Darboux equation of the form $\square_\gamma^m u = f$, $m\in\mathbb{N}$. By constructing an analytic continuation we show that for $\alpha=0$ the potential $I^\alpha_{\square_\gamma}$ is the identity operator $I^0_{\square_\gamma} f = f$ for some class of functions.

Note that (11.22) can be written as

$$
\begin{aligned}
(I^\alpha_{\square_\gamma} f)(x) &= \frac{1}{\mathscr{H}_{n,\gamma}(\alpha)}\int\limits_{K^+}(y_1^2-y_2^2-...-y_n^2)^{\frac{\alpha-n-|\gamma|}{2}}({}^\gamma\mathbf{T}^y f)(x)y^\gamma dy =\\
&\frac{1}{\mathscr{H}_{n,\gamma}(\alpha)}\int\limits_0^\infty y_1^{\gamma_1}dy_1\int\limits_{|y'|<y_1}(y_1^2-|y'|^2)^{\frac{\alpha-n-|\gamma|}{2}}({}^\gamma\mathbf{T}^y f)(x)(y')^{\gamma'}dy' = \{y'=y_1z'\} =\\
&\frac{1}{\mathscr{H}_{n,\gamma}(\alpha)}\int\limits_0^\infty y_1^{\alpha-1}dy_1\int\limits_{|z'|<1}(1-|z'|^2)^{\frac{\alpha-n-|\gamma|}{2}}({}^{\gamma_1,\gamma'}\mathbf{T}^{y_1,y_1z'} f)(x)(z')^{\gamma'}dz'.
\end{aligned}
\tag{11.23}
$$

11.3.3 Variables in Lorentz space

Let $x_1, x_2, ..., x_n$ be the coordinates of the Lorentz space, such that $x_1\geq 0$, $x_2\geq 0$,..., $x_n\geq 0$. The metric in Lorentz space is defined as follows:

$$(x,x)=x_1^2-x_2^2-...-x_n^2.$$

The squared distance between points x and y, $x, y\in\mathbb{R}^n_+$, is

$$r_{xy}^2=(x-y,x-y)=(x_1-y_1)^2-(x_2-y_2)^2-...-(x_n-y_n)^2.$$

The scalar product (a,b) of two vectors a and b, $a, b\in\mathbb{R}^n_+$, are defined as

$$(a,b)=a_1b_1-a_2b_2-...-a_nb_n.$$

Two vectors whose scalar product vanishes are called orthogonal to each other. If the scalar square (a,a) of a vector a is positive, then vector a is called *time-like*. If (a,a) is negative, then vector a is called *space-like*. A *light-like* vector is a vector a such that $(a,a)=0$. A *light cone* or *characteristic cone* with vertex a is given by the equality $(x-a,x-a)=0$.

Consider a fixed time-like unit vector $a=(1,0,...,0)$ and a variable space-like unit vector $v=(0,v_2,...,v_n)$ which is orthogonal to a such that $\sum\limits_{k=2}^n v_k^2=1$. If a vector v

goes from the origin, then its end gives part of the unit sphere $S_1^+(n)$. Also a vector v is orthogonal to a. Therefore,

$$(a,a)=1, \qquad (v,v)=-1, \qquad (a,v)=0.$$

Let $t \geq 0$, $\rho \geq 0$. The arbitrary vector $y \in \overline{\mathbb{R}}^n_+$ can be written as

$$y = ta + \rho v.$$

We rewrite the expression for y as

$$y = \frac{1}{2}(t+\rho)(a+v) + \frac{1}{2}(t-\rho)(a-v).$$

By entering the notation

$$b = \frac{1}{2}(a+v), \qquad c = \frac{1}{2}(a-v),$$

we can write

$$y = (t+\rho)b + (t-\rho)c = (t+\rho)\left(b + \frac{t-\rho}{t+\rho}c\right).$$

Putting now

$$\tau = \frac{t-\rho}{t+\rho}, \qquad \sigma = t+\rho,$$

we obtain

$$y = \sigma(b + \tau c),$$

where b and c are some vectors. Expressing ρ and t in terms of σ and τ, we obtain

$$\rho = \frac{1}{2}\sigma(1-\tau), \qquad t = \frac{1}{2}\sigma(1+\tau).$$

From the definitions of a and v it follows that

$$(b,b)=0, \qquad (c,c)=0, \qquad (b,c)=\frac{1}{2}.$$

The square of the Lorentz distance from the point y to the origin can now be expressed as

$$r^2 = (y,y) = (ta+\rho v, ta+\rho v) = t^2 - \rho^2 = (t+\rho)^2\,\frac{t-\rho}{t+\rho} = \sigma^2\tau.$$

We have

$$y = (y_1, y_2, ..., y_n) = ta + \rho v = (t, \rho v_2, ..., \rho v_n), \qquad |v| = \sqrt{v_2^2 + ... + v_n^2} = 1.$$

So $y_1 = t$, $|y'| = \rho$. Let $\delta > 0$, $y' = (y_2, ..., y_n)$, and let $|y'|^2 = (y_1 - \delta)^2$ be a part of a cone with the vertex $(\delta, 0, ..., 0)$. Consider its lower part $0 < y_1 < \delta$. Then taking into account that $|y'| = \rho$, we can write $y_1 + \rho = \delta$ or $t + \rho = \delta$. Therefore, the lower part of the cone, having the form $|y'| = \delta - y_1$ or $\rho = \delta - t$ $(\rho + t = \delta)$ in new coordinates, will take the form $\sigma = \delta$. On the surface (y, y)=0 $(t = \rho)$, with the exception of the vertex, we obtain τ=0 and σ>0.

The interior $|y'| = \delta - y_1$ ($|y'| = \delta - y_1$ or $t + \rho < \delta$) for $y_1 > \delta/2$ $(\rho < t)$ becomes $\sigma < \delta$ and $\tau > 0$. The interior $|y'| = y_1$ ($|y'| < y_1$ or $\rho < t$) for $y_1 < \delta/2$ becomes $0 < \tau < 1$ and $\sigma > 0$.

The Jacobian is

$$I = \frac{\partial(\rho, t)}{\partial(\sigma, \tau)} = \frac{1}{4}\begin{vmatrix} 1-\tau & -\sigma \\ 1+\tau & \sigma \end{vmatrix} = \frac{1}{4}\sigma(1 - \tau + 1 + \tau) = \frac{1}{2}\sigma.$$

Denoting by K_δ^+ the domain bounded by $|y'|^2$=$(y_1$−$\delta)^2$ or $t + \rho = \delta$ on top and by the part of the cone $(y, y) = 0$ or $t = \rho$ below, we write that $K^+ = K_\delta^+ \cup (K^+ \setminus K_\delta^+)$. Using new variables τ and σ we get $0 \le \tau \le 1$ and $0 \le \sigma \le \delta$ characterizing K_δ^+.

Let $0 < \varepsilon < \delta$. Then the cone $(y_1 - \varepsilon)^2 = |y'|^2$ becomes $\tau \cdot \sigma = \varepsilon$.

11.3.4 Identity operator

Theorem 144. *For the function* $f \in S_{ev}$*, the Riesz hyperbolic B-potential continues analytically to the values* $\alpha > -1$*. Moreover,* $(I^0_{\square_\gamma} f)(x)$ *is the identity operator:*

$$(I^0_{\square_\gamma} f)(x) = f(x). \tag{11.24}$$

Proof. Let first $n + |\gamma| - 2 < \alpha$. We consider the hyperbolic Riesz B-potential at the point $(x_1, ..., x_n) = (0, ..., 0) = O$

$$(I^\alpha_{\square_\gamma} f)(O) = \frac{1}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_{K^+} (y_1^2 - y_2^2 - ... - y_n^2)^{\frac{\alpha - n - |\gamma|}{2}} f(y) y^\gamma dy, \quad y^\gamma = \prod_{i=1}^n y_i^{\gamma_i}.$$

Dividing the domain K^+ into two parts, K_δ^+ and $K^+ \setminus K_\delta^+$, we can write

$$(I^\alpha_{\square_\gamma} f)(O) = I_1^\alpha + I_2^\alpha,$$

where

$$I_1^\alpha = \frac{1}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_{K_\delta^+} (y_1^2 - y_2^2 - ... - y_n^2)^{\frac{\alpha - n - |\gamma|}{2}} f(y) y^\gamma dy,$$

$$I_2^\alpha = \frac{1}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_{K^+ \setminus K_\delta^+} (y_1^2 - y_2^2 - ... - y_n^2)^{\frac{\alpha - n - |\gamma|}{2}} f(y) y^\gamma dy.$$

We show that I_1^α and I_2^α are holomorphic as functions of α for $\alpha > -1$ and $(I_1^0)f(O) = f(O)$, $I_2^0 = 0$, which is equivalent to $(I^0_{\Box_\gamma} f)(O)=f(O)$.

Let us pass in I_1^α to spherical coordinates $y'=(y_2, ..., y_n)$:

$$I_1^\alpha = \frac{1}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_{K_\delta^+} (y_1^2 - |y'|^2)^{\frac{\alpha-n-|\gamma|}{2}} f(y) y^\gamma dy = \{y' = \rho\theta\}$$

$$= \int\limits_{K_\delta^+} \rho^{n+|\gamma'|-2} (y_1^2 - \rho^2)^{\frac{\alpha-n-|\gamma|}{2}} f(y_1, \rho\theta) \theta^{\gamma'} y_1^{\gamma_1} dS d\rho dy_1,$$

where $|y'|=\sqrt{y_2^2 + ... + y_n^2}$, $\gamma'=(\gamma_2, ..., \gamma_n)$, $|\gamma'|=\gamma_2+...+\gamma_n$, $\theta^{\gamma'} = \theta_1^{\gamma_2}...\theta_{n-1}^{\gamma_n}$.

By replacing variables y_1 and ρ by formulas

$$\rho = \frac{1}{2}\sigma(1-\tau), \qquad y_1 = \frac{1}{2}\sigma(1+\tau), \tag{11.25}$$

considering that $\frac{\partial(y_1,\rho)}{\partial(\sigma,\tau)} = \frac{1}{2}\sigma$ and $y = (y_1, \rho\theta) = \sigma(b + \tau c)$ (see Section 11.3.3), we obtain

$$I_1^\alpha =$$

$$\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_{S_1^+(n-1)} \theta^{\gamma'} dS \int\limits_0^\delta \sigma^{\alpha-1} d\sigma \times$$

$$\int\limits_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}} (1+\tau)^{\gamma_1} (1-\tau)^{n+|\gamma'|-2} f(\sigma(b+\tau c)) d\tau.$$

Let

$$A(\alpha,\sigma) = \frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}} (1+\tau)^{\gamma_1} (1-\tau)^{n+|\gamma'|-2} f(\sigma(b+\tau c)) d\tau.$$

Then

$$I_1^\alpha = \int\limits_{S_1^+(n-1)} \theta^{\gamma'} dS \int\limits_0^\delta A(\alpha,\sigma)\sigma^{\alpha-1} d\sigma.$$

We expand $f(\sigma(b+\tau c))$ by the Taylor formula by variable τ:

$$f(y) = f(\sigma(b+\tau c)) = \sum_{p=0}^{N-1} \frac{\tau^p}{p!} F_p(\sigma,\theta) + R_N(\tau),$$

where

$$F_p(\sigma,\theta)=\frac{\partial^p}{\partial\tau^p}f(\sigma(b+\tau c))\bigg|_{\tau=0}$$

and

$$R_N(\tau)=\frac{1}{(N-1)!}\int\limits_0^\tau\frac{\partial^N}{\partial\tau^N}f(\sigma(b+\widetilde{\tau}c))\,(\tau-\widetilde{\tau})^{N-1}d\widetilde{\tau}.$$

We get the following expression for $A(\alpha,\sigma)$:

$$A(\alpha,\sigma)=\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)}\sum_{p=0}^{N-1}\frac{F_p(\sigma,\theta)}{p!}\int\limits_0^1\tau^{\frac{\alpha-n-|\gamma|}{2}+p}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}d\tau+$$
$$\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)}\int\limits_0^1\tau^{\frac{\alpha-n-|\gamma|}{2}}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}R_N(\tau)d\tau.$$

Using the integral representation of the Gauss hypergeometric function for $c-a-b>0$,

$$_2F_1(a,b;c;-1)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1+t)^{-a}\,dt,\tag{11.26}$$

in our case we get $c-a-b=n+|\gamma|-2>0$, $n+|\gamma|-2<\alpha$, and

$$\int\limits_0^1\tau^{\frac{\alpha-n-|\gamma|}{2}+p}(1-\tau)^{n+|\gamma'|-2}(1+\tau)^{\gamma_1}d\tau=$$
$$\frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+p+1\right)\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p\right)}\times$$
$$_2F_1\left(-\gamma_1,\frac{\alpha-n-|\gamma|}{2}+p+1;\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p;-1\right).$$

The integral (11.26) converges for $b>0$, $c-b>0$, and $c-a-b>0$ and has an analytic continuation on the values $b\le 0$, $c-b\le 0$ as a series

$$F(a,b;c;-1)=\sum_{n=0}^\infty\frac{(a)_n(b)_n(-1)^n}{(c)_n n!}.$$

Using this analytic continuation, we continue the integral

$$\int_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}+p}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}d\tau$$

to values $\alpha \leq n+|\gamma|-2$.

Introducing the notation

$$A_p(\alpha)=\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)p!}\frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+p+1\right)\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p\right)}\times$$
$${}_2F_1\left(-\gamma_1,\frac{\alpha-n-|\gamma|}{2}+p+1;\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p;-1\right)=\frac{K_p(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)},$$

where

$$K_p(\alpha)=\frac{\pi 2^{1-|\gamma|-\alpha}}{p!\sin\left(\frac{\gamma_1+1}{2}\pi\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)}\times$$
$$\frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+p+1\right)\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p\right)}\times$$
$${}_2F_1\left(-\gamma_1,\frac{\alpha-n-|\gamma|}{2}+p+1;\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p;-1\right),$$

we can write

$$A(\alpha,\sigma)=$$
$$\sum_{p=0}^{N-1}F_p(\sigma,\theta)A_p(\alpha)+\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)}\int_0^1\tau^{\frac{\alpha-n-|\gamma|}{2}}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}R_N(\tau)d\tau \tag{11.27}$$

and

$$I_1^\alpha=\sum_{p=0}^{N-1}A_p(\alpha)\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int_0^\delta F_p(\sigma,\theta)\sigma^{\alpha-1}d\sigma+$$
$$\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)}\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int_0^\delta\sigma^{\alpha-1}d\sigma\int_0^1\tau^{\frac{\alpha-n-|\gamma|}{2}}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}R_N(\tau)d\tau=$$

$$A_0(\alpha)\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int\limits_0^\delta F_0(\sigma,\theta)\sigma^{\alpha-1}d\sigma+$$
$$\sum_{p=1}^{N-1}A_p(\alpha)\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int\limits_0^\delta F_p(\sigma,\theta)\sigma^{\alpha-1}d\sigma+$$
$$\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)}\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int\limits_0^\delta\sigma^{\alpha-1}d\sigma\int\limits_0^1\tau^{\frac{\alpha-n-|\gamma|}{2}}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}R_N(\tau)d\tau. \tag{11.28}$$

The most important term in (11.28) is $A_0(\alpha)\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int\limits_0^\delta F_0(\sigma,\theta)\sigma^{\alpha-1}d\sigma$. We show that

$$\lim_{\alpha\to0}A_0(\alpha)\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int\limits_0^\delta F_0(\sigma,\theta)\sigma^{\alpha-1}d\sigma=f(0), \tag{11.29}$$
$$\lim_{\alpha\to0}\sum_{p=1}^{N-1}A_p(\alpha)\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int\limits_0^\delta F_p(\sigma,\theta)\sigma^{\alpha-1}d\sigma=0, \tag{11.30}$$

and

$$\lim_{\alpha\to0}\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)}\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\times$$
$$\int\limits_0^\delta\sigma^{\alpha-1}d\sigma\int\limits_0^1\tau^{\frac{\alpha-n-|\gamma|}{2}}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}R_N(\tau)d\tau=0. \tag{11.31}$$

To prove (11.29), let us consider

$$F_0(\sigma,\theta)A_0(\alpha)=\frac{K_0(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)}f(\sigma b).$$

We have

$$K_0(0)=\frac{\pi 2^{1-|\gamma|}}{\sin\left(\frac{\gamma_1+1}{2}\pi\right)\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(1-\frac{n+|\gamma|}{2}\right)}\frac{\Gamma\left(1-\frac{n+|\gamma|}{2}\right)\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{n+|\gamma'|-\gamma_1}{2}\right)}\times$$

$$
{}_2F_1\left(-\gamma_1, 1-\frac{n+|\gamma|}{2}; \frac{n+|\gamma'|-\gamma_1}{2}; -1\right) =
$$
$$
\frac{\pi 2^{1-|\gamma|}\Gamma(n+|\gamma'|-1)}{\sin\left(\frac{\gamma_1+1}{2}\pi\right)\Gamma\left(\frac{n+|\gamma'|-\gamma_1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\times
$$
$$
{}_2F_1\left(-\gamma_1, 1-\frac{n+|\gamma|}{2}; \frac{n+|\gamma'|-\gamma_1}{2}; -1\right).
$$

Using formula (15.1.21) from [2] of the form

$$
{}_2F_1(a,b;a-b+1;-1) = \frac{\sqrt{\pi}\Gamma(a-b+1)}{2^a\Gamma\left(1+\frac{a}{2}-b\right)\Gamma\left(\frac{a+1}{2}\right)},
$$
$$
a-b+1 \neq 0,-1,-2,...,
$$

we get $a=-\gamma_1$, $b=1-\frac{n+|\gamma|}{2}$, $a-b+1=\frac{n+|\gamma'|-\gamma_1}{2}\neq 0,-1,-2,...,$

$$
{}_2F_1\left(-\gamma_1, 1-\frac{n+|\gamma|}{2}; \frac{n+|\gamma'|-\gamma_1}{2}; -1\right) = \frac{2^{\gamma_1}\sqrt{\pi}\Gamma\left(\frac{n+|\gamma'|-\gamma_1}{2}\right)}{\Gamma\left(\frac{n+|\gamma'|}{2}\right)\Gamma\left(\frac{1-\gamma_1}{2}\right)}
$$

and

$$
K_0(0) = \frac{\pi 2^{1-|\gamma|}\Gamma(n+|\gamma'|-1)}{\sin\left(\frac{\gamma_1+1}{2}\pi\right)\Gamma\left(\frac{n+|\gamma'|-\gamma_1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\frac{2^{\gamma_1}\sqrt{\pi}\Gamma\left(\frac{n+|\gamma'|-\gamma_1}{2}\right)}{\Gamma\left(\frac{n+|\gamma'|}{2}\right)\Gamma\left(\frac{1-\gamma_1}{2}\right)}
$$
$$
= \frac{2^{1-|\gamma'|}\pi\sqrt{\pi}\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{n+|\gamma'|}{2}\right)\sin\left(\frac{\gamma_1+1}{2}\pi\right)\Gamma\left(\frac{1-\gamma_1}{2}\right)\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}.
$$

Taking into account formula (1.5), we obtain

$$
K_0(0) = \frac{2^{1-|\gamma'|}\sqrt{\pi}\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{n+|\gamma'|}{2}\right)\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}. \tag{11.32}
$$

Now we construct an analytic continuation of the expression

$$
\int\limits_0^{\delta} A(\alpha,\sigma)\sigma^{\alpha-1}d\sigma.
$$

The most important term in (11.27) is the term containing $F_0(\sigma,\theta)A_0(\alpha)$. Let us write it in the form

$$A_0(\alpha)\int\limits_0^\delta F_0(\sigma,\theta)\sigma^{\alpha-1}d\sigma=\frac{K_0(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^\delta f(\sigma b)\sigma^{\alpha-1}d\sigma.$$

Factor $K_0(\alpha)$ does not have singularity at $\alpha=0$ and for $\gamma_1\neq 2k-1$, $k\in\mathbb{N}$, can be calculated by (11.32). Integrating $\int\limits_0^\delta f(\sigma b)\sigma^{\alpha-1}d\sigma$ by parts we get

$$\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^\delta f(\sigma b)\sigma^{\alpha-1}d\sigma=\frac{1}{\alpha\Gamma\left(\frac{\alpha}{2}\right)}\left(f(\sigma b)\sigma^{\alpha}\Big|_{\sigma=0}^{\delta}-\int\limits_0^\delta f'_\sigma(\sigma b)\sigma^{\alpha}d\sigma\right)$$
$$=\frac{1}{2\Gamma\left(\frac{\alpha}{2}+1\right)}\left(f(\delta b)\delta^{\alpha}-\int\limits_0^\delta f'_\sigma(\sigma b)\sigma^{\alpha}d\sigma\right).$$

Then

$$\lim_{\alpha\to 0}\frac{K_0(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^\delta f(\sigma b)\sigma^{\alpha-1}d\sigma=\frac{K_0(0)}{2}\left(f(\delta b)-\int\limits_0^\delta f'_\sigma(\sigma b)d\sigma\right)=$$
$$\frac{K_0(0)}{2}\left(f(\delta b)-f(\delta b)+f(0)\right)=\frac{K_0(0)}{2}f(0).$$

Thus, we have

$$\lim_{\alpha\to 0}A_0(\alpha)\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS\int\limits_0^\delta F_0(\sigma,\theta)\sigma^{\alpha-1}d\sigma=\frac{K_0(0)}{2}f(0)\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS.$$

Using formula (1.7), we obtain

$$\frac{K_0(0)}{2}\int\limits_{S_1^+(n-1)}\theta^{\gamma'}dS=\frac{2^{-|\gamma'|}\sqrt{\pi}\,\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{n+|\gamma'|}{2}\right)\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}\,\frac{\prod\limits_{i=2}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-2}\Gamma\left(\frac{n-1+|\gamma'|}{2}\right)}=1,$$

which gives (11.29).

To show (11.30) and (11.31), let us consider $A_p(\alpha)\int\limits_0^\delta F_p(\sigma,\theta)\sigma^{\alpha-1}d\sigma$ for $p>0$:

$$A_p(\alpha)=\frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)p!}\frac{\Gamma\left(p+1-\frac{n+|\gamma|-\alpha}{2}\right)\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p\right)}\times$$

$$
{}_2F_1\left(-\gamma_1, \frac{\alpha-n-|\gamma|}{2}+p+1; \frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p; -1\right) =
$$
$$
\frac{2^{1-|\gamma|-\alpha}\pi}{p!\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(1-\frac{n+|\gamma|-\alpha}{2}\right)} \frac{\Gamma\left(p+1-\frac{n+|\gamma|-\alpha}{2}\right)\Gamma(n+|\gamma'|-1)}{\Gamma\left(\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p\right)}\times
$$
$$
{}_2F_1\left(-\gamma_1, \frac{\alpha-n-|\gamma|}{2}+p+1; \frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p; -1\right).
$$

Applying (1.4),

$$
\Gamma\left(p+1-\frac{n+|\gamma|-\alpha}{2}\right) =
$$
$$
\left(1-\frac{n+|\gamma|-\alpha}{2}\right)\left(2-\frac{n+|\gamma|-\alpha}{2}\right)\ldots
$$
$$
\ldots\left(p-\frac{n+|\gamma|-\alpha}{2}\right)\Gamma\left(1-\frac{n+|\gamma|-\alpha}{2}\right)
$$

and

$$
A_p(\alpha) =
$$
$$
\frac{2^{1-|\gamma|-\alpha}\pi\Gamma(n+|\gamma'|-1)\left(1-\frac{n+|\gamma|-\alpha}{2}\right)\left(2-\frac{n+|\gamma|-\alpha}{2}\right)\ldots\left(p-\frac{n+|\gamma|-\alpha}{2}\right)}{p!\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p\right)}\times
$$
$$
{}_2F_1\left(-\gamma_1, \frac{\alpha-n-|\gamma|}{2}+p+1; \frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p; -1\right) = \frac{K_p(\alpha)}{\Gamma\left(\frac{\alpha}{2}\right)},
$$

where

$$
K_p(\alpha) =
$$
$$
\frac{2^{1-|\gamma|-\alpha}\pi\Gamma(n+|\gamma'|-1)\left(1-\frac{n+|\gamma|-\alpha}{2}\right)\left(2-\frac{n+|\gamma|-\alpha}{2}\right)\ldots\left(p-\frac{n+|\gamma|-\alpha}{2}\right)}{p!\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p\right)}\times
$$
$$
{}_2F_1\left(-\gamma_1, \frac{\alpha-n-|\gamma|}{2}+p+1; \frac{\alpha+n+|\gamma'|-\gamma_1}{2}+p; -1\right).
$$

We obtain that the expression

$$
K_p(0) = \frac{2^{1-|\gamma|}\pi\Gamma(n+|\gamma'|-1)\left(1-\frac{n+|\gamma|}{2}\right)\left(2-\frac{n+|\gamma|}{2}\right)\ldots\left(p-\frac{n+|\gamma|}{2}\right)}{p!\prod_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{n+|\gamma'|-\gamma_1}{2}+p\right)}\times
$$

$$
{}_2F_1\left(-\gamma_1, p+1-\frac{n+|\gamma|}{2}; \frac{n+|\gamma'|-\gamma_1}{2}+p; -1\right)
$$

is finite for $n+|\gamma'|-\gamma_1 \neq -2k$, $k \in \mathbb{N}$. For any entire p we get

$$
\frac{\partial^p}{\partial \tau^p} f(\sigma(b+\tau c)) = \sigma^p \left(\sum_{k=1}^n c^k \partial_k\right)^p f, \tag{11.33}
$$

where $\partial_k = \frac{\partial}{\partial y_k}$, $y = \sigma(b+\tau c)$. Considering (11.33) we note that $\frac{\partial^p}{\partial \tau^p} f(\sigma(b+\tau c))$ has a factor σ^p, $p = 1, 2, ...$ Therefore all integrals

$$
\int\limits_0^\delta F_p(\sigma,\theta)\sigma^{\alpha-1}d\sigma, \qquad p = 1, 2, ...,
$$

and

$$
\int\limits_0^\delta \sigma^{\alpha-1}d\sigma \int\limits_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}R_N(\tau)d\tau
$$

converge for $\alpha > -1$. Taking into account that $K_p(0)$ is a finite number and $\lim \alpha \to 0\frac{1}{\Gamma(\frac{\alpha}{2})} = 0$, (11.30) and (11.31) are proved.

Let us now consider I_2^α:

$$
\begin{aligned}
I_2^\alpha &= \frac{1}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_{K^+\setminus K_\delta^+} r^{\alpha-n-|\gamma|}(y) f(y) y^\gamma dy = \{y' = \rho\theta\} \\
&= \frac{1}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_{K^+\setminus K_\delta^+} \rho^{n+|\gamma'|-2}(y_1^2-\rho^2)^{\frac{\alpha-n-|\gamma|}{2}} f(y_1,\rho\theta)\theta^{\gamma'} y_1^{\gamma_1} dS d\rho dy_1.
\end{aligned}
$$

Passing in the last expression to variables (11.25), we obtain

$$
I_2^\alpha = \frac{2^{1-n-|\gamma|}}{\mathscr{H}_{n,\gamma}(\alpha)} \int\limits_{S_1^+(n-1)} \theta^{\gamma'} dS \int\limits_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}d\tau \times
$$

$$
\int\limits_\delta^\infty \sigma^{\alpha-1} f(\sigma(b+\tau c))d\sigma.
$$

Since $f \in S_{ev}$ and $\delta > 0$, the function $G(\tau,\theta,\alpha) = \int\limits_{\delta}^{\infty} \sigma^{\alpha-1} f(\sigma(b+\tau c))d\sigma$ belongs to S_{ev} by (τ,θ) and is holomorphic by α. Putting

$$\frac{2^{1-|\gamma|-\alpha}\pi}{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}(1+\tau)^{\gamma_1}(1-\tau)^{n+|\gamma'|-2}G(\tau,\theta,\alpha) = W(\tau),$$

we obtain

$$I_2^{\alpha} = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)} \int\limits_{S_1^+(n-1)} \theta^{\gamma'} dS \int\limits_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}} W(\tau)d\tau.$$

The expression $\frac{1}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)} \int\limits_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}} W(\tau)d\tau$, using integration by parts, can be continued analytically as a holomorphic function of α for all $\alpha > \alpha_0$, where α_0 is an arbitrary number. So

$$\frac{1}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)} \int\limits_{S_1^+(n-1)} \theta^{\gamma'} dS \int\limits_0^1 \tau^{\frac{\alpha-n-|\gamma|}{2}} W(\tau)d\tau$$

is a holomorphic function for $\frac{1}{\Gamma\left(\frac{\alpha}{2}\right)}$ and I_2^{α} vanishes for $\alpha \to 0$. Therefore, it is shown that $(I^0_{\Box_\gamma} f)(O) = f(O)$. If we take $g(O) = {}^{\gamma}\mathbf{T}_y^O f(y)$ instead of $f(O)$, we can write $(I^0_{\Box_\gamma,\gamma} f)(x) = f(x)$, which means that $I^0_{\Box_\gamma}$ is an identity operator. $\Box$

Based on the proved theorem and equality (10.28) for hyperbolic Riesz B-potential for $f \in S_{ev}$, $0 < \alpha$ and $k \in \mathbb{N}$, the following formula is valid:

$$(\Box_\gamma)^k I^{\alpha+2k}_{\Box_\gamma} f = I^{\alpha}_{\Box_\gamma} f, \tag{11.34}$$

where $\Box_\gamma = B_{\gamma_1} - \sum\limits_{i=2}^{n} B_{\gamma_i}$.

In addition, when $0 < \alpha$ and $f \in S_{ev}$ such that $x_i^{\gamma_i} \frac{\partial}{\partial x_i} f|_{x_i=0} = 0$, $i = 1, ..., n$, the equality

$$I^{\alpha+2}_{\Box_\gamma} \Box_\gamma f = I^{\alpha}_{\Box_\gamma} f \tag{11.35}$$

is true. If function f such that $x_i^{\gamma_i} \frac{\partial}{\partial x_i} (\Box_\gamma)^j f|_{x_i=0} = 0$, $j = 0, .., m-1$, $i = 1, ..., n$, then the equality

$$I_{\square_\gamma}^{\alpha+2m}(\square_\gamma)^k f = I_{\square_\gamma}^{\alpha} f \tag{11.36}$$

is true (see Theorems 130 and 131).

Due to density S_{ev} in L_p^γ, equalities (11.34)–(11.36) extend to functions from L_p^γ for $1 < p < \frac{n+|\gamma|}{\alpha}$, in the case when the integral $I_{\square_\gamma}^{\alpha} f$ converges absolutely for $f \in L_p^\gamma$.

11.4 The Riesz potential method for solving nonhomogeneous equations of Euler–Poisson–Darboux type

Riesz potentials are generalized convolutions with fractional powers of a certain distance (Euclidean, Lorentz, or other). From the point of view of application, such potentials are tools for solving differential equations of mathematical physics and inverse problems. For example, M. Riesz used such operators to obtain a solution to the Cauchy problem for the wave equation. The modern theory of Radon transforms is based on Riesz potentials. In this section, we use Riesz potentials constructed using generalized convolution to solve wave equations with Bessel operators. First, we describe the general Riesz potential method, introduce solvable equations, and compare each equation with a suitable potential. Then, using the connection of the Riesz hyperbolic B-potentials with the d'Alembert type operators with Bessel operators instead of the second derivatives, we solve some singular hyperbolic equations.

11.4.1 General nonhomogeneous iterated Euler–Poisson–Darboux equation

In this subsection, we consider the nonhomogeneous iterated Euler–Poisson–Darboux equation, with the Bessel operator acting on each of the variables, of the form

$$(\square_{k,\gamma})_{t,x}^m u(x,t) = f(x,t), \qquad u = u(x,t;k), \qquad (x,t) \in \mathbb{R}_+^{n+1}, \qquad m \in \mathbb{N}, \tag{11.37}$$

where

$$(\square_{k,\gamma})_{t,x} = (B_k)_t - (\Delta_\gamma)_x, \qquad (B_k)_t = \frac{\partial^2}{\partial t^2} + \frac{k}{t}\frac{\partial}{\partial t},$$
$$(\Delta_\gamma)_x = \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right),$$

$f(x,t) \in S_{ev}(\mathbb{R}_+^{n+1})$ and $u(x,t;k) \in S_{ev}(\mathbb{R}_+^{n+1})$. If necessary, f can be taken from a wider class of functions, such that the corresponding Riesz B-potential exists and the solution u has the desired properties. To solve Eq. (11.37), we will use the potential

built and studied in Section 10.1. Also, if function f is taken from a wider class of functions, it should be such that $t^k \frac{\partial}{\partial t}(\square_{k,\gamma})^j f|_{t=0}=0$, $x_i^{\gamma_i} \frac{\partial}{\partial x_i}(\square_{k,\gamma})^j f|_{x_i=0}=0$, $j=0,..,m-1$, $i=1,...,n$.

As a potential inverting $(\square_{k,\gamma})^m_{t,x}$ we will use the first hyperbolic B-potential (10.3) having a form (10.4), but with a more suitable constant. Namely, for $n+|\gamma|+k-1<\alpha$,

$$(I^{\alpha}_{\square_{k,\gamma}} f)(x,t)=\frac{1}{H_{n,k,\gamma}(\alpha)}\int\limits_{K^+}(\tau^2-|y|^2)^{\frac{\alpha-n-1-k-|\gamma|}{2}}({}^kT_t^{\tau}\,{}^{\gamma}\mathbf{T}_x^y f(x,t))\tau^k y^{\gamma}\,d\tau dy, \tag{11.38}$$

where $y=(y_1,...,y_n)$, $|y|=\sqrt{\sum\limits_{i=1}^n y_i^2}$, $y^{\gamma}=\prod\limits_{i=1}^n y_i^{\gamma_i}$, $K^+=\{(t,y)\in\mathbb{R}^{n+1}_+ : t^2\geq|y|^2\}$,

$$H_{n,k,\gamma}(\alpha)=$$
$$\frac{2^{\alpha-n-1}}{\pi}\sin\left(\frac{k+1}{2}\pi\right)\Gamma\left(\frac{k+1}{2}\right)\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+1-n-k-|\gamma|}{2}\right),$$

$k\neq 2m-1$, $m\in\mathbb{N}$. It is known that for $f\in S_{ev}$ and $\alpha>n+|\gamma|+k-1$, the integrals $(I^{\alpha}_{\square_{k,\gamma}} f)(x,t)$ converge absolutely for $x\in\mathbb{R}^n_+$, $t>0$. For $0\leq\alpha\leq n+|\gamma|+k-1$ hyperbolic B-potentials $I^{\alpha}_{P\pm i0,\gamma}$ can be defined as

$$(I^{\alpha}_{\square_{k,\gamma}} f)(x,t)=(\square_{k,\gamma})^q_{t,x}(I^{\alpha+2q}_{\square_{k,\gamma}} f)(x,t),$$

where $q=\left[\frac{n+|\gamma|+k-\alpha+1}{2}\right]$.

Applying the hyperbolic B-potential $I^{\alpha+2m}_{\square_{k,\gamma}}$ to both sides of Eq. (11.37), using (11.36), we obtain

$$I^{\alpha+2m}_{\square_{k,\gamma}}(\square_{k,\gamma})^m_{t,x}u(x,t)=I^{\alpha+2m}_{\square_{k,\gamma}} f(x,t)$$

or

$$I^{\alpha}_{\square_{k,\gamma}}u(x,t)=I^{\alpha+2m}_{\square_{k,\gamma}} f(x,t).$$

Now putting $\alpha=0$ and taking into account (11.24), we get a solution to (11.37) in the form

$$u(x,t)=I^{2m}_{\square_{k,\gamma}} f(x,t).$$

This solution is not unique and we should note that for this solutions we have conditions

$$\left.\frac{\partial^{2s}u}{\partial t^{2s}}\right|_{t=0}=\varphi_{2s}(x),\qquad \left.\frac{\partial^{2s-1}u}{\partial t^{2s-1}}\right|_{t=0}=0,\qquad \left.\frac{\partial^{2s-1}u}{\partial x_i^{2s-1}}\right|_{x_i=0}=0,\qquad i=1,...,n, \tag{11.39}$$

where $\varphi_{2s}(x)=\frac{\partial^{2s}}{\partial t^{2s}}(I^{2m}_{\square_{k,\gamma}}f)(x,t)\Big|_{t=0}$, for $s=0,1,...,m-1$.

Let us formulate the result as a theorem.

Theorem 145. *Let $(\square_{k,\gamma})_{t,x}=(B_k)_t-(\Delta_\gamma)_x$. For $f\in S_{ev}(\mathbb{R}^{n+1}_+)$ and $k\in\mathbb{N}$ the expression $u=(I^{2m}_{\square_{k,\gamma}}f)(x,t)f$ is the solution to the iterated equation*

$$(\square_{k,\gamma})^m_{t,x}u=f(x,t),\qquad u=u(x,t;k),\qquad (x,t)\in\mathbb{R}^{n+1}_+,\qquad m\in\mathbb{N},\tag{11.40}$$

such that

$$\frac{\partial^{2s}u}{\partial t^{2s}}\bigg|_{t=0}=\psi_{2s}(x),\qquad \frac{\partial^{2s-1}u}{\partial t^{2s-1}}\bigg|_{t=0}=0,\qquad \frac{\partial^{2s-1}u}{\partial x_i^{2s-1}}\bigg|_{x_i=0}=0,\qquad i=1,...,n,\tag{11.41}$$

where $\psi_{2s}(x)=\frac{\partial^{2s}}{\partial t^{2s}}(I^{2k}_{\square_{k,\gamma}}f)(x)\Big|_{t=0}$, for $s=0,1,...,2k-1$.

11.4.2 Mixed truncated hyperbolic Riesz B-potential

In this subsection we consider the application of the transmutation operator method to solve the equation

$$(\square_{k,\gamma})^m_{t,x}u=f(x,t),\qquad u=u(x,t;k),\qquad (x,t)\in\mathbb{R}^{n+1}_+,\qquad m\in\mathbb{N},\tag{11.42}$$

where

$$(\square_{k,\gamma})_{t,x}=(B_k)_t-(\Delta_\gamma)_x,\quad (B_k)_t=\frac{\partial^2}{\partial t^2}+\frac{k}{t}\frac{\partial}{\partial t},$$
$$(\Delta_\gamma)_x=\sum_{i=1}^n\left(\frac{\partial^2}{\partial x_i^2}+\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right),$$

$f(x,t)\in S_{ev}(\mathbb{R}^{n+1}_+)$, and $u(x,t;k)\in S_{ev}(\mathbb{R}^{n+1}_+)$ with homogeneous conditions

$$\frac{\partial^s u}{\partial t^s}\bigg|_{t=0}=0,\ \frac{\partial^{2l-1}u}{\partial x_i^{2l-1}}\bigg|_{x_i=0}=0,\ i=1,...,n,\ s=0,1,...,2m-1,\ l=1,2,....\tag{11.43}$$

If necessary, f can be taken from a wider class of functions, such that the corresponding solution u will exist and have the desired properties.

As the transmutation operator, we use the Poisson operator (3.120):

$$\mathcal{P}^k_t f(t)=\frac{2^{\frac{1-k}{2}}t^{1-k}}{\Gamma\left(\frac{k+1}{2}\right)}\int\limits_0^t\left(t^2-\tau^2\right)^{\frac{k}{2}-1}f(\tau)\,d\tau,\qquad k>0,$$

with the intertwining property

$$\mathcal{P}_t^k D^2 = B_k \mathcal{P}_t^k.$$

This approach was used in [218,219]. The inverse operator $(\mathcal{P}_t^k)^{-1}$ is defined by (3.124).

First, we construct the potential that gives a solution to the problem

$$\left(\frac{\partial^2 u}{\partial t^2} - (\Delta_\gamma)_x\right)^m_{t,x} u = f(x,t), \quad u = u(x,t;k), \quad (x,t) \in \mathbb{R}^{n+1}_+, \quad m \in \mathbb{N}, \tag{11.44}$$

where

$$(\Box_{k,\gamma})_{t,x} = (B_k)_t - (\Delta_\gamma)_x, \quad (B_k)_t = \frac{\partial^2}{\partial t^2} + \frac{k}{t}\frac{\partial}{\partial t},$$
$$(\Delta_\gamma)_x = \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}\right),$$
$$\left.\frac{\partial^s u}{\partial t^s}\right|_{t=0} = 0, \; \left.\frac{\partial^{2l-1} u}{\partial x_i^{2l-1}}\right|_{x_i=0} = 0, \; i = 1, ..., n, \; s = 0, 1, ..., 2m-1, \; l = 1, 2, ... \tag{11.45}$$

Then using $\mathcal{P}_t^k$ we obtain the solution to (11.42)–(11.43).

For $x \in \mathbb{R}^n_+$, $t > 0$, $\lambda \in C$ we define a function s^λ by the formula

$$\mathbf{S}^\lambda(x,t) = \begin{cases} \frac{(t^2 - |x|^2)^\lambda}{N(\alpha,\gamma,n)} & (t^2 \geq |x|^2) \wedge (t \geq 0), \\ 0 & (t^2 < |x|^2) \vee (t < 0), \end{cases} \tag{11.46}$$

where

$$N(\alpha,\gamma,n) = \frac{2^{\alpha-n-1}}{\sqrt{\pi}} \prod_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right) \Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right) \Gamma\left(\frac{\alpha}{2}\right). \tag{11.47}$$

Consider a mixed truncated hyperbolic Riesz B-potential (a similar operator where the external integral is taken in the range from 0 to ∞ is considered in [504])

$$(I^\alpha_{trun,\gamma} f)(x,t) = \int_0^t d\tau \int_{\{|y|<\tau\}^+} \mathbf{S}^{\frac{\alpha-n-|\gamma|-1}{2}}(\tau,y)(^\gamma\mathbf{T}_x^y) f(x, t-\tau) y^\gamma dy, \tag{11.48}$$

where $\{|y| < \tau\}^+ = \{y \in \mathbb{R}^n_+ : |y| < \tau\}$, $y^\gamma = \prod_{i=1}^n y_i^{\gamma_i}$, and $^\gamma\mathbf{T}_x^y$ is the multi-dimensional generalized translation (3.169). Integral (11.48) converges absolutely

for $n+|\gamma|-1<\alpha$ for the integrable with weight y^γ on the part of the cone $\{|y|<\tau\}^+=\{y\in\mathbb{R}^n:|y|<\tau\}$, $0<\tau<t$, function $f(\tau,y)$ (this is shown similarly to Theorem 127).

For $0\leq\alpha\leq n+|\gamma|-1$,

$$(I^{\alpha}_{trun,\gamma}f)(x,t)=(\Box_{k,\gamma})^q_{t,x}(I^{\alpha+2q}_{trun,\gamma}f)(x,t),$$

where $q=\left[\frac{n+|\gamma|-\alpha+1}{2}\right]$.

Theorem 146. *For function* $f\in S_{ev}$ *such that* $\left.\frac{\partial^s f}{\partial t^s}\right|_{t=0}=0$, $\left.\frac{\partial^{2l-1}f}{\partial x_i^{2l-1}}\right|_{x_i=0}=0$, $i=1,...,n$, $s=0,1,...,2m-1$, $l=1,2,...$, *the following formula is valid:*

$$I^{\alpha+2m}_{trun,\gamma}(\Box_{k,\gamma})^m_{t,x}f=I^{\alpha}_{trun,\gamma}f. \tag{11.49}$$

Proof. Applying formula (3.146) of the form ${}^{\gamma_i}T^{y_i}_{x_i}(B_{\gamma_i})_{x_i}=(B_{\gamma_i})^{\gamma_i}_{x_i}T^{y_i}_{x_i}$ and the fact that $\frac{\partial^2}{\partial t^2}f(x,t-\tau)=\frac{\partial^2}{\partial\tau^2}f(x,t-\tau)$, we obtain

$$I^{\alpha+2}_{trun,\gamma}(\Box_{k,\gamma})_{t,x}f=\left(I^{\alpha+2}_{trun,\gamma}\left(\frac{\partial^2}{\partial t^2}-\Delta_\gamma\right)f\right)(x)=$$
$$\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}s^{\frac{\alpha-n-|\gamma|+1}{2}}(\tau,y)({}^\gamma\mathbf{T}^y_x)\left(\frac{\partial^2}{\partial t^2}-\Delta_\gamma\right)f(x,t-\tau)y^\gamma dy=I_1-I_2,$$

where

$$I_1=\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}\mathbf{S}^{\frac{\alpha-n-|\gamma|+1}{2}}(\tau,y)\frac{\partial^2}{\partial\tau^2}({}^\gamma\mathbf{T}^y_x)f(x,t-\tau)y^\gamma dy,$$
$$I_2=\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}\mathbf{S}^{\frac{\alpha-n-|\gamma|+1}{2}}(\tau,y)\Delta_\gamma({}^\gamma\mathbf{T}^y_x)f(x,t-\tau)y^\gamma dy.$$

Let us consider I_1. Turning to spherical coordinates $y=r\theta$ in I_1, we get

$$I_1=\frac{1}{N(\alpha+2,\gamma,n)}\int\limits_0^t d\tau\int\limits_0^\tau r^{n+|\gamma|-1}(\tau^2-r^2)^{\frac{\alpha-n-|\gamma|+1}{2}}dr\times$$
$$\int\limits_{S_1^+(n)}\frac{\partial^2}{\partial\tau^2}({}^\gamma\mathbf{T}^{r\theta}_x)f(x,t-\tau)\theta^\gamma dS.$$

Let us denote

$$\int\limits_{S_1^+(n)} ({}^{\gamma}\mathbf{T}_x^{r\theta})f(x,t-\tau)\theta^{\gamma}dS = g(r,t-\tau).$$

Replacing the order of integration in I_1, we obtain

$$I_1 = \frac{1}{N(\alpha+2,\gamma,n)}\int\limits_0^t r^{n+|\gamma|-1}dr\int\limits_r^t(\tau^2-r^2)^{\frac{\alpha-n-|\gamma|+1}{2}}\frac{\partial^2}{\partial\tau^2}g(r,t-\tau)d\tau.$$

Integrating by parts twice, we get

$$I_1 = \frac{(\alpha-n-|\gamma|-1)(\alpha-n-|\gamma|+1)}{N(\alpha+2,\gamma,n)}\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}\tau^2(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-3}{2}}\times$$

$$({}^{\gamma}\mathbf{T}_x^y)f(x,t-\tau)y^{\gamma}dy+$$

$$\frac{\alpha-n-|\gamma|+1}{N(\alpha+2,\gamma,n)}\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}({}^{\gamma}\mathbf{T}_x^y)f(x,t-\tau)y^{\gamma}dy.$$

Let us consider I_2:

$$I_2 = \sum_{j=1}^n\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}\mathbf{S}^{\frac{\alpha-n-|\gamma|+1}{2}}(\tau,y)(B_{\gamma_j})_{y_j}({}^{\gamma}\mathbf{T}_x^y)f(x,t-\tau)y^{\gamma}dy.$$

Let $y'=(y_1,...,y_{j-1},y_{j+1},...,y_n)$ and $|y'|=\sqrt{y_1^2+...+y_{j-1}^2+y_{j+1}^2+...+y_n^2}$. For $j=1,...,n$, integrating by parts by y_i we can write

$$\int\limits_0^{\sqrt{\tau^2-|y'|^2}}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|+1}{2}}\left[(B_{\gamma_j})_{y_j}({}^{\gamma}\mathbf{T}_x^y)f(x,t-\tau)\right]y_j^{\gamma_j}dy_j =$$

$$\int\limits_0^{\sqrt{\tau^2-|y'|^2}}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|+1}{2}}\left[\frac{\partial}{\partial y_j}y_j^{\gamma_j}\frac{\partial}{\partial y_j}({}^{\gamma}\mathbf{T}_x^y)f(x,t-\tau)\right]dy_j =$$

$$\left\{u=(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|+1}{2}},\ dv=\left[\frac{\partial}{\partial y_j}y_j^{\gamma_j}\frac{\partial}{\partial y_j}({}^{\gamma}\mathbf{T}_x^y)f(x,t-\tau)\right]dy_j\right\} =$$

$$(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|+1}{2}}\left[y_j^{\gamma_j}\frac{\partial}{\partial y_j}({}^{\gamma}\mathbf{T}_x^y)f(x,t-\tau)\right]_{y_j=0}^{\sqrt{\tau^2-|y'|^2}}+$$

$$(\alpha-n-|\gamma|+1)\int\limits_0^{\sqrt{\tau^2-|y'|^2}} y_j^{\gamma_j+1}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\left[\frac{\partial}{\partial y_j}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)\right]dy_j=$$

$$(\alpha-n-|\gamma|+1)\int\limits_0^{\sqrt{\tau^2-|y'|^2}} y_j^{\gamma_j+1}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\left[\frac{\partial}{\partial y_j}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)\right]dy_j=$$

$$\left\{u=y_j^{\gamma_j+1}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}},\ dv=\left[\frac{\partial}{\partial y_j}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)\right]dy_j\right\}=$$

$$(\alpha-n-|\gamma|+1)y_j^{\gamma_j+1}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)\Big|_{y_j=0}^{\sqrt{\tau^2-|y'|^2}}-$$

$$(\gamma_j+1)(\alpha-n-|\gamma|+1)\int\limits_0^{\sqrt{\tau^2-|y'|^2}} y_j^{\gamma_j}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)dy_j+$$

$$(\alpha-n-|\gamma|-1)(\alpha-n-|\gamma|+1)\times$$

$$\int\limits_0^{\sqrt{\tau^2-|y'|^2}} y_j^{\gamma_j+2}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-3}{2}}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)dy_j=$$

$$-(\gamma_j+1)(\alpha-n-|\gamma|+1)\int\limits_0^{\sqrt{\tau^2-|y'|^2}} y_j^{\gamma_j}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)dy_j+$$

$$(\alpha-n-|\gamma|-1)(\alpha-n-|\gamma|+1)\times$$

$$\int\limits_0^{\sqrt{\tau^2-|y'|^2}} y_j^{\gamma_j+2}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-3}{2}}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)dy_j.$$

Summarizing by j from 1 to n, we get

$$I_2=$$

$$-\frac{\alpha-n-|\gamma|+1}{N(\alpha+2,\gamma,n)}(|\gamma|+n)\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)y^\gamma dy+$$

$$\frac{(\alpha-n-|\gamma|-1)(\alpha-n-|\gamma|+1)}{N(\alpha+2,\gamma,n)}\times$$

$$\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}|y|^2(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-3}{2}}\,({}^\gamma\mathbf{T}_x^y)f(x,t-\tau)y^\gamma dy.$$

Then

$$I_1 - I_2 =$$

$$\frac{(\alpha-n-|\gamma|-1)(\alpha+2)}{N(\alpha+2,\gamma,n)}\int\limits_0^t d\tau \int\limits_{\{|y|<\tau\}^+} (\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}} ({}^\gamma \mathbf{T}_x^y) f(x,t-\tau) y^\gamma dy.$$

Since

$$\frac{(\alpha-n-|\gamma|-1)(\alpha+2)}{N(\alpha+2,\gamma,n)} = \frac{(\alpha-n-|\gamma|-1)(\alpha+2)}{\frac{2^{\alpha-n+1}}{\sqrt{\pi}}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|+3}{2}\right)\Gamma\left(\frac{\alpha}{2}+1\right)} =$$

$$\frac{1}{\frac{2^{\alpha-n-1}}{\sqrt{\pi}}\prod\limits_{i=1}^n \Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)} = \frac{1}{N(\alpha,\gamma,n)},$$

formula (11.49) is proved for $m=1$. Repeating the calculations, we get (11.49) for $m \in \mathbb{N}$. $\square$

11.4.3 Nonhomogeneous general Euler–Poisson–Darboux equation with homogeneous conditions

From Section 11.3 it follows that for suitable function $f(x,t)$ we have $(I^0_{trun,\gamma} f)(x,t) = f(x,t)$. Therefore,

$$v(x,t) = (I^{2m}_{trun,\gamma} F)(x,t),$$

where $\{|y|<\tau\}^+ = \{y \in \mathbb{R}^n_+ : |y| < \tau\}$ is a solution to the mixed problem

$$\left(\frac{\partial^2}{\partial t^2} - (\Delta_\gamma)_x\right)^m v = F(x,t), \qquad x \in \mathbb{R}^n_+, \qquad t>0, \tag{11.50}$$

$$\left.\frac{\partial^s v}{\partial t^s}\right|_{t=0} = 0, \ \left.\frac{\partial^{2l-1} v}{\partial x_i^{2l-1}}\right|_{x_i=0} = 0, \ i=1,...,n, \ s=0,1,...,2m-1, \ l=1,2,... \tag{11.51}$$

for $F \in S_{ev}$. When $n+|\gamma|-1<2m$, we have

$$v(x,t) = (I^{2m}_{trun,\gamma} F)(x,t) = \int\limits_0^t d\tau \int\limits_{\{|y|<\tau\}^+} \mathbf{S}^{\frac{2m-n-|\gamma|-1}{2}}(\tau,y)({}^\gamma \mathbf{T}_x^y) F(x,t-\tau) y^\gamma dy.$$

For $0 \le 2m \le n+|\gamma|-1$,

$$v(x,t) = (I^{2m}_{trun,\gamma} f)(x,t) = (\square_{k,\gamma})^q_{t,x} (I^{2m+2q}_{trun,\gamma} f)(x,t),$$

where $q=\left[\frac{n+|\gamma|-2m+1}{2}\right]$.

When $m=1$, we have

$$v(x,t)=(I^2_{trun,\gamma}F)(x,t)=\int\limits_0^t d\tau\int\limits_{\{|y|<\tau\}^+}\mathbf{S}^{\frac{1-n-|\gamma|}{2}}(\tau,y)({}^\gamma\mathbf{T}^y_x)F(x,t-\tau)y^\gamma dy.$$

Let us construct a solution $u(x,t;k)$ of the problem

$$\begin{aligned}
&(\Box_{k,\gamma})_{t,x}u=f(x,t),\qquad u=u(x,t;k),\quad x\in\mathbb{R}^n_+,\quad t>0\quad f\in S_{ev},\\
&u(x,0;k)=0,\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=0,\ \left.\frac{\partial^{2l-1}u}{\partial x_i^{2l-1}}\right|_{x_i=0}=0,\\
&i=1,...,n,\ s=0,1,...,2m-1,\ l=1,2,\dots.
\end{aligned}\tag{11.52}$$

Application of the one-dimensional Poisson operator (3.120) gives the solution

$$u(x,t;k)=\mathcal{P}^k_t v(x,t),\qquad x\in\mathbb{R}^n_+,\qquad t>0.$$

Here functions f and F connected by the equality

$$F(x,t)=\left(\mathcal{P}^k_t\right)^{-1}f(x,t).\tag{11.53}$$

Introducing the notation $f_1(x,t)=\mathcal{P}^k_t v(x,t)$, we obtain for the iterated equation

$$(\Box_{k,\gamma})^m_{t,x}u=\Box_{k,\gamma}(\Box_{k,\gamma})^{m-1}_{t,x}u=\Box_{k,\gamma}u_1=f(x,t),$$

where $u_1=(\Box_{k,\gamma})^{m-1}_{t,x}u$ and

$$(\Box_{k,\gamma})^{m-1}_{t,x}u=f_1.$$

Next, putting $f_2(x,t)=\mathcal{P}^k_t v_1(x,t)$, where v_1 is a solution to (11.50)–(11.51) where instead of F function F_1 is connected with f_1 by equality (11.53), taking into account (11.43), we get

$$(\Box_{k,\gamma})^{m-2}_{t,x}u=f_2$$

with homogeneous conditions (11.52). Continuing like this further, we obtain the solution to (11.42)–(11.43).

11.4.4 Examples

In this subsection we present some examples of solutions to nonhomogeneous singular wave equations.

Example 1. *First we consider the problem*

$$\big((B_{2\beta})_t-(B_\gamma)_x\big)\,u=t^2 j_{\frac{\gamma-1}{2}}(x),\qquad u=u(x,t;2\beta),\quad x>0,\quad t>0,\quad \gamma>0,$$
$$u(x,0;2\beta)=0,\qquad u_t(x,0;2\beta)=0,\qquad u_x(0,t;2\beta)=0,$$

where $0<2\beta<1$, $0<\gamma<2$, *and* $j_\nu(x)$ *is function* (1.19).

We can find $F(x,t)$ *by formula* (11.53)*:*

$$F(x,t)=\frac{j_{\frac{\gamma-1}{2}}(x)}{\Gamma(1-\beta)}\frac{d}{dt}\int\limits_0^t(t^2-s^2)^{-\beta}s^{2\beta+2}ds=$$
$$\frac{j_{\frac{\gamma-1}{2}}(x)}{\Gamma(1-\beta)}\frac{2\Gamma(1-\beta)\Gamma\left(\beta+\frac{3}{2}\right)}{3\sqrt{\pi}}\frac{d}{dt}t^3=\frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\sqrt{\pi}}t^2 j_{\frac{\gamma-1}{2}}(x).$$

The solution to the problem

$$(D_t^2-(B_\gamma)_x)v(x,t)=\frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\sqrt{\pi}}t^2 j_{\frac{\gamma-1}{2}}(x),$$
$$v(x,0)=0,\qquad v_t(x,0)=0,\qquad v_x(0,t)=0$$

is

$$v(x,t)=\frac{1}{N(2,\gamma,1)}\frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\sqrt{\pi}}\int\limits_0^t(t-\tau)^2d\tau\int\limits_0^\tau(\tau^2-y^2)^{-\frac{\gamma}{2}}\,{}^\gamma T_x^y j_{\frac{\gamma-1}{2}}(x)y^\gamma dy.$$

Taking into account the equality ${}^\gamma T_x^y j_{\frac{\gamma-1}{2}}(x)=j_{\frac{\gamma-1}{2}}(x)j_{\frac{\gamma-1}{2}}(y)$, *we obtain*

$$v(x,t)=\frac{j_{\frac{\gamma-1}{2}}(x)}{N(2,\gamma,1)}\frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\sqrt{\pi}}\int\limits_0^t(t-\tau)^2d\tau\int\limits_0^\tau(\tau^2-y^2)^{-\frac{\gamma}{2}}j_{\frac{\gamma-1}{2}}(y)y^\gamma dy=$$
$$\frac{2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)j_{\frac{\gamma-1}{2}}(x)}{N(2,\gamma,1)}\frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\sqrt{\pi}}\int\limits_0^t(t-\tau)^2d\tau\int\limits_0^\tau(\tau^2-y^2)^{-\frac{\gamma}{2}}J_{\frac{\gamma-1}{2}}(y)y^{\frac{\gamma+1}{2}}dy.$$

Using formula (2.12.4.6) from [456] of the form

$$\int\limits_0^a x^{\nu+1}(a^2-x^2)^{\beta-1}J_\nu(cx)dx=\frac{2^{\beta-1}a^{\beta+\nu}}{c^\beta}\Gamma(\beta)J_{\beta+\nu}(ac),\tag{11.54}$$
$$a>0,\qquad \operatorname{Re}\beta>0,\qquad \operatorname{Re}\nu>-1$$

and the fact that $J_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi}} \frac{\sin(t)}{\sqrt{t}}$, *we obtain*

$$v(x,t) = \frac{2^{\frac{\gamma-1}{2}} \Gamma\left(\frac{\gamma+1}{2}\right) j_{\frac{\gamma-1}{2}}(x)}{N(2,\gamma,1)} \frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\sqrt{\pi}} \frac{2^{\frac{1-\gamma}{2}} \Gamma\left(1-\frac{\gamma}{2}\right)}{\sqrt{\pi}} \int\limits_0^t (t-\tau)^2 \sin\tau d\tau =$$

$$\frac{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(1-\frac{\gamma}{2}\right)}{N(2,\gamma,1)} \frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\pi} j_{\frac{\gamma-1}{2}}(x)[t^2+2\cos(t)-2].$$

Finally,

$$u(x,t;2\beta) = I^{(t)}_{-\frac{1}{2},\beta} v(x,t) =$$

$$\frac{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(1-\frac{\gamma}{2}\right)}{N(2,\gamma,1)} \frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\pi} \frac{2t^{-2(\beta-\frac{1}{2})}}{\Gamma(\beta)} j_{\frac{\gamma-1}{2}}(x) \times$$

$$\int\limits_0^t (t^2-s^2)^{\beta-1}[s^2+2\cos(s)-2]ds =$$

$$\frac{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(1-\frac{\gamma}{2}\right)}{N(2,\gamma,1)} \frac{2\Gamma\left(\beta+\frac{3}{2}\right)}{\pi} \frac{2t^{-2(\beta-\frac{1}{2})}}{\Gamma(\beta)} j_{\frac{\gamma-1}{2}}(x) \times$$

$$\frac{\sqrt{\pi}\Gamma(\beta)t^{2\beta-1}\left(-4\beta+(4\beta+2)\,{}_0F_1\left(;\beta+\frac{1}{2};-\frac{t^2}{4}\right)+t^2-2\right)}{4\Gamma\left(\beta+\frac{3}{2}\right)}.$$

Noting that

$$N(2,\gamma,1) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\gamma+1}{2}\right) \Gamma\left(1-\frac{\gamma}{2}\right),$$

we get

$$u(x,t) = \left(-4\beta+(4\beta+2)\,{}_0F_1\left(;\beta+\frac{1}{2};-\frac{t^2}{4}\right)+t^2-2\right) j_{\frac{\gamma-1}{2}}(x).$$

Let us check the result:

$$(B_{2\beta})_t u(x,t) = j_{\frac{\gamma-1}{2}}(x)(B_{2\beta})_t\left(-4\beta+(4\beta+2)\,{}_0F_1\left(;\beta+\frac{1}{2};-\frac{t^2}{4}\right)+t^2-2\right)=$$

$$-(4\beta+2)\left({}_0F_1\left(;\beta+\frac{1}{2};-\frac{t^2}{4}\right)-1\right) j_{\frac{\gamma-1}{2}}(x),$$

$$\left(-4\beta+(4\beta+2)\,{}_0F_1\left(;\beta+\frac{1}{2};-\frac{t^2}{4}\right)+t^2-2\right)(B_\gamma)_x j_{\frac{\gamma-1}{2}}(x) =$$

$$\left(4\beta-(4\beta+2)\,{}_0F_1\left(;\beta+\frac{1}{2};-\frac{t^2}{4}\right)-t^2+2\right)j_{\frac{\gamma-1}{2}}(x),$$

$$\big((B_{2\beta})_t-(B_\gamma)_x\big)\left(-4\beta+(4\beta+2)\,{}_0F_1\left(;\beta+\frac{1}{2};-\frac{t^2}{4}\right)+t^2-2\right)j_{\frac{\gamma-1}{2}}(x)=$$

$$t^2 j_{\frac{\gamma-1}{2}}(x).$$

Example 2. *Next we consider*

$$\big((B_2)_t-(B_\gamma)_x\big)\,u(x,t)=t^2e^{-t}j_{\frac{\gamma-1}{2}}(x),\qquad (x,t)\in\mathbb{R}^2_+,\qquad \gamma>0,\tag{11.55}$$

$$u(x,0)=3j_{\frac{\gamma-1}{2}}(x),\qquad u_t(x,0)=0,\qquad u_x(0,t)=0.\tag{11.56}$$

The solution to (11.55)–(11.56) *is*

$$u(x,t)=\frac{1}{2}e^{-t}(t^2+3t+3)\,j_{\frac{\gamma-1}{2}}(x).$$

Checking, we obtain

$$(B_2)_t\,\frac{1}{2}e^{-t}(t^2+3t+3)\,j_{\frac{\gamma-1}{2}}(x)=\frac{1}{2}e^{-t}(t^2-3t-3)\,j_{\frac{\gamma-1}{2}}(x),$$

$$(B_\gamma)_x\frac{1}{2}e^{-t}e^{-t}(t^2+3t+3)\,j_{\frac{\gamma-1}{2}}(x)=-\frac{1}{2}e^{-t}(t^2+3t+3)\,j_{\frac{\gamma-1}{2}}(x),$$

and

$$\big((B_2)_t-(B_\gamma)_x\big)\,\frac{1}{2}e^{-t}(t^2+3t+3)\,j_{\frac{\gamma-1}{2}}(x)=t^2e^{-t}j_{\frac{\gamma-1}{2}}(x),$$

$$u(x,0;2)=3j_{\frac{\gamma-1}{2}}(x),\qquad u_t(x,0;2)=0.$$

Example 3. *Let us consider the problem*

$$((B_2)_t-\Delta_\gamma)u=t^2e^{-t}\mathbf{j}_\gamma(x;b),\qquad u=(x,t;2),\qquad (x,t)\in\mathbb{R}^{n+1}_+,$$

$$u(x,0;2)=\frac{3}{2}\mathbf{j}_\gamma(x;b),\qquad u_t(x,0;2)=0,\qquad u_{x_i}(x,0;2)=0,\qquad i=1,...,n,$$

where $\gamma=(\gamma_1,...,\gamma_n)$, $\gamma_1>0,...,\gamma_n>0$, $b=(b_1,...,b_n)$, *and* $\mathbf{j}_\gamma$ *is* (1.30). *The solution is*

$$u=-\mathbf{j}_\gamma(x;b)\frac{1}{2t}\int\limits_0^\infty h(\tau)e^{-\tau}\left(\sin(t+\tau)-\sin|t-\tau|\right)\tau^3d\tau=$$

$$\frac{1}{2}\mathbf{j}_\gamma(x;b)e^{-t}(t^2+3t+3).$$

Checking, we have

$$(B_2)_t\mathbf{j}_\gamma(x;b)e^{-t}(t^2+3t+3)=\frac{1}{2}\mathbf{j}_\gamma(x;b)e^{-t}(t^2-3t-3),$$

$$\Delta_\gamma \mathbf{j}_\gamma(x;b)e^{-t}(t^2+3t+3) = -\frac{1}{2}\mathbf{j}_\gamma(x;b)e^{-t}(t^2+3t+3),$$
$$((B_2)_t - \Delta_\gamma)\mathbf{j}_\gamma(x;b)e^{-t}(t^2+3t+3) = t^2e^{-t}\mathbf{j}_\gamma(x;b).$$

The conditions $u(x,0;2) = \frac{3}{2}\mathbf{j}_\gamma(x;b)$ *and* $u_t(x,0;2) = 0$ *are obvious. The condition* $u_{x_i}(x,t;2)|_{x_i=0} = 0$ *for* $i = 1,...,n$ *follows from the properties of* $\mathbf{j}_\gamma(x;b)$.

Example 4. *Let us consider the problem*

$$((B_2)_t - \Delta_\gamma)u = h(t)\mathbf{j}_\gamma(x;b), \qquad u = u(x,t;2),$$
$$u(x,0;2) = \left(\pi\left(3-\frac{\pi^2}{8}\right)-6\right)\mathbf{j}_\gamma(x;b),$$
$$u_t(x,0;2) = 0, \qquad u_{x_i}(x,0;2) = 0, \qquad i = 1,...,n,$$

where $h(t) = t^2$ *for* $0 \le t \le \frac{\pi}{2}$ *and* $h(t) = 0$ *for* $x_1 > \frac{\pi}{2}$. *Then the solution is*

$$u = -\mathbf{j}_\gamma(x;b)\frac{1}{2t}\int\limits_0^{\frac{\pi}{2}}(\sin(t+\tau)-\sin|t-\tau|)\,\tau^3 d\tau =$$
$$\left(t^2-6+\frac{\pi\sin t}{t}\left(3-\frac{\pi^2}{8}\right)\right)\mathbf{j}_\gamma(x;b).$$

Checking, we get

$$(B_2)_t\left(t^2-6+\frac{\pi\sin t}{t}\left(3-\frac{\pi^2}{8}\right)\right)\mathbf{j}_\gamma(x;b) =$$
$$\left(6+\frac{\pi\sin t}{t}\left(\frac{\pi^2}{8}-3\right)\right)\mathbf{j}_\gamma(x;b),$$
$$\Delta_\gamma\left(t^2-6+\frac{\pi\sin t}{t}\left(3-\frac{\pi^2}{8}\right)\right)\mathbf{j}_\gamma(x;b) =$$
$$-\left(t^2-6+\frac{\pi\sin t}{t}\left(3-\frac{\pi^2}{8}\right)\right)\mathbf{j}_\gamma(x;b),$$
$$((B_2)_t - \Delta_\gamma)\left(t^2-6+\frac{\pi\sin t}{t}\left(3-\frac{\pi^2}{8}\right)\right)\mathbf{j}_\gamma(x;b) = t^2\mathbf{j}_\gamma(x;b).$$

The conditions $u(x,0;2) = \frac{3}{2}\mathbf{j}_\gamma(x;b)$ *and* $u_t(x,0;2) = 0$ *are obvious. The condition* $u_{x_i}(x,t;2)|_{x_i=0} = 0$ *for* $i = 1,...,n$ *follows from the properties of* $\mathbf{j}_\gamma(x;b)$.

Conclusion

12

Transmutation operators theory is an intrinsic part of mathematics used for problem solving, investigation, estimation, numerical analysis, and statistics.

The methods presented in this book, which are mostly applied to problems with the Bessel operator, can be generalized to the case when instead of the Bessel operator we take some other operator L for which the generalized shift operator can be constructed. We present formal algorithms for creating tools for solving problems with the operator L.

We consider the generalization of the translation operator proposed by J. Delsarte in [83] (see also [317]).

If f is a function defined on the real axis, then the shift operator T_x^y, $y \in \mathbb{R}$, is determined by the equality

$$T_x^y f(x) = f(x+y). \tag{12.1}$$

Let now $f \in C^\infty(\mathbb{R})$. The approach of J. Delsarte was to find a generalization of the Taylor formula

$$T_x^y f(x) = f(x+y) = \sum_{n=0}^{\infty} \frac{y^n}{n!} \left(\frac{d}{dx}\right)^n f(x), \tag{12.2}$$

which gives the expansion of the translation operator T_x^y in powers of the differentiation operator $\frac{d}{dx}$. For the translation operator (12.1), using (12.2), J. Delsarte mapped the function $\varphi_n(y) = \frac{y^n}{n!}$ to the differential operator $L_x = \frac{d}{dx}$ in some special sense. Namely, he proceeded from the fact that the solution $\varphi(y,\lambda)$, $y \in \mathbb{R}$, $\lambda \in \mathbb{C}$, to the problem

$$L_y \varphi = \lambda \varphi, \qquad \varphi(0,\lambda) = 1 \tag{12.3}$$

is $\varphi(y,\lambda) = e^{\lambda y}$. For any real y, this function is entire of λ and

$$\varphi(y,\lambda) = \sum_{n=0}^{\infty} \varphi_n(y)\lambda^n \qquad \text{or} \qquad e^{\lambda y} = \sum_{n=0}^{\infty} \frac{y^n}{n!}\lambda^n. \tag{12.4}$$

Functions $\varphi_n(y) = \frac{y^n}{n!}$, $n = 0, 1, 2, \ldots$, satisfy the conditions

$$\begin{aligned} &L_y \varphi_0 = 0, \qquad \varphi_0(1) = 1, \\ &L_y \varphi_n = \varphi_{n-1}, \qquad \varphi_n(0) = 0, \qquad n = 1, 2, \ldots . \end{aligned}$$

According to (12.3), Delsarte generalizes Taylor's formula (9.48) as follows:

$$T_x^y f(x) = \sum_{n=0}^{\infty} \varphi_n(y)(L_x)^n f(x), \tag{12.5}$$

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where L_x is some operator. Since $L_y\varphi_0 = 0$ and $L_y\varphi_n = \varphi_{n-1}$, formally

$$L_y T_x^y f(x) = \sum_{n=0}^{\infty} L_y\varphi_n(y)(L_x)^n f(x) = \sum_{n=1}^{\infty} \varphi_{n-1}(y)(L_x)^n f(x)$$
$$= \sum_{n=0}^{\infty} \varphi_n(y)(L_x)^{n+1} f(x) = L_x T^y f(x),$$

i.e., $T_x^y f(x)$ formally satisfies the equation

$$L_x T_x^y f(x) = L_y T_x^y f(x) \tag{12.6}$$

under initial conditions

$$T_x^y f(x)|_{y=0} = f(x), \qquad \left.\frac{\partial}{\partial y} T_x^y f(x)\right|_{y=0} = 0. \tag{12.7}$$

Delsarte called the operators T_x^y generalized translation operators and established a number of properties for them.

In this book, we have studied in detail the generalized translation operator corresponding to the Bessel operator

$$L_x = (B_\gamma)_x = \frac{\partial^2}{\partial x^2} + \frac{\gamma}{x}\frac{\partial}{\partial x}.$$

However, using this scheme and methods developed in this book, one can construct harmonic analysis for any suitable operator L. In this case, the generalized translation operator can be obtained either by formula (12.5) or as a solution to the problem (12.6)–(12.7). Using this translation, we can introduce a generalized convolution, a generalized spherical mean, and corresponding potentials, and solve problems with the operator L. The integral transformation $\mathscr{F}_L$, convenient for working with expressions containing L, is constructed as an integral operator with the kernel φ satisfying the equation $L\varphi = \lambda\varphi$, the condition $\varphi(0, \lambda) = 1$, and a suitable weight function.

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